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DIFFERENCE BETWEEN THE INTEGRAL MEANS ARISING FROM MONTGOMERY'S IDENTITY AND APPLICATIONS

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Abstract. Some new estimates of the difference between the integral means for convex, s-convex in the second sense, and quasi-convex functions are established. New estimates of errors in approximating probability density function involving general moments are also given.

1. Introduction

In [1], Cerone and Dragomir established an identity of Montgomery to obtain the following result.

Theorem 1. *Let $f : [a, b] \to R$ be an absolutely continuous mapping and as is also $g : [x, y] \to R$ with $[x, y] \subseteq [a, b]$. Then the following inequalities hold,*

$$
(1.1)\quad \left|\int_x^y g(s)f(s)ds - \frac{1}{b-a}\int_a^b f(s)ds\int_x^y g(s)ds\right|
$$

$$
\leq \begin{cases} \frac{\|f'\|_\infty}{(b-a)}\left\{\frac{\int_x^y g(s)ds}{2}\left[(x-a)^2+(b-y)^2\right] + \int_x^y |\phi(s;x,y;g)|ds\right\}, & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(b-a)}\left\{\frac{\left|\int_x^y g(s)ds\right|^q}{q+1}\left[(x-a)^{q+1}+(b-y)^{q+1}\right] + \int_x^y |\phi(s;x,y;g)|^q ds\right\}^{\frac{1}{q}}, & \text{if } f' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{(b-a)}\max\left\{\Theta\int_x^y |g(s)|ds, \sup_{s\in[x,y]}|\phi(s;x,y;g)|\right\}, & \text{if } f' \in L_1[a,b]; \end{cases}
$$

where, for $s \in [a, b]$,

$$
\phi(s;x,y;g) = (s-a)\int_s^y g(u)du - (b-s)\int_x^s g(u)du,
$$

$$
\Theta = \frac{b-a}{2} - \frac{y-x}{2} + \left|\frac{a+b}{2} - \frac{x+y}{2}\right|,
$$

and $\|\cdot\|_p, p \geq 1$ are the usual Lebesgue norms on $[a, b]$. More precisely, $\|g\|_\infty := ess\sup_{t\in[a,b]}|g(t)|$ and $\|g\|_p = \left(\int_a^b |g(s)|^p ds\right)^{\frac{1}{p}}, 1 \leq p < \infty$.

The above results are obtained for a generalized Chebychev functional, see [2, Ch. IX] invoving the integral mean of functions over different intervals. The special

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case of Theorem 1, it produces a generalization of the following Mahajani type inequalities, see [3, p.474]. If f has a bounded derivative on [a,b] and $\int_a^b f(x)dx = 0$ then, for $x \in [a,b]$, the following inequlity holds :

$$\Big| \int_a^x f(t)dt \Big| \leq \frac{(b-a)^2}{8} \|f'\|_\infty.$$

In [4], Fink has given some generalization of Mahajani type inequality as well.

In what follow, we recall the definition of s-convex function in the second sense, usually denoted by K_s^2, that was introduced by Hudzik and Maligranda [5]. This class is defined in the following way : $f : [0,\infty) \to R$ is said to be s-convex function in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0,\infty), \alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. For example, the function $f : [0,1] \to [0,1]$ defined by $f(t) = t^s, s \in (0,1]$, is a s-convex function in the second sense. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [6], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense. Suppose that $f : [0,\infty) \to [0,\infty)$ is a s-convex function in the second sense and $a, b, \in [0,\infty)$ with $a < b$. If $f \in L^1[a,b]$, then the following inequality holds:

$$2^{s-1} f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant is the best possible in the second inequality (1.2).

Recently, Alomari et al. [7] have established some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are s-convex functions in the second sense.

In the following, we recall the definition of quasi-convex functions. In [8], this class is defined in the following way : $f : [a,b] \to R$ is a quasi-convex functions if

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in [a,b], \alpha \in [0,1]$. Clearly, any convex function is a quasi-convex function, and there exist a quasi-convex function which is not convex, see [9].

The purpose of this article is to establish some new results related to the inequality (1.1) for the functions whose absolute value of the first derivative are convex. The corresponding versions in the case that the power of the absolute value of the first derivative is s-convex in the second sense, s-concave in the second sense and quasi-convex, respectively, are also obtained. Applying the obtained results, some new Mahajani type inequalities over any subinterval and some new inequalities for the probability density functions involving moments will be also given.

For convenience, for $a \le x < y \le b$, we use the following notations throughtout this paper:

$$
\begin{aligned}
&A(x,y;g) = \int_x^y g(s)ds;\\
&\phi(s;x,y;g) = (s-a)A(s,y;g) - (b-s)A(x,s;g), s \in [x,y];\\
&I(x,y;g) = \frac{(x-a)^2}{3(b-a)} + \frac{1}{|A(x,y;g)|(b-a)(y-x)} \int_x^y |\phi(s;x,y;g)|(y-s)ds;\\
&J(x,y;g) = \frac{(b-y)^2}{3(b-a)} + \frac{1}{|A(x,y;g)|(b-a)(y-x)} \int_x^y |\phi(s;x,y;g)|(s-x)ds;\\
&K(x,y,p;g) = \frac{y-x}{|A(x,y;g)|}\left(\frac{p+1}{y-x}\int_x^y |\phi(s;x,y;g)|^p ds\right)^{\frac{1}{p}}, p > 1.
\end{aligned}
$$

where $f : [a,b] \to R$ is an absolutely continuous mapping and as is also $g : [x,y] \to R$.

2. Difference between the integral means

Theorem 2. *Let $f : [a,b] \to R$ be an absolutely continuous mapping and as is also $g : [x,y] \to R$ with $[x,y] \subseteq [a,b]$. Then the following inequalities hold,*

$$
(2.1)\quad \Big|\frac{1}{A(x,y;g)}\int_x^y g(s)f(s)ds - \frac{1}{b-a}\int_a^b f(s)ds\Big|
$$

$$
\le \begin{cases}
\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;g)|f'(x)| + J(x,y;g)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|, \\
\qquad \text{if } |f'| \text{ is convex on } [a,b]; \\
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}} \frac{\|f'\|_\infty}{b-a}\big[(x-a)^2 + K(x,y,p;g) + (b-y)^2\big], \\
\qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty), q > 1, \frac{1}{p}+\frac{1}{q} = 1 \ ; \\
\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \frac{2^{\frac{s-1}{q}}}{b-a}\Big[(x-a)^2\big|f'(\frac{a+x}{2})\big| + K(x,y,p;g)\big|f'(\frac{x+y}{2})\big| \\
+(b-y)^2\big|f'(\frac{y+b}{2})\big|\Big], \\
\qquad \text{if } |f'|^q \text{ is s-concave on } [a,b] \subseteq [0,\infty), q > 1, \frac{1}{p}+\frac{1}{q} = 1; \\
\frac{(x-a)^2}{2(b-a)}\big(\max\{|f'(a)|^q, |f'(x)|^q\}\big)^{\frac{1}{q}} + \frac{(b-y)^2}{2(b-a)}\big(\max\{|f'(y)|^q, |f'(b)|^q\}\big)^{\frac{1}{q}} \\
+\frac{1}{|A(x,y;g)|(b-a)}\int_x^y |\phi(s;x,y;g)|ds\big(\max\{|f'(x)|^q, |f'(y)|^q\}\big)^{\frac{1}{q}}, \\
\qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q \ge 1.
\end{cases}
$$

Proof. The following first identity has been obtained in [1]. By suitable substitution of variables, we get the following identities.

$$
\begin{aligned}
(2.2)\quad & \frac{1}{A(x,y;g)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds \\
&=\int_a^x \frac{(x-a)}{b-a}f'(s)ds-\frac{1}{(b-a)A(x,y;g)}\int_x^y \phi(s;x,y;g)f'(s)ds+\int_0^1 \frac{(b-y)}{b-a}f'(s)ds \\
&=\frac{(x-a)^2}{b-a}\int_0^1 tf'((1-t)a+tx)dt \\
&-\frac{y-x}{(b-a)A(x,y;g)}\int_0^1 \phi((1-t)x+ty;x,y;g)f'((1-t)x+ty)dt \\
&+\frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt.
\end{aligned}
$$

Firstly, from (2.2), we obtain

$$
\begin{aligned}
(2.3)\quad & \left|\frac{1}{A(x,y;g)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
&+\frac{y-x}{(b-a)|A(x,y;g)|}\int_0^1 \left|\phi((1-t)x+ty;x,y;g)\right|\cdot\left|f'((1-t)x+ty)\right|dt \\
&+\frac{(b-y)^2}{b-a}\int_0^1 (1-t)\left|f'((1-t)y+tb)\right|dt.
\end{aligned}
$$

Using the convexity of $|f'|$, we get

$$
\begin{aligned}
(2.4)\quad & \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 \Big[t(1-t)|f'(a)|+t^2|f'(x)|\Big]dt \\
&=\frac{(x-a)^2}{6(b-a)}|f'(a)|+\frac{(x-a)^2}{3(b-a)}|f'(x)|,
\end{aligned}
$$

$$\begin{aligned}
(2.5)\qquad & \frac{y-x}{(b-a)|A(x,y;g)|}\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|\cdot\big|f'((1-t)x+ty)\big|dt \\
&\le \frac{y-x}{(b-a)|A(x,y;g)|}\left[\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|(1-t)dt\right)\big|f'(x)\big|\right. \\
&\left.+\left(\int_0^1 \big|\phi((1-t)x+ty;x,y,g)\big|tdt\right)\big|f'(y)\big|\right] \\
&\le \frac{1}{|A(x,y;g)|(b-a)(y-x)}\left(\int_x^y |\phi(s;x,y;g)|(y-s)ds\right)|f'(x)| \\
&+ \frac{1}{|A(x,y;g)|(b-a)(y-x)}\left(\int_x^y |\phi(s;x,y;g)|(s-x)ds\right)|f'(y)|
\end{aligned}$$

and

$$\begin{aligned}
(2.6)\qquad & \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt \\
&\le \frac{(b-y)^2}{b-a}\int_0^1 \Big[(1-t)^2|f'(y)|+(1-t)t|f'(b)|\Big]dt \\
&= \frac{(b-y)^2}{3(b-a)}|f'(y)|+\frac{(b-y)^2}{6(b-a)}|f'(b)|.
\end{aligned}$$

By combining inequalities (2.3), (2.4), (2.5) and (2.6), we obtain the inequality (2.1) for $|f'|$ is convex.

Secondly, continuing (2.3) and using the Hölder inequality, we obtain that

(2.7)

$$\begin{aligned}
&\left|\frac{1}{A(x,y)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds\right| \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)a+tx)|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(y-x)}{(b-a)|A(x,y)|}\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)x+ty)|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(b-y)^2}{b-a}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'((1-t)y+tb)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, using the s-convexity of $|f'|^q$ and $f'\in L_\infty[a,b]$, we get

$$\begin{aligned}
(2.8)\qquad & \int_0^1 |f'((1-t)a+tx)|^q dt \\
&\le \int_0^1 \Big[(1-t)^s|f'(a)|^q+t^s|f'(x)|^q\Big]dt \\
&= \frac{1}{s+1}\Big(|f'(a)|^q+|f'(x)|^q\Big)\le \frac{2\|f'\|_\infty^q}{s+1},
\end{aligned}$$

$$\begin{aligned}
\text{(2.9)} \qquad & \int_0^1 |f'((1-t)x+ty)|^q dt \\
& \leq \int_0^1 \Big[(1-t)^s|f'(x)|^q + t^s|f'(y)|^q\Big] dt \\
& = \frac{1}{s+1}\Big(|f'(x)|^q + |f'(y)|^q\Big) \leq \frac{2\|f'\|_\infty^q}{s+1}
\end{aligned}$$

and

$$\begin{aligned}
\text{(2.10)} \qquad & \int_0^1 |f'((1-t)y+tb)|^q dt \\
& \leq \int_0^1 \Big[(1-t)^s|f'(y)|^q + t^s|f'(b)|^q\Big] dt \\
& = \frac{1}{s+1}\Big(|f'(y)|^q + |f'(b)|^q\Big) \leq \frac{2\|f'\|_\infty^q}{s+1}.
\end{aligned}$$

By simple computation, we obtain

$$\int_0^1 t^p dt = \frac{1}{p+1}, \tag{2.11}$$

$$\int_0^1 (1-t)^p dt = \frac{1}{p+1} \tag{2.12}$$

and

$$\int_0^1 \big|\phi((1-t)x+ty)\big|^p dt = \frac{1}{y-x}\int_x^y |\phi(s)|^p ds, \tag{2.13}$$

and combining inequalities (2.7)-(2.13), we get the inequality (2.1) for $|f'|^q$ is s-convex.

Thirdly, using the s-concavity of $|f'|^q$ and the first inequality of (1.2), we obtain

$$\int_0^1 |f'((1-t)a+tx)|^q dt \leq 2^{s-1}\Big|f'\big(\frac{a+x}{2}\big)\Big|^q, \tag{2.14}$$

$$\int_0^1 |f'((1-t)x+ty)|^q dt \leq 2^{s-1}\Big|f'\big(\frac{x+y}{2}\big)\Big|^q, \tag{2.15}$$

and

$$\int_0^1 |f'((1-t)y+tb)|^q dt \leq 2^{s-1}\Big|f'\big(\frac{y+b}{2}\big)\Big|^q. \tag{2.16}$$

By combining inequalities (2.7), (2.11)-(2.16), we have the inequality (2.1) for the case $|f'|^q$ is s-concave.

Finally, by (2.3), and using Hölder inequality and the quasi-convexity of $|f'|^q$, we have

$$
\begin{aligned}
&\left|\frac{1}{A(x,y)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds\right| \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{\frac{q-1}{q}}\left(\int_0^1 t|f'((1-t)a+tx)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{y-x}{(b-a)|A(x,y;g)|}\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|dt\right)^{\frac{q-1}{q}} \\
&\times\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big||f'((1-t)x+ty)|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-y)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{\frac{q-1}{q}}\left(\int_0^1 |f'((1-t)y+tb)|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)\max\{|f'(a)|^q,|f'(x)|^q\}^{\frac{1}{q}} \\
&+\frac{(b-y)^2}{b-a}\left(\int_0^1 (1-t)dt\right)\max\{|f'(y)|^q,|f'(b)|^q\}^{\frac{1}{q}} \\
&+\frac{y-x}{|A(x,y;g)|(b-a)}\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|dt\right)\max\{|f'(x)|^q,|f'(y)|^q\}^{\frac{1}{q}}.
\end{aligned}
$$

By combining above inequalities and (2.11)-(2.16) for $p=1$, we have the inequality (2.1) for the case $|f'|^q$ is quasi-convex. This completes the proofs of Theorem 2. ∎

Remark 1. *Using the definitions of* $\|f'\|_\infty, I(x,y;g)$ *and* $J(x,y;g)$ *, we obtain*

$$
\begin{aligned}
&\frac{(x-a)^2}{6(b-a)}|f'(a)|+I(x,y;g)|f'(x)|+J(x,y;g)|f'(y)|+\frac{(b-y)^2}{6(b-a)}|f'(b)|, \\
&\le \|f'\|_\infty\left[\frac{(x-a)^2}{6(b-a)}+\frac{(x-a)^2}{3(b-a)}+\frac{1}{|A(x,y;g)|(b-a)(y-x)}\int_x^y|\phi(s;x,y;g)|(y-s)ds\right. \\
&\left.+\frac{(b-y)^2}{3(b-a)}+\frac{1}{|A(x,y;g)|(b-a)(y-x)}\int_x^y|\phi(s;x,y;g)|(s-x)ds+\frac{(b-y)^2}{6(b-a)}\right] \\
&=\frac{\|f'\|_\infty}{(b-a)}\left\{\frac{\int_x^y g(s)ds}{2}\left[(x-a)^2+(b-y)^2\right]+\int_x^y|\phi(s;x,y;g)|ds\right\}.
\end{aligned}
$$

Therefore, for the strick convex mapping, we get the bound in the first inequality of (2.1) *is smaller than the one in the first inequality* (1.1)*. By the similar computation, we also have the bound in the last inequality of* (2.1) *is smaller than the one in the first inequality* (1.1)*.*

The following corollary gives an new estimate for the difference between weighted and unweighted integral means.

Corollary 1. *Assume that hypotheses in Theorem 2 hold. Then we have*

$$\left|\frac{1}{\int_a^b g(u)du}\int_a^b g(u)f(u)du - \frac{1}{b-a}\int_a^b f(u)du\right| \tag{2.17}$$
$$\leq \begin{cases} I(a,b;g)|f'(a)| + J(a,b;g)|f'(b)|, \\ \qquad\qquad\qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;g), \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b]\subseteq[0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\left|K(a,b,p;g)\right|f'(\frac{a+b}{2})\Big|, \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b]\subseteq[0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{|A(a,b;g)|(b-a)}\int_a^b|\phi(s;x,y;g)\big|ds\big(\max\{|f'(x)|^q,|f'(y)|^q\}\big)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1. \end{cases}$$

Proof. Set $x=a$ and $y=b$ in Theorem 2 produces the result (2.17). ∎

The following corollary gives bounds for the difference between the mean of a function compared to its mean over a subinterval.

Corollary 2. *Let $f:[a,b]\to R$ be an absolutely continuous mapping and $a\leq x< y\leq b$. Then we have*

$$\left|\frac{1}{y-x}\int_x^y f(u)du - \frac{1}{b-a}\int_a^b f(u)du\right| \tag{2.18}$$
$$\leq \begin{cases} \frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;1)|f'(x)| + J(x,y;1)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|, \\ \qquad\qquad\qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\Big[(x-a)^2 + K(x,y,p;1) + (b-y)^2\Big], \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b]\subseteq[0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\Big[(x-a)^2\big|f'(\frac{a+x}{2})\big| + K(x,y,p;1)\big|f'(\frac{x+y}{2})\big| \\ +(b-y)^2\big|f'(\frac{y+b}{2})\big|\Big], \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b]\subseteq[0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(x-a)^2}{2(b-a)}\big(\max\{|f'(a)|^q,|f'(x)|^q\}\big)^{\frac{1}{q}} \\ +\frac{(b-y)^2}{2(b-a)}\big(\max\{|f'(y)|^q,|f'(b)|^q\}\big)^{\frac{1}{q}} \\ +\frac{1}{(y-x)(b-a)}\int_x^y|\phi(s;x,y;1)|ds\big(\max\{|f'(x)|^q,|f'(y)|^q\}\big)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1. \end{cases}$$

Proof. Taking $g\equiv 1$ in Theorem 2, we have the (2.18) immediately. ∎

Remark 2. *The type of inequality in* (2.18) *has been applied to probability density functions, special means, Jeffreys divergence in Information Theorem and the sampling of continuous streams in Statistics, see* [10, 11, 12, 13].

Corollary 3. *Assume that the hypotheses in Corollary 2 hold and* $\int_a^b f(u)du = 0$. *Then we have*

$$\left|\int_x^y f(u)du\right| \tag{2.19}$$
$$\leq \begin{cases} (y-x)\Big[\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;1)|f'(x)| + J(x,y;1)|f'(y)| \\ +\frac{(b-y)^2}{6(b-a)}|f'(b)|\Big], \qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ \\ (y-x)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\Big[(x-a)^2 + K(x,y,p;1) \\ +(b-y)^2\Big], \text{if } |f'|^q \in K_s^2 \text{ on } [a,b]\subseteq[0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}(y-x)}{b-a}\Big[(x-a)^2\left|f'(\frac{a+x}{2})\right| + K(x,y,p;1) \\ \times\left|f'(\frac{x+y}{2})\right| + (b-y)^2\left|f'(\frac{y+b}{2})\right|\Big], \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b]\subseteq[0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ (y-x)\Big[\frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|^q, |f'(x)|^q\}^{\frac{1}{q}} \\ +\frac{(b-y)^2}{2(b-a)}\max\{|f'(y)|^q, |f'(b)|^q\}^{\frac{1}{q}}\Big] \\ +\frac{1}{(b-a)}\int_x^y|\phi(s;x,y;1)|ds\big(\max\{|f'(x)|^q, |f'(y)|^q\}\big)^{\frac{1}{q}}, \\ \qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1. \end{cases}$$

Proof. From Corollary 2, putting $\int_a^b f(t)dt = 0$ and multiplying both sides by $y-x$, we have the desire inequality (2.19) immediately. ∎

Remark 3. *The inequality in* (2.19) *is Mahajani type inequality over any subinterval, and if* $\int_a^b f(t)dt = 0$ *in* (2.1) *and* (2.17) *then they may be looked upon as weighted Mahajani type inequalities over arbitrary subintervals.*

3. Applications involving moments

In this section we investigate inequalities involving moments.

Theorem 3. *Let $f : [a,b] \to R$ be an absolutely continuous mapping, $\gamma \in R$ and $[x,y] \subseteq [a,b]$. Then the following inequalities hold,*

$$\left|\int_x^y (t-\gamma)^n f(t)dt - \frac{1}{b-a}\int_a^b f(s)ds \int_x^y (t-\gamma)^n dt\right| \tag{3.1}$$

$$\leq \begin{cases} A(x,y;(t-\gamma)^n)\Big[\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;(t-\gamma)^n)|f'(x)| \\ +J(x,y;(t-\gamma)^n)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|\Big], \\ \qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ A(x,y;(t-\gamma)^n)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\Big[(x-a)^2 \\ +K(x,y,p;(t-\gamma)^n) + (b-y)^2\Big], \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\ A(x,y;(t-\gamma)^n)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\Big[(x-a)^2\big|f'(\frac{a+x}{2})\big| \\ +K(x,y,p;(t-\gamma)^n)\big|f'(\frac{x+y}{2})\big| + (b-y)^2\big|f'(\frac{y+b}{2})\big|\Big], \\ \qquad \text{if } |f'|^q \text{ is s-concave on } [a,b] \subseteq [0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(x-a)^2}{2(b-a)}\Big(\max\{|f'(a)|^q, |f'(x)|^q\}\Big)^{\frac{1}{q}} \\ +\frac{(b-y)^2}{2(b-a)}\Big(\max\{|f'(y)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}} \\ +\frac{1}{(b-a)}\int_x^y |\phi(s;x,y;(t-\gamma)^n)|ds \\ \times\Big(\max\{|f'(x)|^q, |f'(y)|^q\}\Big)^{\frac{1}{q}}, \\ \qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q \geq 1. \end{cases}$$

Proof. Taking $g(t) = (t-\gamma)^n$ in (2.1) and multiplying both sides by $A(x,y;(t-\gamma)^n)$, we have the desire results. ∎

Corollary 4. *Let $f : [a,b] \to R$ be an absolutely continuous mapping, $\gamma \in R$ and $[x,y] \subseteq [a,b]$. Then we have*

$$\left|\int_a^b (t-\gamma)^n f(t)dt - \frac{1}{b-a}\int_a^b f(s)ds \int_a^b (t-\gamma)^n dt\right| \tag{3.2}$$

$$\leq \begin{cases} A(a,b;(t-\gamma)^n)\big[I(a,b;(t-\gamma)^n)|f'(a)| \\ +J(a,b;(t-\gamma)^n)|f'(b)|\big], \qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ A(a,b;(t-\gamma)^n)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;(t-\gamma)^n), \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\ A(a,b;(t-\gamma)^n)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}| + K(a,b,p;(t-\gamma)^n)\Big|f'(\frac{a+b}{2})\Big|, \\ \qquad \text{if } |f'|^q \text{ is s-concave on } [a,b] \subseteq [0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(b-a)}\int_a^b |\phi(s;a,b;(t-\gamma)^n)|ds\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \\ \qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q \geq 1. \end{cases}$$

Proof. Taking $x = a$ and $y = b$ in Theorem 3, inequality (3.1) reduce to inequality (3.2) obviously. ∎

Remark 4. *The Theorem 3 and Corollary 4 give the new estimates of errors for the general moments over a interral and subinterval, and we note that the first and the last inequality in* (3.2) *are better than the first inequality of (3.10) in* [1].

Corollary 5. *Let* $f : [a, b] \to R$ *be an absolutely continuous p.d.f. associated with a random variable* X. *Then the expectation* $E(X)$ *satisfies the inequalities*

$$\left|E(X) - \frac{a+b}{2}\right| \tag{3.3}$$

$$\leq \begin{cases} \frac{b^2-a^2}{2}\Big[I(a,b;t)|f'(a)| + J(a,b;t)|f'(b)|\Big], \\ \qquad\qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ \frac{b^2-a^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;t), \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b^2-a^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a} + K(a,b,p;t)\left|f'(\frac{a+b}{2})\right|, \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b] \subseteq [0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{b-a}\int_a^b |\phi(s;a,b;t)|ds\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1. \end{cases}$$

Proof. Taking $n = 1$ and $\gamma = 0$ in Corollary 4, gives (3.3). ∎

Remark 5. *The Corollary 5 give a new estimates of errors for expectation, and the bound of the first and the last inequality in* (3.3) *are both smaller than one in the first inequality of (3.22) in* [1].

Corollary 6. *Let* $f : [a, b] \to R$ *be an absolutely continuous p.d.f. associated with a random variable* X. *Then the variance* $\sigma^2(X)$ *satisfies the inequalities*

$$\left|\sigma^2(X) - \frac{(b-E(X))^3 - (a-E(X))^3}{3(b-a)}\right| \tag{3.4}$$

$$\leq \begin{cases} A(a,b;(t-E(X)^2)\big[I(a,b;(t-E(X))^2)|f'(a)| \\ +J(a,b;(t-E(X))^2)|f'(b)|\big], \qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ A(a,b;(t-E(X)^2)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;(t-E(X))^2), \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ A(a,b;(t-E(X)^2)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}K(a,b,p;(t-E(X))^2)\left|f'(\frac{a+b}{2})\right|, \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b] \subseteq [0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{b-a}\int_a^b |\phi(s;a,b;(t-E(X))^2)|ds\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \\ \qquad\qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1. \end{cases}$$

Proof. Taking $n = 2$ and $\gamma = E(X)$ in Corollary 4, gives (3.4). ∎

Remark 6. *The* (3.4) *give a new estimates of errors for variance, and the first and the last inequality in* (3.4) *are both better than the first inequality of (3.23) in* [1].

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