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NEW INEQUALITIES FOR THE p-ANGULAR DISTANCE IN NORMED SPACES WITH APPLICATIONS

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Abstract. For nonzero vectors x and y in the normed linear space $(X, \|\cdot\|)$ we can define the *p-angular distance* by

$$\alpha_p[x,y] := \left\| \|x\|^{p-1} x - \|y\|^{p-1} y \right\|.$$

In this paper we show among others that for $p \geq 2$

$$\begin{aligned}\alpha_p[x,y] &\leq p\|y-x\| \int_0^1 \|(1-t)x+ty\|^{p-1} dt \\ &\leq p\|y-x\| \left[\frac{\|x\|^{p-1} + \|y\|^{p-1}}{2} + \left\| \frac{x+y}{2} \right\|^{p-1} \right] \\ &\leq p\|y-x\| \frac{\|x\|^{p-1}+\|y\|^{p-1}}{2} \leq p\|y-x\| [\max\{\|x\|,\|y\|\}]^{p-1},\end{aligned}$$

for any $x, y \in X$. This improve a result of Maligranda from [Simple norm inequalities, *Amer. Math. Month.* **113**(2006), 256-260] in which he proved the inequality between the first and last term above.

Applications for functions f defined by power series in estimating the more general "distance" $\|f(\|x\|)x - f(\|y\|)y\|$ for certain $x, y \in X$ are provided as well.

1. Introduction

Following [2, p. 403] or [11], for nonzero vectors x and y in the normed linear space $(X, \|\cdot\|)$ we define the *angular distance* $\alpha[x,y]$ between x and y by

$$\alpha[x,y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

In 1958, Massera and Schäffer [11, Lemma 5.1] showed that

$$\alpha[x,y] \leq \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}}, \tag{1.1}$$

which is better than the *Dunkl-Williams inequality* [6]

$$\alpha[x,y] \leq \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1.2}$$

We notice that the *Massera-Schäffer inequality* was rediscovered by Gurariĭ in [7] (see also [12, p. 516]).

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In [10], Maligranda obtained the double inequality

$$\frac{\|x-y\|-|\|x\|-\|y\||}{\min\{\|x\|,\|y\|\}}\leq\alpha[x,y]\leq\frac{\|x-y\|+|\|x\|-\|y\||}{\max\{\|x\|,\|y\|\}}. \tag{1.3}$$

The second inequality in (1.3) is better than Massera-Schäffer's inequality (1.1).

In the recent paper [10], L. Maligranda has also considered the *p-angular distance*

$$\alpha_p[x,y]:=\left\|\|x\|^{p-1}x-\|y\|^{p-1}y\right\|$$

between the vectors x and y in the normed linear space $(X,\|.\|)$ over the real or complex number field $\mathbb{K}$ and showed that

$$\alpha_p[x,y]\leq\|x-y\| \times\begin{cases}(2-p)\cdot\frac{\max\{\|x\|^p,\|y\|^p\}}{\max\{\|x\|,\|y\|\}} & \text{if } p\in(-\infty,0)\text{ and } x,y\neq 0;\\ (2-p)\cdot\frac{1}{[\max\{\|x\|,\|y\|\}]^{1-p}} & \text{if } p\in[0,1]\text{ and } x,y\neq 0;\\ p\cdot[\max\{\|x\|,\|y\|\}]^{p-1} & \text{if } p\in(1,\infty).\end{cases} \tag{1.4}$$

The constants $2-p$ and p in (1.1) are best possible in the sense that they cannot be replaced by smaller quantities.

As pointed out in [10], the inequality (1.1) for $p\in[1,\infty)$ is better than the Bourbaki inequality obtained in 1965, [1, p. 257] (see also [12, p. 516]):

$$\alpha_p[x,y]\leq 3p\|x-y\|[\|x\|+\|y\|]^{p-1},\quad x,y\in X. \tag{1.5}$$

The following results concerning upper bounds for the p-angular distance have been obtained by the author in [4]:

$$\alpha_p[x,y]\leq\begin{cases}\|x-y\|[\max\{\|x\|,\|y\|\}]^{p-1}+\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\min\{\|x\|,\|y\|\} \\ \qquad\text{if } p\in(1,\infty);\\ \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}}+\left|\|x\|^{1-p}-\|y\|^{1-p}\right|\min\left\{\frac{\|x\|^p}{\|y\|^{1-p}},\frac{\|y\|^p}{\|x\|^{1-p}}\right\} \\ \qquad\text{if } p\in[0,1];\\ \frac{\|x-y\|}{[\min\{\|x\|,\|y\|\}]^{1-p}}+\frac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\max\left\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\right\}} \quad \text{if } p\in(-\infty,0);\end{cases} \tag{1.6}$$

and

$$\alpha_p[x,y]\leq\begin{cases}\|x-y\|[\min\{\|x\|,\|y\|\}]^{p-1}+\left|\|x\|^{p-1}-\|y\|^{p-1}\right|\max\{\|x\|,\|y\|\} \\ \qquad\text{if } p\in(1,\infty);\\ \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}}+\left|\|x\|^{1-p}-\|y\|^{1-p}\right|\max\left\{\frac{\|x\|^p}{\|y\|^{1-p}},\frac{\|y\|^p}{\|x\|^{1-p}}\right\} \\ \qquad\text{if } p\in[0,1];\\ \frac{\|x-y\|}{[\max\{\|x\|,\|y\|\}]^{1-p}}+\frac{\left|\|x\|^{1-p}-\|y\|^{1-p}\right|}{\min\left\{\|x\|^{-p}\|y\|^{1-p},\|y\|^{-p}\|x\|^{1-p}\right\}} \quad \text{if } p\in(-\infty,0);\end{cases} \tag{1.7}$$

for any two nonzero vectors x, y in the normed linear space $(X, \|.\|)$.

The upper bounds for $\alpha_p[x,y]$ provided by (1.4), (1.6) and (1.7) have been compared in [4] to conclude that some of the later ones are better in certain cases. The details are omitted here.

The following result which provides a lower bound for the *p-angular distance* was stated without a proof by Gurariĭ in [7] (see also [12, p. 516]):

$$2^{-p}\|x-y\|^p \leq \alpha_p[x,y] \tag{1.8}$$

where $p \in [1,\infty)$ and $x, y \in X$. The proof of the inequality (1.8) is still an open question for the author.

Finally, we recall the results of G. N. Hile from [3]:

$$\alpha_p[x,y] \leq \frac{\|x\|^p - \|y\|^p}{\|x\| - \|y\|} \cdot \|x-y\|, \tag{1.9}$$

for $p \in [1,\infty)$ and $x, y \in X$ with $\|x\| \neq \|y\|$, and

$$\alpha_{-p-1}[x,y] \leq \frac{\|x\|^p - \|y\|^p}{\|x\| - \|y\|} \cdot \frac{\|x-y\|}{\|x\|^p\|y\|^p}, \tag{1.10}$$

for $p \in [1,\infty)$ and $x, y \in X \setminus \{0\}$ with $\|x\| \neq \|y\|$.

2. Integral Bounds for p-Angular Distance

The following result holds.

Theorem 1. *Let $(X; \|\cdot\|)$ be a normed linear space and $p \geq 1$. Then for any $x, y \in X$ we have the inequality*

$$\alpha_p[x,y] \leq p\|y-x\| \int_0^1 \|(1-t)x + ty\|^{p-1} dt. \tag{2.1}$$

If the vectors $x, y \in X$ are linearly independent and $p < 1$, then we have the inequality

$$\alpha_p[x,y] \leq (2-p)\|y-x\| \int_0^1 \|(1-t)x + ty\|^{p-1} dt. \tag{2.2}$$

Proof. Assume that $x \neq y$. For $p \geq 2$, consider the function $f_p : [0,1] \to [0,\infty)$ given by $f_p(t) = \|(1-t)x + ty\|^{p-1}$. The function f_p is convex on the interval $[0,1]$ for all $p \geq 2$. Therefore the lateral derivatives f'_{p+} and f'_{p-} exist on each point of the interval $[0,1)$ and $(0,1]$, respectively, and they are equal except a countably number of points in the interval $(0,1)$. The function f_p is absolutely continuos on $[0,1]$, the derivative f'_p exists almost everywhere on $[0,1]$ and

$$f'_p(t) = (p-1)\|(1-t)x + ty\|^{p-2}\tau_{+(-)}((1-t)x + ty, y-x) \tag{2.3}$$

almost everywhere on $[0,1]$, where the *tangent functional* $\tau_{+(-)}$ is defined by

$$\tau_{+(-)}(u,v) := \begin{cases} \lim_{s\to 0+(-)} \frac{\|u+sv\|-\|u\|}{s} & \text{if } u \neq 0, \\ \\ +(-)\|v\| & \text{if } u = 0. \end{cases} \tag{2.4}$$

Now, if we consider the vector valued function $g_p : [0,1] \to X$ given by

$$g_p(t) := f_p(t)[(1-t)x + ty]$$

then we observe that g_p is strongly differentiable almost everywhere on $[0,1]$ and

$$\begin{aligned} g_p'(t) &= f_p'(t)\left[(1-t)x+ty\right]+f_p(t)(y-x) \\ &= (p-1)\left\|(1-t)x+ty\right\|^{p-2}\tau_{+(-)}\left((1-t)x+ty,y-x\right) \\ &\times\left[(1-t)x+ty\right]+\left\|(1-t)x+ty\right\|^{p-1}(y-x) \end{aligned}$$

for almost every $t\in[0,1]$.

Since for any $u,v\in H$ with $u\neq 0$ we have

$$\left|\tau_{+(-)}(u,v)\right|\leq\|v\|$$

then

$$\begin{aligned} \left\|g_p'(t)\right\| &\leq (p-1)\left\|(1-t)x+ty\right\|^{p-1}\left|\tau_{+(-)}\left((1-t)x+ty,y-x\right)\right| \\ &+\left\|(1-t)x+ty\right\|^{p-1}\|y-x\| \\ &\leq (p-1)\left\|(1-t)x+ty\right\|^{p-1}\|y-x\|+\left\|(1-t)x+ty\right\|^{p-1}\|y-x\| \\ &= p\left\|(1-t)x+ty\right\|^{p-1}\|y-x\| \end{aligned}$$

for almost every $t\in[0,1]$.

By the norm inequality for the vector-valued integral we have

$$\begin{aligned} \left\|\|y\|^{p-1}y-\|x\|^{p-1}x\right\| &= \left\|g_p(1)-g_p(0)\right\| \\ &= \left\|\int_0^1 g_p'(t)\,dt\right\|\leq\int_0^1\left\|g_p'(t)\right\|dt \\ &\leq p\|y-x\|\int_0^1\left\|(1-t)x+ty\right\|^{p-1}dt \end{aligned}$$

and the proof of (2.1) is complete.

Let $p\in(1,2)$. The function $f_p:[0,1]\to[0,\infty)$ given by $f_p(t)=\left\|(1-t)x+ty\right\|^{p-1}$ is absolutely continuous on $[0,1]$ and the equality (2.3) also holds almost everywhere on $[0,1]$. The above argument can then be extended to this case as well and the inequality (2.1) also holds.

If the vectors $x,y\in X$ are linearly independent and $p<1$ then $\left\|(1-t)x+ty\right\|>0$ for any $t\in[0,1]$ and the function $h_p:[0,1]\to[0,\infty)$ given by $h_p(t)=\left\|(1-t)x+ty\right\|^{p-1}$ is absolutely continuous on $[0,1]$ and

$$h_p'(t)=(p-1)\left\|(1-t)x+ty\right\|^{p-2}\tau_{+(-)}\left((1-t)x+ty,y-x\right) \tag{2.5}$$

almost everywhere on $[0,1]$.

If we consider the vector valued function $m_p:[0,1]\to X$ given by

$$m_p(t):=h_p(t)\left[(1-t)x+ty\right]$$

then we observe that m_p is strongly differentiable almost everywhere on $[0,1]$ and

$$\begin{aligned} m_p'(t) &= h_p'(t)\left[(1-t)x+ty\right]+h_p(t)(y-x) \\ &= (p-1)\left\|(1-t)x+ty\right\|^{p-2}\tau_{+(-)}\left((1-t)x+ty,y-x\right) \\ &\times\left[(1-t)x+ty\right]+\left\|(1-t)x+ty\right\|^{p-1}(y-x) \end{aligned}$$

for almost every $t\in[0,1]$.

As above we have

$$\begin{aligned}\left\|m_p'(t)\right\| &\leq (1-p)\left\|(1-t)x+ty\right\|^{p-1}\left\|y-x\right\| \\ &+\left\|(1-t)x+ty\right\|^{p-1}\left\|y-x\right\| \\ &=(2-p)\left\|(1-t)x+ty\right\|^{p-1}\left\|y-x\right\|\end{aligned}$$

for almost every $t\in[0,1]$, which implies the desired inequality (2.2). □

Remark 1. *If the vectors x and y are linearly dependent and $y=\lambda x$ with $\lambda\in\mathbb{K}$, then the p-angular distance between x and y reduces to*

$$\alpha_p[x,y]=\|x\|^p\left|1-|\lambda|^{p-1}\lambda\right|=\|x\|^p\alpha_p[1,\lambda].$$

The study of $\alpha_p[1,\lambda]$ with $\lambda\in\mathbb{K}$ may be done in a similar way, however the details are omitted.

Remark 2. *If $p\geq 2$, then the function $f_p:[0,1]\to[0,\infty)$ given by $f_p(t)=\|(1-t)x+ty\|^{p-1}$ is convex and by the Hermite-Hadamard type inequality for the convex function $g:[a,b]\to\mathbb{R}$ (see for instance*

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(s)\,ds &\leq \frac{1}{2}\left[\frac{g(a)+g(b)}{2}+g\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{g(a)+g(b)}{2}\leq\max\{g(a),g(b)\}\end{aligned}\tag{2.6}$$

we have the following chain of inequalities

$$\begin{aligned}\alpha_p[x,y] &\leq p\|y-x\|\int_0^1\|(1-t)x+ty\|^{p-1}dt \\ &\leq p\|y-x\|\left[\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}+\left\|\frac{x+y}{2}\right\|^{p-1}\right] \\ &\leq p\|y-x\|\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\leq p\|y-x\|\left[\max\{\|x\|,\|y\|\}\right]^{p-1},\end{aligned}\tag{2.7}$$

which provides a refinement of Maligranda's inequality (1.4).

If $p\geq 1$ and since, by the triangle inequality we have

$$\|(1-t)x+ty\|\leq(1-t)\|x\|+t\|y\|,$$

then

$$\|(1-t)x+ty\|^{p-1}\leq[(1-t)\|x\|+t\|y\|]^{p-1}$$

for any $t\in[0,1]$. Integrating on $[0,1]$ we get

$$\begin{aligned}\int_0^1\|(1-t)x+ty\|^{p-1}dt &\leq \int_0^1[(1-t)\|x\|+t\|y\|]^{p-1}dt \\ &= \frac{1}{p}\frac{\|y\|^p-\|x\|^p}{\|y\|-\|x\|}\end{aligned}$$

if $\|y\|\neq\|x\|$, and by (2.1) we obtain the chain of inequalities

$$\begin{aligned}\alpha_p[x,y] &\leq p\|y-x\|\int_0^1\|(1-t)x+ty\|^{p-1}dt \\ &\leq \frac{\|y\|^p-\|x\|^p}{\|y\|-\|x\|}\|y-x\|,\end{aligned}\tag{2.8}$$

which provides a refinement of Hile's inequality (1.9).

For $p \geq 2$, by the Hermite-Hadamard's type inequalities (2.6) we also have

$$\begin{aligned}\frac{1}{p}\frac{\|y\|^p-\|x\|^p}{\|y\|-\|x\|} &= \int_0^1 [(1-t)\|x\|+t\|y\|]^{p-1}dt \\ &\leq \frac{1}{2}\left[\left(\frac{\|x\|+\|y\|}{2}\right)^{p-1}+\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\right] \\ &\leq \frac{\|x\|^{p-1}+\|y\|^{p-1}}{2} \leq [\max\{\|x\|,\|y\|\}]^{p-1}\end{aligned}$$

which implies the following sequence of inequalities

$$\begin{aligned}\alpha_p[x,y] &\leq p\|y-x\|\int_0^1 \|(1-t)x+ty\|^{p-1}dt \\ &\leq \frac{\|y\|^p-\|x\|^p}{\|y\|-\|x\|}\|y-x\| \\ &\leq \frac{1}{2}p\|y-x\|\left[\left(\frac{\|x\|+\|y\|}{2}\right)^{p-1}+\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2}\right] \\ &\leq p\|y-x\|\frac{\|x\|^{p-1}+\|y\|^{p-1}}{2} \leq p\|y-x\|[\max\{\|x\|,\|y\|\}]^{p-1}\end{aligned} \tag{2.9}$$

for $\|y\| \neq \|x\|$ and $p \geq 2$.

In particular, the inequality (2.9) shows that in the case $p \geq 2$, Hiles's inequality (1.9) is better than Maligranda's inequality (1.4).

Remark 3. *The case $p = 0$ is of interest, since by (2.2) we have the following upper bound for the angular distance $\alpha[x,y]$*

$$\alpha[x,y] \leq 2\|y-x\|\int_0^1 \|(1-t)x+ty\|^{-1}dt, \tag{2.10}$$

provided the vectors x and y are linearly independent.

Since for any $t \in [0,1]$

$$\begin{aligned}\|(1-t)x+ty\| &= \|x-t(x-y)\| \geq |\|x\|-t\|x-y\|| \\ &\geq \|x\|-t\|x-y\| \geq \|x\|\end{aligned}$$

and, similarly

$$\|(1-t)x+ty\| \geq \|y\|$$

then we have

$$\|(1-t)x+ty\| \geq \max\{\|x\|,\|y\|\},$$

which implies that

$$\int_0^1 \|(1-t)x+ty\|^{-1}dt \leq \frac{1}{\max\{\|x\|,\|y\|\}}. \tag{2.11}$$

Therefore, we have the following refinement of the Massera-Schäffer's inequality (1.1)

$$\alpha[x,y] \leq 2\|y-x\|\int_0^1 \|(1-t)x+ty\|^{-1}dt \leq \frac{2\|y-x\|}{\max\{\|x\|,\|y\|\}}.$$

Remark 4. *In [8], the authors introduced the concept of p-HH-norm on the Cartesian product of two copies of a normed space, namely*

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)\,x + ty\|^p \,dt\right)^{1/p},$$

where $(x,y) \in X \times X := X^2$ and $p \geq 1$. They showed that $\|\cdot\|_{p-HH}$ is a norm on X^2 equivalent with the usual p-norms

$$\|(x,y)\|_p := \left(\|x\|^p + \|y\|^p\right)^{1/p}.$$

They also showed that completeness, reflexivity, smoothness, strict convexity etc. is inherited by X^2 with this norm.

In [9] the authors proved the following interesting lower bound for $\|(x,y)\|_{p-HH}$

$$\left(\frac{\|x\|^p + \|y\|^p}{2\,(p+1)}\right)^{1/p} \leq \|(x,y)\|_{p-HH} \tag{2.12}$$

for any $(x,y) \in X^2$ and $p \geq 1$.

Now, we observe that, by (2.1) we also have

$$\alpha_{p+1}\,[x,y] \leq (p+1)\,\|y-x\|\,\|(x,y)\|_{p-HH}^p. \tag{2.13}$$

for any $(x,y) \in X^2$ and $p \geq 1$.

For $x \neq y$ this is equivalent with

$$\left(\frac{\|\|x\|^p\,x - \|y\|^p\,y\|}{(p+1)\,\|y-x\|}\right)^{1/p} \leq \|(x,y)\|_{p-HH} \tag{2.14}$$

where $p \geq 1$.

It is natural to ask which lower bound from (2.12) and (2.14) for the p-HH-norm is better?

If we take $X = \mathbb{C}$, $\|\cdot\| = |\cdot|$ and $p = 2$, then by plotting the difference d given by

$$d\,(x,y) := \left(\frac{\left||x|^2\,x - |y|^2\,y\right|}{3\,|y-x|}\right)^{1/2} - \left(\frac{|x|^2 + |y|^2}{6}\right)^{1/2}$$

for $x, y \in \mathbb{R}$ and $x \neq y$, we observe that d is nonnegative, showing that the new bound (2.14) is better than (2.12). The plot is depicted in Figure 1 as follows:

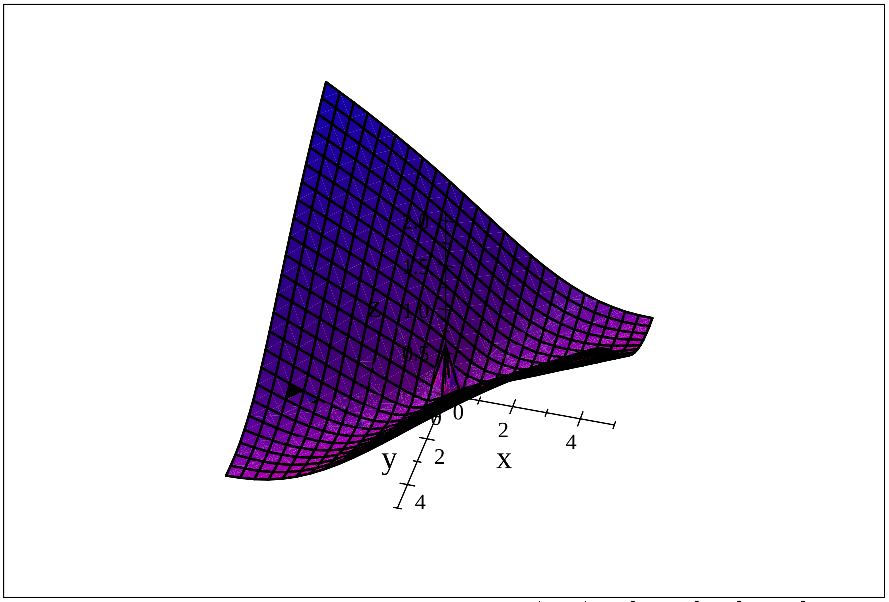

Figure 1: The variation of d in the box $(x, y) \in [-4, 4] \times [-4, 4]$.

Problem 1. *Is the inequality*

$$\frac{\|x\|^p + \|y\|^p}{2} \leq \frac{\| \|x\|^p x - \|y\|^p y\|}{\|y - x\|} \tag{2.15}$$

true for any $(x, y) \in X^2$ with $x \neq y$ and $p \geq 1$?

3. Applications for Power Series

For power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients we can naturally construct another power series which have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series have the same radius of convergence as the original series, and that if all coefficients $a_n \geq 0$, then $f_a = f$.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \ z \in D(0,1); \\ g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \ z \in \mathbb{C}; \\ h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \ z \in \mathbb{C}; \\ l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \ z \in D(0,1); \end{aligned} \tag{3.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1); \tag{3.2}$$
$$g_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h_A(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$l_A(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n \qquad z\in\mathbb{C}, \tag{3.3}$$
$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \qquad z\in D(0,1);$$
$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}, \qquad z\in D(0,1);$$
$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \qquad z\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is Gamma function.

Theorem 2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If $(X;\|\cdot\|)$ is a normed linear space and $x,y\in X$ with $\|x\|,\|y\|<R$, then*

$$\begin{aligned}&\|f(\|x\|)x-f(\|y\|)y\|\\&\le\|y-x\|\\&\times\int_0^1[f_a(\|(1-t)x+ty\|)+\|(1-t)x+ty\|f_a'(\|(1-t)x+ty\|)]\,dt.\end{aligned} \tag{3.4}$$

Proof. From the inequality (2.1) for $p=n+1$, n a natural number with $n\ge1$, we have

$$\left\|\|x\|^nx-\|y\|^ny\right\|\le(n+1)\|y-x\|\int_0^1\|(1-t)x+ty\|^n\,dt. \tag{3.5}$$

We notice that the above inequality also holds for $n=0$, reducing to an equality.

Let $m \geq 1$. Then we have, by the generalized triangle inequality and by (3.5), that

$$\left\| \left(\sum_{n=0}^{m} a_n \|x\|^n \right) x - \left(\sum_{n=0}^{m} a_n \|y\|^n \right) y \right\| \tag{3.6}$$
$$\leq \sum_{n=0}^{m} |a_n| \left\| \|x\|^n x - \|y\|^n y \right\|$$
$$\leq \|y - x\| \sum_{n=0}^{m} (n+1) |a_n| \int_0^1 \|(1-t) x + ty\|^n dt$$
$$= \|y - x\| \int_0^1 \left(\sum_{n=0}^{m} (n+1) |a_n| \|(1-t) x + ty\|^n \right) dt.$$

Since $\|x\|, \|y\| < R$ the series

$$\sum_{n=0}^{\infty} a_n \|x\|^n, \sum_{n=0}^{\infty} a_n \|y\|^n$$

and

$$\sum_{n=0}^{\infty} (n+1) |a_n| \|(1-t) x + ty\|^n$$

are convergent.

Moreover, we have

$$\sum_{n=0}^{\infty} a_n \|x\|^n = f(\|x\|), \sum_{n=0}^{\infty} a_n \|y\|^n = f(\|y\|)$$

and

$$\sum_{n=0}^{\infty} (n+1) |a_n| \|(1-t) x + ty\|^n$$
$$= \sum_{n=0}^{\infty} |a_n| \|(1-t) x + ty\|^n + \sum_{n=0}^{\infty} n |a_n| \|(1-t) x + ty\|^n$$
$$= f_a(\|(1-t) x + ty\|) + \|(1-t) x + ty\| f_a'(\|(1-t) x + ty\|)$$

for any $\|x\|, \|y\| < R$.

Taking the limit over $m \to \infty$ in (3.6) we get the desired result (3.4). □

Remark 5. *If we take* $f(z) := \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ *then we have from (3.4) the following inequality*

$$\|\exp(\|x\|) x - \exp(\|y\|) y\| \tag{3.7}$$
$$\leq \|y - x\| \int_0^1 \exp(\|(1-t) x + ty\|) (1 + \|(1-t) x + ty\|) dt.$$

for any $x, y \in X$.

If we apply the inequality (3.4) for the functions $f(z) := \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ *and* $f(z) := \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ *then we have*

$$\left\| \frac{x}{1 \pm \|x\|} - \frac{y}{1 \pm \|y\|} \right\| \leq \|y - x\| \int_0^1 \frac{dt}{(1 - \|(1-t) x + ty\|)^2} \tag{3.8}$$

for any $x, y \in X$ *with* $\|x\|, \|y\| < 1$.

Utilising the Hile's inequality, we can also prove the following divided difference inequality:

Proposition 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a function defined by power series with complex coefficients and convergent on the open disk* $D(0, R) \subset \mathbb{C}$, $R > 0$. *If* $(X; \|\cdot\|)$ *is a normed linear space and* $x, y \in X$ *with* $\|x\|, \|y\| < R$ *and* $\|x\| \neq \|y\|$, *then*

$$\frac{\|f(\|x\|)x - f(\|y\|)y\|}{\|y - x\|} \leq \frac{f_a(\|x\|)\|x\| - f_a(\|y\|)\|y\|}{\|x\| - \|y\|}. \tag{3.9}$$

Proof. The proof goes along the line of the one from Theorem 2 by utilizing Hile's inequality (1.9)

$$\frac{\|\|x\|^n x - \|y\|^n y\|}{\|y - x\|} \leq \frac{\|x\|^{n+1} - \|y\|^{n+1}}{\|x\| - \|y\|}$$

for any n a natural number. □

Remark 6. *If we write the inequality (3.9) for the exponential function, then we get*

$$\frac{\|\exp(\|x\|)x - \exp(\|y\|)y\|}{\|y - x\|} \leq \frac{\exp(\|x\|)\|x\| - \exp(\|y\|)\|y\|}{\|x\| - \|y\|}. \tag{3.10}$$

for any $x, y \in X$ *with* $\|x\| \neq \|y\|$.

If we apply the inequality (3.9) for the functions $f(z) := \frac{1}{1-z}$ *and* $f(z) := \frac{1}{1+z}$ *then we get*

$$\left\| \frac{x}{1 \pm \|x\|} - \frac{y}{1 \pm \|y\|} \right\| \leq \frac{\|y - x\|}{(1 - \|x\|)(1 - \|y\|)} \tag{3.11}$$

for any $x, y \in X$ *with* $\|x\| \neq \|y\|$ *and* $\|x\|, \|y\| < 1$.

References

[1] N. Bourbaki, *Integration*, Herman, Paris, 1965.

[2] J. A. Clarkson, Uniform convex spaces, *Trans. Amer. Math. Soc.* **40**(1936), 396-414.

[3] G. N. Hile, Entire solutions of linear elliptic equations with Laplacian principal part, *Pacific J. Math.* **62**(1976), 124-140.

[4] S. S. Dragomir, Inequalities for the p-angular distance in normed linear spaces. *Math. Inequal. Appl.* **12** (2009), no. 2, 391–401.

[5] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Momographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[6] C. F. Dunkl and K. S. Williams, A simple norm inequality, *Amer. Math. Month.* **71**(1964), 53-54.

[7] V. I. Gurariĭ, Strengthening the Dunkl-Williams inequality on the norms of elements of Banach spaces (Ukrainian), *Dopovidi Akad. Nauk Ukrain. RSR* **1966**(1966), 35-38.

[8] Eder Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space. *Math. Inequal. Appl.* **13** (2010), no. 1, 1–32.

[9] Eder Kikianty, G. Sinnamon, The p-HH norms on Cartesian powers and sequence spaces. *J. Math. Anal. Appl.* **359** (2009), no. 2, 765–779.

[10] L. Maligranda, Simple norm inequalities, *Amer. Math. Month.* **113**(2006), 256-260.

[11] J. L. Massera and J. J. Schäffer, Linear differential equations and functional analysis. I., *Ann. of Math.* **67**(1958), 517-573.

[12] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*,, Dordrecht, 1993. Kluwer

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