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NEW HADAMARD-TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE $(\alpha, m)-$CONVEX FUNCTIONS

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Abstract. In this paper some new inequalities are proved related to left hand side of Hermite-Hadamard inequality for the classes of functions whose derivatives of absolute values are $(\alpha, m)-$convex.

1. INTRODUCTION

The classical Hermite-Hadamard inequality gives us an estimate of the mean value of a convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ which is well-known in the literature as following;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The concept of $m-$convexity has been introduced by Toader in [5], as following:

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m-$convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

For recent results based on $m-$convexity see the papers [2], [3], [4], [5], [6], [7], [8], [9], [10] and [11].

In [12], Miheşan gave definition of $(\alpha, m)-$convexity as following;

Definition 2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. For the recent results based on the above definition see the papers [2], [3], [10], [13], [14], and [15].

Recently, in [15], Özdemir *et al.* proved the following inequalities for $(\alpha, m)-$convex functions;

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Theorem 1. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is $(\alpha, m)-$convex on $[a, b]$ for $(\alpha, m) \in [0, 1]^2$, $q \geq 1$, then the following inequality holds;*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x)dx \right| \\ \leq & \frac{(mb - a)^2}{2} \left(\frac{1}{6} \right)^{1 - \frac{1}{q}} \\ & \times \left[|f''(a)|^q \frac{1}{(\alpha + 2)(\alpha + 3)} + m |f'(b)|^q \left(\frac{1}{6} - \frac{1}{(\alpha + 2)(\alpha + 3)} \right) \right]^{\frac{1}{q}}. \end{aligned}$$

Theorem 2. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is $(\alpha, m)-$convex on $[a, b]$ for $(\alpha, m) \in [0, 1]^2$, $q > 1$, then the following inequality holds;*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x)dx \right| \\ \leq & \frac{(mb - a)^2}{8} \left(\frac{\Gamma(1 + p)}{\Gamma\left(\frac{3}{2} + p\right)} \right)^{\frac{1}{p}} \left[|f''(a)|^q \frac{1}{\alpha + 1} + m |f'(b)|^q \left(\frac{\alpha}{\alpha + 1} \right) \right]^{\frac{1}{q}}. \end{aligned}$$

Theorem 3. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f'' \in L[a, b]$, where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is $(\alpha, m)-$convex on $[a, b]$ for $(\alpha, m) \in [0, 1]^2$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds;*

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb - a} \int_a^{mb} f(x)dx \right| \leq \frac{(mb - a)^2}{2} \left\{ |f''(a)|^q \left[\left(\frac{q}{\alpha + q + 1} \right) \frac{\Gamma(\alpha + 1)\Gamma(q)}{\Gamma(\alpha + q + 1)} \right] \right. \\ & \left. + m |f'(b)|^q \left[\frac{1}{q + 1} - \left(\frac{q}{\alpha + q + 1} \right) \frac{\Gamma(\alpha + 1)\Gamma(q)}{\Gamma(\alpha + q + 1)} \right] \right\}^{\frac{1}{q}}. \end{aligned}$$

The main aim of this paper is to prove some new Hadamard-type inequalities for functions whose derivatives of absolute values are $(\alpha, m)-$convex functions.

2. MAIN RESULTS

To prove our main results, we use following Lemma which was used by Alomari *et al.* (see [1]).

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I where $a, b \in I$, with $a < b$. Let $f' \in L[a, b]$, then the following equality holds;*

$$\begin{aligned} & f\left(\frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x)dx \\ = & \frac{b - a}{4} \left[\int_0^1 t f'\left(t \frac{a + b}{2} + (1 - t)a \right) dt + \int_0^1 (t - 1) f'\left(tb + (1 - t) \frac{a + b}{2} \right) dt \right]. \end{aligned}$$

Theorem 4. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f'|$ is $(\alpha, m)-$convex on $[a,b]$ for $(\alpha, m) \in [0,1] \times (0,1]$, then the following inequality holds;*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{b-a}{4} \min\{W_1, W_2\} \tag{2.1}$$

where

$$\begin{aligned} W_1 &= \frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a}{m}\right)\right| \\ &\quad + \frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right| + m\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2m}\right)\right| \end{aligned}$$

and

$$\begin{aligned} W_2 &= \frac{1}{\alpha+2}\left|f'(a)\right| + \frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a+b}{2m}\right)\right| \\ &\quad + \frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right| + m\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|. \end{aligned}$$

Proof. From Lemma 1 and by using the properties of modulus, we can write

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\le \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right]. \end{aligned} \tag{2.2}$$

Since $|f'|$ is $(\alpha, m)-$convex on $[a,b]$, we know that for any $t \in [0,1]$ and $(\alpha, m) \in [0,1]^2$;

$$\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right| \le t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right| + m(1-t^\alpha)\left|f'\left(\frac{a}{m}\right)\right| \tag{2.3}$$

and

$$\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right| \le t^\alpha\left|f'(b)\right| + m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|. \tag{2.4}$$

By the inequalities (2.3) and (2.4), rewriting the inequality (2.2), we obtain;

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\le \frac{b-a}{4}\left[\int_0^1 t\left(t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right| + m(1-t^\alpha)\left|f'\left(\frac{a}{m}\right)\right|\right)dt \right. \\ &\quad \left. + \int_0^1 (1-t)\left(t^\alpha\left|f'(b)\right| + m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|\right)dt\right]. \end{aligned}$$

By calculating the above integrals, we get the following inequality;

(2.5)
$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{b-a}{4}\left\{\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a}{m}\right)\right|\right.\\ &\left.+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|+m\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2m}\right)\right|\right\}.\end{aligned}$$

Analogously, we obtain

(2.6)
$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{b-a}{4}\left\{\frac{1}{\alpha+2}\left|f'(a)\right|+\frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a+b}{2m}\right)\right|\right.\\ &\left.+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|+m\frac{(\alpha+1)(\alpha+2)-2}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|\right\}.\end{aligned}$$

Which completes the proof. □

Corollary 1. *If we choose $\alpha=m=1$ in (2.1), we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{12}\min\left\{K_1,K_2\right\}.$$

$$K_1=\frac{|f'(a)|+|f'(b)|}{2}+2\left|f'\left(\frac{a+b}{2}\right)\right|$$

and

$$K_2=|f'(a)|+|f'(b)|+\left|f'\left(\frac{a+b}{2}\right)\right|.$$

Theorem 5. *Let $f:I\subset[0,b^*]\to\mathbb{R}$ be a differentiable mapping on I^0 such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$, $b^*>0$. If $|f'|^{\frac{p}{p-1}}$ is $(\alpha,m)-$convex on $[a,b]$ for $(\alpha,m)\in[0,1]\times(0,1]$ and $p>1$, then the following inequality holds;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4(p+1)^{\frac{1}{p}}}\min\left\{Z_1,Z_2\right\} \tag{2.7}$$

where $\frac{1}{q}+\frac{1}{p}=1$ *and*

$$Z_1 = \left(\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}$$

$$Z_2 = \left(\frac{1}{\alpha+2}\left|f'(a)\right|^q + m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q + m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}.$$

Proof. By a similar argument to the proof of Theorem 4, we have

(2.8)

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right].$$

By using the well-known Hölder integral inequality to the inequality (2.8), we get

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

It is easy to observe that;

$$\int_0^1 t^p dt = \int_0^1 (1-t)^p dt = \frac{1}{p+1}.$$

Since $|f'|^{\frac{p}{p-1}}$ is $(\alpha,m)-$convex on $[a,b]$, we know that for any $t\in[0,1]$ and $(\alpha,m)\in[0,1]^2$;

$$\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right| \le t^\alpha\left|f'\left(\frac{a+b}{2}\right)\right| + m(1-t^\alpha)\left|f'\left(\frac{a}{m}\right)\right|$$

and

$$\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right| \le t^\alpha\left|f'(b)\right| + m(1-t^\alpha)\left|f'\left(\frac{a+b}{2m}\right)\right|.$$

Therefore, we obtain the inequality;

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.9}$$

By a similar argument to the proof of Theorem 3, analogously, we obtain the following inequalities;

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{1}{\alpha+2}\left|f'(a)\right|^q+m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q+m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{2.10}$$

From the inequalities (2.9)-(2.10), we obtain the inequality (2.7). □

Corollary 2. *Under the assumptions of Theorem 5, if we choose $\alpha=m=1$, we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left(\frac{1}{6}\right)^{\frac{1}{q}}\min\{L_1,L_2\}$$

where $\frac{1}{q}+\frac{1}{p}=1$ *and*

$$\begin{aligned}
L_1 &= \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{2}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} \\
L_2 &= \left(\left|f'(a)\right|^q+\frac{1}{2}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{1}{2}\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 3. *Let $f: I\subset[0,b^*]\to\mathbb{R}$ be a differentiable mapping on I^0 such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$, $b^*>0$. If $|f'|^{\frac{p}{p-1}}$ is $(\alpha,m)-$convex on $[a,b]$*

for $(\alpha, m) \in [0,1] \times (0,1]$ *and* $p > 1$*, then the following inequality holds;*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{b-a}{4(p+1)^{\frac{1}{p}}} \min\{Z_1', Z_2'\} \le \frac{b-a}{4(p+1)^{\frac{1}{p}}} \min\{Z_1, Z_2\}.$$

where $\frac{1}{q} + \frac{1}{p} = 1$ *and*

$$Z_1' = \left(\frac{1}{\alpha+1} \left| f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}}$$

$$Z_2' = \left(\frac{1}{\alpha+2}|f'(a)|^q + \frac{1}{(\alpha+1)(\alpha+2)}|f'(b)|^q + \frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q \right)^{\frac{1}{q}}$$

Z_1 *and* Z_2 *as in Theorem 5.*

Proof. Here $0 < \frac{1}{q} < 1$, for $q > 1$. By using the fact that;

$$\sum_{i=1}^n (a_i + b_i)^r \le \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r$$

for $0 < r < 1$, $a_1, a_2, ..., a_n \ge 0$ and $b_1, b_2, ..., b_n \ge 0$, from the inequality (2.7), if we set $a_1 = \frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q$ and $b_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q$, we obtain the inequality;

(2.11)
$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\le \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left(\frac{1}{\alpha+1} \left| f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q \right. \\ &\left. \quad + m\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \end{aligned}$$

analogously, we obtain

(2.12)
$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\le \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left(\frac{1}{\alpha+2}|f'(a)|^q + \frac{1}{(\alpha+1)(\alpha+2)}|f'(b)|^q + \frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q \right)^{\frac{1}{q}} \end{aligned}$$

by choosing $a_1 = \frac{1}{\alpha+2}|f'(a)|^q + m\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q$ and $b_1 = \frac{1}{(\alpha+1)(\alpha+2)}|f'(b)|^q + m\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q$. From the inequalities (2.11) and (2.12), we get the desired result. □

Theorem 6. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f'|^q$ is $(\alpha, m)-$convex on $[a,b]$ for $(\alpha, m) \in [0,1] \times (0,1]$ and $p \geq 1$, then the following inequality holds;*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \min\{U_1, U_2\} \tag{2.13}$$

where

$$\begin{aligned} U_1 &= \left(\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} \\ &\quad + \left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} \\ U_2 &= \left(\frac{1}{\alpha+2}\left|f'(a)\right|^q + m\left(\frac{1}{2} - \frac{1}{\alpha+2}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}} \\ &\quad + \left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q + m\left(\frac{1}{2} - \frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}. \end{aligned}$$

Proof. From Lemma 1, we can write

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right]. \end{aligned}$$

By applying the Power-mean inequality, we get

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{b-a}{4}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

By using $(\alpha, m)-$convexity of $|f'|^q$ on $[a,b]$ and by simple calculations, we obtain the following inequality;

(2.14)

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{\alpha+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q + m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Hence, by a similar argument to the proofs of Theorem 4-5, analogously, we obtain the following inequalities;

(2.15)

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq & \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\left(\frac{1}{\alpha+2}\left|f'(a)\right|^q + m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q + m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2m}\right)\right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

By the inequalities (2.14)-(2.15), we obtain the inequality (2.13). □

Corollary 4. *Under the assumptions of Theorem 6, if we choose $\alpha = m = 1$, we obtain the inequality;*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{4}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \min\{M_1, M_2\}$$

where

$$\begin{aligned} M_1 &= \left(\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{2}\left|f'(a)\right|^q \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'(b)\right|^q \right)^{\frac{1}{q}} \\ M_2 &= \left(\left|f'(a)\right|^q + \frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{1}{2}\left|f'(b)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 5. *Let $f : I \subset [0, b^*] \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f'|^q$ is $(\alpha, m)-$convex on $[a,b]$ for*

$(\alpha,m)\in[0,1]\times(0,1]$ and $p\geq 1$, then the following inequality holds;

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\min\{U_1',U_2'\}\leq\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\min\{U_1,U_2\}\end{aligned}$$

where

$$\begin{aligned}U_1' &= \left(\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+m\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)\left|f'\left(\frac{a}{m}\right)\right|^q\right.\\ &\left.+m\left(\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\\ U_2' &= \left(\frac{1}{\alpha+2}\left|f'(a)\right|^q+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{\frac{1}{q}}\end{aligned}$$

U_1 and U_2 as in Theorem 6.

Proof. By a similar argument to the proof of Corollary 3, the result is immediately follows. □

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