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A VARIANT OF SUBDIVIDING OF HOLDER'S INEQUALITY

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Abstract. We use a refinement of Holder's inequality for $1 < p < \infty$ and also for $0 < p < 1$ to find an improvement of a subdividing of the Holder's inequality for integrals.

1. Introduction

In the following we use several results from [5], [4], [1] and [6] which will be stated below:

Theorem 1. *([4]) If $f(x) \geq 0,\ g(x) \geq 0$ and $f(x) \in L^p[a,b],\ g(x) \in L^q[a,b]$ and $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}}. \tag{1}$$

Theorem 2. *([5]) If $f(x),\ g(x) \geq 0$ and $s,\ t \in \mathcal{R}$ and let $p = \frac{s-t}{1-t},\ q = \frac{s-t}{s-1}$.*

(i): *If $s < 1 < t$ or $s > t > 1$, then*

$$\int f(x)g(x)dx \leq \left(\int f(x)^{sp}dx\right)^{\frac{1}{p^2}} \left(\int g(x)^{tq}dx\right)^{\frac{1}{q^2}}$$

$$\times \left(\int f(x)^{tp}dx \cdot \int g(x)^{sq}dx\right)^{\frac{1}{pq}},$$

with equality if and only if f and g are proportional.

(ii): *If $s > t > 1$ or $s < t < 1$; $t > s > 1$ or $t < s < 1$, then*

$$\int f(x)g(x)dx \geq \left(\int f(x)^{sp}dx\right)^{\frac{1}{p^2}} \left(\int g(x)^{tq}dx\right)^{\frac{1}{q^2}}$$

$$\times \left(\int f(x)^{tp}dx \cdot \int g(x)^{sq}dx\right)^{\frac{1}{pq}},$$

with equality if and only if f and g are proportional.

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Theorem 3. *([6]) Let $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$ with $||f||_p > 0$, $||g||_q > 0$ then*

$$\frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{||fg||_1}{||f||_p ||g||_q} + \frac{1}{\max\{p,q\}}[\frac{1}{p}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q}\left(\frac{t}{s}\right)^{1/p} -$$

$$-\frac{2}{\sqrt{pq}}\left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}] \le \frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p},$$

where $s, t > 0$.

Theorem 4. *([6]) Let $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$ with $||f||_p > 0$, $||g||_q > 0$ then*

$$\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p} \le \frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{\int_\Omega |f||g|d\mu}{||f||_p ||g||_q} + \frac{1}{\min\{p,q\}}[\frac{1}{p}\left(\frac{s}{t}\right)^{1/q} + \frac{1}{q}\left(\frac{t}{s}\right)^{1/p} -$$

$$-\frac{2}{\sqrt{pq}}\left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}],$$

where $s, t > 0$.

Remark 1. *([6]) (i) Under the above conditions if we put $s = t$ in Theorem 3 and then in Theorem 6 respectively, we will have:*

$$\frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{||fg||_1}{||f||_p ||g||_q} + \frac{1}{\max\{p,q\}}[1 - \frac{2}{\sqrt{pq}} \frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}] \le 1 - \frac{2}{pq}$$

and

$$1 - \frac{2}{pq} \le \frac{1}{p^{1/p}} \frac{1}{q^{1/q}} \frac{\int_\Omega |f||g|d\mu}{||f||_p ||g||_q} + \frac{1}{\min\{p,q\}}[1 - \frac{2}{\sqrt{pq}} \frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}]$$

respectively.

(ii) Under the conditions of Theorem 3 and using its proof we also obtain:

$$||fg||_1 \le C(p,s,t)||f||_p||g||_q,$$

where

$$C(p,s,t) = \frac{q^{\frac{1}{q}}}{p^{1+\frac{1}{q}}}\left(\frac{s}{t}\right)^{1/q} + \frac{p^{\frac{1}{p}}}{q^{1+\frac{1}{p}}}\left(\frac{t}{s}\right)^{1/p}.$$

(iii) Taking now in (ii), $\frac{s}{t} = \frac{p}{q}$ we will obtain the inequality from Theorem 2 [1] *which is a generalization of Holder's inequality, see* [3] *i.e.*

$$1 - \frac{1}{\min\{p,q\}}||\frac{|f|^{p/2}}{||f||_p^{p/2}} - \frac{|g|^{q/2}}{||g||_q^{q/2}}||^2 \le \frac{||fg||_1}{||f||_p||g||_q} \le$$

$$\le 1 - \frac{2}{\max\{p,q\}}\left(1 - \frac{\int_\Omega |f|^{p/2}|g|^{q/2}d\mu}{||f||_p^{p/2}||g||_q^{q/2}}\right) = 1 - \frac{1}{\max\{p,q\}}||\frac{|f|^{p/2}}{||f||_p^{p/2}} - \frac{|g|^{q/2}}{||g||_q^{q/2}}||^2.$$

(iv) If we consider in (ii) the one variable functions f and g defined on $[a,b]$ and satisfying the conditions of Theorem 2.1, [4] *(Theorem 4) then we will obtain the inequality from Theorem 4.*

(v) If we consider in Theorem 3 and Theorem 4 respectively, the two variable functions $f(x,y)$ and $g(x,y)$ defined on $[a,b]\times[c,d]$ then the inequality become:

$$\frac{1}{p^{1/p}}\frac{1}{q^{1/q}}\frac{\int_a^b\int_c^d f(x,y)g(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}+\frac{1}{\max\{p,q\}}\cdot$$

$$\cdot[\frac{1}{p}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q}\left(\frac{t}{s}\right)^{1/p}-\frac{2}{\sqrt{pq}}\left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\frac{\int_a^b\int_c^d f^{p/2}(x,y)g^{q/2}(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}]\le$$

$$\le\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p},$$

and the reverses

$$\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p}\le$$

$$\le\frac{1}{p^{1/p}}\frac{1}{q^{1/q}}\frac{\int_a^b\int_c^d f(x,y)g(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}+\frac{1}{\min\{p,q\}}\cdot$$

$$\cdot[\frac{1}{p}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q}\left(\frac{t}{s}\right)^{1/p}-\frac{2}{\sqrt{pq}}\left(\frac{s}{t}\right)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\frac{\int_a^b\int_c^d f^{p/2}(x,y)g^{q/2}(x,y)dxdy}{(\int_a^b\int_c^d f^p(x,y)dxdy)^{\frac{1}{p}}(\int_a^b\int_c^d g^q(x,y)dxdy)^{\frac{1}{q}}}]$$

respectively.

(vi) Theorem 3 for example (and also Theorem 4) can be rewritten as below:

$$\frac{1}{p^{1/p}q^{1/q}}\frac{||fg||_1}{||f||_p||g||_q}\le\frac{1}{p^2}\left(\frac{s}{t}\right)^{1/q}+\frac{1}{q^2}\left(\frac{t}{s}\right)^{1/p}-$$

$$-\frac{1}{\max\{p,q\}}||\frac{1}{\sqrt{p}}\left(\frac{s}{t}\right)^{1/(2q)}\frac{|f|^{p/2}}{||f||_p^{p/2}}-\frac{1}{\sqrt{q}}\left(\frac{t}{s}\right)^{1/(2p)}\frac{|g|^{q/2}}{||g||_q^{q/2}}||_2^2.$$

Theorem 5. *([2]) Let $0<r<1$ and let $s=\frac{r}{r-1}$ be its conjugate exponent. If $k\in L^s$, $hk\in L^1$, $||h||_r$, $||k||_s>0$ and $\frac{1}{2}\le r<1$ then*

$$||hk||_1\left(1-r\left|\left|\frac{h^{\frac{1}{2}}k^{\frac{1}{2}}}{||h^{\frac{1}{2}}k^{\frac{1}{2}}||_2}-\frac{k^{\frac{s}{2}}}{||k^{\frac{s}{2}}||_2}\right|\right|_2^2\right)_+^{\frac{1}{r}}\le$$

$$\le||h||_r||k||_s\le||hk||_1\left(1-(1-r)\left|\left|\frac{h^{\frac{1}{2}}k^{\frac{1}{2}}}{||h^{\frac{1}{2}}k^{\frac{1}{2}}||_2}-\frac{k^{\frac{s}{2}}}{||k^{\frac{s}{2}}||_2}\right|\right|_2^2\right)^{\frac{1}{r}}$$

while if $0<r\le\frac{1}{2}$, the terms r and $1-r$ exchange their positions in the preceding inequalities.

2. A subdividing of Holder's inequality

Using the method from Theorem 1.2, [5] two new subdividing variants of Holder's inequality corresponding to generalizations from [1] and [2] respectively will be stated below.

Theorem 6. *Let $s,\ t \in \mathcal{R}_+$ and $p = \frac{s-t}{1-t},\ q = \frac{s-t}{s-1}$. Let f and g be nonnegative functions definite on $[a,b]$ with $f(x) > 0,\ g(x) > 0$.*

If $s < 1 < t,\ f \in L^{pt}[a,b],\ g \in L^{tq}[a,b]$ or $s > 1 > t,\ f \in L^{sp}[a,b]$ $g \in L^{sq}[a,b]$ then

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^{sp}(x)dx\right)^{\frac{1}{p^2}} \cdot \left(\int_a^b g^{tq}(x)dx\right)^{\frac{1}{q^2}} \cdot \left(\int_a^b f^{tp}(x)dx \int_a^b g^{sq}(x)dx\right)^{\frac{1}{pq}}$$

$$\times \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} \cdot \left(1 - \frac{\int_a^b f^{s\frac{p}{2}}(x)g^{s\frac{q}{2}}(x)dx}{\left(\int_a^b f^{sp}(x)dx \int_a^b g^{sq}(x)dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}} \times$$

$$\times \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} \cdot \left(1 - \frac{\int_a^b f^{t\frac{p}{2}}(x)g^{t\frac{q}{2}}(x)dx}{\left(\int_a^b f^{tp}(x)dx \int_a^b g^{tq}(x)dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{q}} \times$$

$$\times \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} \cdot \left(1 - \frac{\int_a^b (f(x)g)(x))^{\frac{s+t}{2}} dx}{\left(\int_a^b (f(x)g(x))^s dx \int_a^b (f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right]$$

and moreover,

$$\int_a^b f(x)g(x)dx \geq \left(\int_a^b f^{sp}(x)dx\right)^{\frac{1}{p^2}} \cdot \left(\int_a^b g^{tq}(x)dx\right)^{\frac{1}{q^2}} \cdot \left(\int_a^b f^{tp}(x)dx \int_a^b g^{sq}(x)dx\right)^{\frac{1}{pq}}$$

$$\times \left[1 - 2\max\{\frac{1}{p}, \frac{1}{q}\} \cdot \left(1 - \frac{\int_a^b f^{s\frac{p}{2}}(x)g^{s\frac{q}{2}}(x)dx}{\left(\int_a^b f^{sp}(x)dx \int_a^b g^{sq}(x)dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}} \times$$

$$\times \left[1 - 2\max\{\frac{1}{p}, \frac{1}{q}\} \cdot \left(1 - \frac{\int_a^b f^{t\frac{p}{2}}(x)g^{t\frac{q}{2}}(x)dx}{\left(\int_a^b f^{tp}(x)dx \int_a^b g^{tq}(x)dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{q}} \times$$

$$\times \left[1 - 2\max\{\frac{1}{p}, \frac{1}{q}\} \cdot \left(1 - \frac{\int_a^b (f(x)g)(x))^{\frac{s+t}{2}} dx}{\left(\int_a^b (f(x)g(x))^s dx \int_a^b (f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right]_+ .$$

Proof. It is well known that if $0 < p_1 < p_2$ then $L^{p_2}[a,b] \subset L^{p_1}[a,b]$, therefore if $f \in L^{pt}[a,b]$ then $f \in L^{sp}[a,b]$ and if $g \in L^{tq}[a,b]$ then $g \in L^{sq}[a,b]$ when $s < 1 < t$. Moreover, $\int_a^b (f(x)g(x))^t dx = \int_a^b f^t(x)g^t(x)dx \leq \left(\int_a^b f^{tp}(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^{tq}(x)dx\right)^{\frac{1}{q}} <$

∞ and $fg \in L^s[a,b]$, $fg \in L^{\frac{s+t}{2}}[a,b]$. Also $\int_a^b f^{\frac{pt}{2}}(x)g^{\frac{qt}{2}}(x)dx \leq \frac{1}{2}\int_a^b [f^{pt}(x) + g^{tq}(x)]dx < \infty$.

Taking into account that for $p = \frac{s-t}{1-t}$ by hypothesis we have, $\frac{s-t}{1-t} > 1$ and using then the second inequality form Remark 2, (iii) or Theorem 2, [1], as in [5] we obtain

$$\int_a^b f(x)g(x)dx = \int_a^b [(f(x)g(x))^s]^{\frac{1-t}{s-t}}[(f(x)g(x))^t]^{\frac{s-1}{s-t}}dx \leq \tag{2}$$

$$\leq \left(\int_a^b (f(x)g(x))^s dx\right)^{\frac{1-t}{s-t}} \left(\int_a^b (f(x)g(x))^t dx\right)^{\frac{s-1}{s-t}} \times$$

$$\times \left[1 - 2\min\{\frac{1-t}{s-t}, \frac{s-1}{s-t}\}\left(1 - \frac{\int_a^b (f(x)g(x))^{\frac{s}{2}}(f(x)g(x))^{\frac{t}{2}}dx}{\left(\int_a^b (f(x)g(x))^s dx \int_a^b (f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right].$$

We will use again Remark 2, (iii) or Theorem 2, [1], for $\frac{s-t}{1-t} > 1$, we have

$$\int_a^b (f(x)g(x))^s dx \leq \left(\int_a^b f^{s\frac{s-t}{1-t}}(x)dx\right)^{\frac{1-t}{s-t}} \left(g^{s\frac{s-t}{s-1}}(x)dx\right)^{\frac{s-1}{s-t}} \times \tag{3}$$

$$\times \left[1 - 2\min\{\frac{1-t}{s-t}, \frac{s-1}{s-t}\}\left(1 - \frac{\int_a^b f^{s\frac{s-t}{2(1-t)}}(x)g^{s\frac{s-t}{2(s-1)}}(x)dx}{(\int_a^b f^{s\frac{s-t}{2(1-t)}}(x)dx \int_a^b g^{s\frac{s-t}{2(s-1)}}(x)dx)^{\frac{1}{2}}}\right)\right]$$

and

$$\int_a^b (f(x)g(x))^t dx \leq \left(\int_a^b f^{t\frac{s-t}{1-t}}(x)dx\right)^{\frac{1-t}{s-t}} \left(g^{t\frac{s-t}{s-1}}(x)dx\right)^{\frac{s-1}{s-t}} \times \tag{4}$$

$$\times \left[1 - 2\min\{\frac{1-t}{s-t}, \frac{s-1}{s-t}\}\left(1 - \frac{\int_a^b f^{t\frac{s-t}{2(1-t)}}(x)g^{t\frac{s-t}{2(s-1)}}(x)dx}{(\int_a^b f^{t\frac{s-t}{2(1-t)}}(x)dx \int_a^b g^{t\frac{s-t}{2(s-1)}}(x)dx)^{\frac{1}{2}}}\right)\right].$$

From inequalities (2), (3) and (4) we obtain by calculus first inequality.

For second inequality, we use the first inequality from Remark 2, (iii), or Theorem 2, [1], as in [5], for $p = \frac{s-t}{1-t} > 1$.

∎

Now if we consider $(\Omega, \mathcal{F}, \mu)$ a measure space, a finite positive measure space, and p a real number with $p \geq 1$ then the space $L^p = L^p(\Omega, \mathcal{F}, \mu)$ is the collection of all complex-valued Borel measurable functions f such that $\int_\Omega |f|^p d\mu < \infty$ and $||f||_p = (\int_\Omega |f|^p d\mu)^{1/p}$, $f \in L^p$.

Remark 2. *Let* $s,\ t \in \mathbb{R}_+$ *and* $p = \frac{s-t}{1-t}$, $q = \frac{s-t}{s-1}$. *If* $s < 1 < t$, $f \in L^{pt}[a,b]$, $g \in L^{tq}[a,b]$ *with* $f,\ g$ *nonnegative and* $||f||_p,\ ||g||_q > 0$ *then*

$$||fg||_1 \leq ||f^s||_p^{\frac{1}{p}} ||g^t||_q^{\frac{1}{q}} ||f^t||_p^{\frac{1}{q}} ||g^s||_q^{\frac{1}{p}} \times \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}\left(1 - \frac{||f^{s\frac{p}{2}} g^{s\frac{q}{2}}||_1}{||f^s||_p^{\frac{p}{2}} ||g^s||_q^{\frac{q}{2}}}\right)\right]^{\frac{1}{p}} \times$$

$$\left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}\left(1 - \frac{||f^{t\frac{p}{2}} g^{t\frac{q}{2}}||_1}{||f^t||_p^{\frac{p}{2}} ||g^t||_q^{\frac{q}{2}}}\right)\right]^{\frac{1}{q}} \times \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}\left(1 - \frac{||(fg)^{\frac{s+t}{2}}||_1}{||(fg)^s||_1^{\frac{1}{2}} ||(fg)^t||_1^{\frac{1}{2}}}\right)\right]$$

and moreover,

$$||fg||_1 \geq ||f^s||_p^{\frac{1}{p}} ||g^t||_q^{\frac{1}{q}} ||f^t||_p^{\frac{1}{q}} ||g^s||_q^{\frac{1}{p}} \times \left[1 - 2\max\{\frac{1}{p}, \frac{1}{q}\} \left(1 - \frac{||f^{s\frac{p}{2}} g^{s\frac{q}{2}}||_1}{||f^s||_p^{\frac{p}{2}} ||g^s||_q^{\frac{q}{2}}}\right)\right]^{\frac{1}{p}} \times$$

$$\left[1 - 2\max\{\frac{1}{p}, \frac{1}{q}\} \left(1 - \frac{||f^{t\frac{p}{2}} g^{t\frac{q}{2}}||_1}{||f^t||_p^{\frac{p}{2}} ||g^t||_q^{\frac{q}{2}}}\right)\right]^{\frac{1}{q}} \times \left[1 - 2\max\{\frac{1}{p}, \frac{1}{q}\} \left(1 - \frac{||(fg)^{\frac{s+t}{2}}||_1}{||(fg)^s||_1^{\frac{1}{2}} ||(fg)^t||_1^{\frac{1}{2}}}\right)\right].$$

In Theorem 1, [2], the corresponding refinement of stability version of Holder's inequality when $0 < p < 1$ is established.

The following result is an adaptation of Theorem 1, see [2], such that to have a similar form to Remark 2, (iii) given when $1 < p < \infty$.

Lemma 1. *Let* $0 < r < 1$, *and* $s = \frac{r}{r-1}$. *If* $k \in L^s[a,b]$, $hk \in L^1[a,b]$, $h(x) > 0$, $k(x) > 0$ *then*

$$\int_a^b h(x)k(x)dx \left[1 - \frac{2}{\min\{\frac{1}{r}, \frac{1}{1-r}\}} \left(1 - \frac{\int_a^b h^{\frac{1}{2}}(x) k^{\frac{s+1}{2}}(x)dx}{(\int_a^b h(x)k(x)dx \int_a^b k^s(x)dx)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{r}} \leq$$

$$\leq \left(\int_a^b h^r(x)dx\right)^{\frac{1}{r}} \left(\int_a^b k^s(x)dx\right)^{\frac{1}{s}} \leq$$

$$\leq \int_a^b h(x)k(x)dx \left[1 - \frac{2}{\max\{\frac{1}{r}, \frac{1}{1-r}\}} \left(1 - \frac{\int_a^b h^{\frac{1}{2}}(x) k^{\frac{s+1}{2}}(x)dx}{(\int_a^b h(x)k(x)dx \int_a^b k^s(x)dx)^{\frac{1}{2}}}\right)\right]^{\frac{1}{r}}.$$

Proof. As in [2] we set $p = \frac{1}{r}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we can apply Remark 2, (iii) to the functions $f = h^r k^r$ and $g = k^{-r}$ because $\int_a^b f^p dx = \int_a^b hk dx < \infty$ and $\int_a^b g^q dx = \int_a^b k^s dx < \infty$ and this means that $f \in L^p[a,b]$ and $g \in L^q[a,b]$ respectively. By calculus we obtain the above inequality.
∎

Theorem 7. *Let* $s,\ t \in \mathcal{R}$ *and* $p = \frac{s-t}{1-t}$, $q = \frac{s-t}{s-1}$. *Let* f *and* g *be nonnegative functions definite on* $[a,b]$ *with* $f(x) > 0$, $g(x) > 0$.

If $s > t > 1$ *or* $s < t < 1$; $t > s > 1$ *or* $t < s < 1$ *and* $fg \in L^{\max\{s,t,1\}}[a,b]$, $g \in L^{\max\{sq,tq\}}[a,b]$, $f \in L^{\max\{sp,tp\}}[a,b]$. *then*

$$\int_a^b f(x)g(x)dx \left[1 - \frac{2}{\max\{\frac{1}{p}, \frac{1}{1-p}\}} \left(1 - \frac{\int_a^b (f(x)g(x))^{\frac{t+1}{2}} dx}{\left(\int_a^b f(x)g(x)dx \int_a^b (f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}} \times$$

$$\times \left[1 - \frac{2}{\max\{\frac{1}{p}, \frac{1}{1-p}\}} \left(1 - \frac{\int_a^b f^{\frac{s}{2}}(x) g^{s\frac{q+1}{2}}(x)dx}{\left(\int_a^b (f(x)g(x))^s dx \int_a^b (f(x)g(x))^{sq} dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p^2}} \times$$

$$\times\left[1-\frac{2}{\max\{\frac{1}{p},\frac{1}{1-p}\}}\left(1-\frac{\int_a^b f^{\frac{t}{2}}(x)g^{t\frac{q+1}{2}}(x)dx}{\left(\int_a^b(f(x)g(x))^t dx\int_a^b(f(x)g(x))^{tq}dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{pq}}\geq$$

$$\geq\left(\int_a^b f^{sp}(x)dx\right)^{\frac{1}{p^2}}\left(\int_a^b g^{tq}(x)dx\right)^{\frac{1}{q^2}}\left(\int_a^b g^{sq}(x)dx\int_a^b f^{tp}(x)dx\right)^{\frac{1}{pq}}$$

and moreover

$$\left(\int_a^b f^{sp}(x)dx\right)^{\frac{1}{p^2}}\left(\int_a^b g^{tq}(x)dx\right)^{\frac{1}{q^2}}\left(\int_a^b g^{sq}(x)dx\int_a^b f^{tp}(x)dx\right)^{\frac{1}{pq}}\geq$$

$$\geq\int_a^b f(x)g(x)dx\left[1-\frac{2}{\min\{\frac{1}{p},\frac{1}{1-p}\}}\left(1-\frac{\int_a^b (f(x)g(x))^{\frac{t+1}{2}}dx}{\left(\int_a^b f(x)g(x)dx\int_a^b(f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}}\times$$

$$\times\left[1-\frac{2}{\min\{\frac{1}{p},\frac{1}{1-p}\}}\left(1-\frac{\int_a^b f^{\frac{s}{2}}(x)g^{s\frac{q+1}{2}}(x)dx}{\left(\int_a^b(f(x)g(x))^s dx\int_a^b(f(x)g(x))^{sq}dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p^2}}\times$$

$$\times\left[1-\frac{2}{\min\{\frac{1}{p},\frac{1}{1-p}\}}\left(1-\frac{\int_a^b f^{\frac{t}{2}}(x)g^{t\frac{q+1}{2}}(x)dx}{\left(\int_a^b(f(x)g(x))^t dx\int_a^b(f(x)g(x))^{tq}dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{pq}}.$$

Proof. We can see that $\int_a^b f^{\frac{t}{2}}g^{\frac{t}{2}(q+1)}dx=\int_a^b(fg)^{\frac{t}{2}}g^{\frac{tq}{2}}dx\leq$
$\leq\frac{1}{2}\int_a^b[(fg)^t+g^{tq}]dx=\frac{1}{2}\left[\int_a^b f^t dx+\int_a^b g^{tq}dx\right]<\infty$.

Then we write $fg=[(fg)^s]^{\frac{1-t}{s-t}}[(fg)^t]^{\frac{s-1}{s-t}}=(fg)^{s\frac{1}{p}}(fg)^{t\frac{1}{q}}$ and we use the inequalities from Lemma 1 for $h=(fg)^{s\frac{1}{p}}$ and $k=(fg)^{t\frac{1}{q}}$. Taking into account that for $p=\frac{s-t}{1-t}$ from hypothesis we have

$$s>t>1 \text{ or } s<t<1\ \Rightarrow\ \frac{s-t}{1-t}<0$$

and

$$t>s>1 \text{ or } t<s<1\Rightarrow 0<\frac{s-t}{1-t}<1$$

we find

$$\int_a^b f(x)g(x)dx\left[1-\frac{2}{\min\{\frac{1}{p},\frac{1}{1-p}\}}\left(1-\frac{\int_a^b(f(x)g(x))^{\frac{s}{2p}}(f(x)g(x))^{\frac{t}{2}\frac{q+1}{q}}dx}{\left(\int_a^b f(x)g(x)dx\int_a^b(f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}}\leq$$

$$\leq\left(\int_a^b(f(x)g(x))^s dx\right)^{\frac{1}{p}}\left(\int_a^b(f(x)g(x))^t(x)dx\right)^{\frac{1}{q}}\leq$$

$$\leq\int_a^b f(x)g(x)dx\left[1-\frac{2}{\max\{\frac{1}{p},\frac{1}{1-p}\}}\left(1-\frac{\int_a^b(f(x)g(x))^{\frac{s}{2p}}(f(x)g(x))^{\frac{t}{2}\frac{q+1}{q}}dx}{\left(\int_a^b f(x)g(x)dx\int_a^b(f(x)g(x))^t dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}}.$$

Now applying again the corresponding inequality from Lemma 1 for $0 < \frac{s-t}{1-t} < 1$ or $\frac{s-t}{1-t} < 0$, first when $h = f^s$ and $k = g^s$ and then when $h = f^t$ and $k = g^t$ we obtain

$$\int_a^b (f(x)g(x))^s \, dx \left[1 - \frac{2}{\min\{\frac{1}{p}, \frac{1}{1-p}\}} \left(1 - \frac{\int_a^b f^{\frac{s}{2}}(x) g^{s\frac{q+1}{2}}(x)dx}{\left(\int_a^b (f(x)g(x))^s dx \int_a^b g^{sq}(x)dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}} \leq$$

$$\leq \left(\int_a^b f^{sp}(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^{sq}(x)dx\right)^{\frac{1}{q}} \leq$$

$$\leq \int_a^b (f(x)g(x))^s \, dx \left[1 - \frac{2}{\max\{\frac{1}{p}, \frac{1}{1-p}\}} \left(1 - \frac{\int_a^b f^{\frac{s}{2}}(x) g^{s\frac{q+1}{2}}(x)dx}{\left(\int_a^b (f(x)g(x))^s dx \int_a^b g^{sq}(x)dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}},$$

and

$$\int_a^b (f(x)g(x))^t \, dx \left[1 - \frac{2}{\min\{\frac{1}{p}, \frac{1}{1-p}\}} \left(1 - \frac{\int_a^b f^{\frac{t}{2}}(x) g^{t\frac{q+1}{2}}(x)dx}{\left(\int_a^b (f(x)g(x))^t dx \int_a^b g^{tq}(x)dx\right)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}} \leq$$

$$\leq \left(\int_a^b f^{tp}(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^{tq}(x)dx\right)^{\frac{1}{q}} \leq$$

$$\leq \int_a^b (f(x)g(x))^t \, dx \left[1 - \frac{2}{\max\{\frac{1}{p}, \frac{1}{1-p}\}} \left(1 - \frac{\int_a^b f^{\frac{t}{2}}(x) g^{t\frac{q+1}{2}} dx}{\left(\int_a^b (f(x)g(x))^t dx \int_a^b g^{tq}(x)dx\right)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}},$$

respectively.

From last three inequalities the theorem is proved.

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