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NEW DEFINITIONS AND THEOREMS VIA DIFFERENT KINDS OF CONVEX DOMINATED FUNCTIONS

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Abstract. In this paper, we establish several new convex dominated functions and then we obtain new Hadamard type inequalities.

1. Introduction

The inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f:[a,b]\to\mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality.

In [6], Dragomir and Ionescu introduced the following class of functions.

Definition 1. *Let $g:I\to\mathbb{R}$ be a convex function on the interval I. The function $f:I\to\mathbb{R}$ is called $g-$convex dominated on I if the following condition is satisfied:*

$$|\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)|\leq\lambda g(x)+(1-\lambda)g(y)-g(\lambda x+(1-\lambda)y)$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

In [7], Dragomir et al. proved the following theorem for $g-$convex dominated functions related to (1.1).

Theorem 1. *Let $g:I\to\mathbb{R}$ be a convex fuction and $f:I\to\mathbb{R}$ be a $g-$convex dominated mapping. Then, for all $a,b\in I$ with $a<b$,*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{1}{b-a}\int_a^b g(x)\,dx-g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)\,dx.$$

In [14], G. Toader defined $m-$convexity as the following:

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Definition 2. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that $f$is m-concave if $(-f)$ is m-convex.

Here, let us notice that $m-$convexity with $(\alpha,m)=(1,1)$, we have ordinary convex functions on $[0,b]$.

For the recent results based on the above definition see the papers [2], [3] and [12].

In [4], Dragomir proved the following theorem.

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be an $m-$convex function with $m\in(0,1]$ and $0\leq a<b$. If $f\in L_1[a,b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\leq\frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right].\tag{1.2}$$

In [8], Hudzik and Maligranda considered the class of functions which are $s-$convex in the second sense among others. This class is defined in the following way:

Definition 3. *A function $f:\mathbb{R}^+\to\mathbb{R}$, where $\mathbb{R}^+=[0,\infty)$, is said to be s-convex in the second sense if*

$$f(\alpha x+\beta y)\leq\alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\geq0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$.

This class of $s-$convex functions in the second sense is usually denoted by K_s^2. It can be easily seen that for $s=1$, $s-$convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

For the recent results based on the above definition see the papers [1], [5], [8] and [9].

In [5], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense.

Theorem 3. *Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ is an $s-$convex mapping in the second sense, $s\in(0,1)$ and $a,b\in\mathbb{R}_+$ with $a<b$. If $f\in L_1[a,b]$, then one has the inequalities:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}.\tag{1.3}$$

In [11], S. Varošanec introduced the following class of functions.

I and J are intervals in $\mathbb{R}$, $(0,1)\subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

Definition 4. *Let $h:J\to\mathbb{R}$ be a non-negative function, $h\not\equiv0$. We say that $f:I\to\mathbb{R}$ is an $h-$convex function, or that f belongs to the class $SX(h,I)$, if f is non-negative and for all $x,y\in I$, $\alpha\in(0,1]$, we have*

$$f(\alpha x+(1-\alpha)y)\leq h(\alpha)f(x)+h(1-\alpha)f(y).\tag{1.4}$$

If the inequality 1.4 is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$.

In [13], Sarıkaya et al. proved a variant of Hadamard inequality which holds for $h-$convex functions.

Theorem 4. *Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1([a, b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le [f(a) + f(b)]\int_0^1 h(\alpha)\,d\alpha. \tag{1.5}$$

In the following sections our main results are given: We establish several new convex dominated functions and then we obtain new Hadamard type inequalities.

2. $(g, m)-$convex dominated functions

Definition 5. *Let $g : [0, b] \to \mathbb{R}$ be a given $m-$convex function on the interval $[0, b]$. The real function $f : [0, b] \to \mathbb{R}$ is called $(g, m)-$convex dominated on $[0, b]$ if the following condition is satisfied*

$$\begin{aligned} &|\lambda f(x) + m(1-\lambda)f(y) - f(\lambda x + m(1-\lambda)y)| \\ \le\ &\lambda g(x) + m(1-\lambda)g(y) - g(\lambda x + m(1-\lambda)y) \end{aligned} \tag{2.1}$$

for all $x, y \in [0, b]$, $\lambda \in [0, 1]$ and $m \in [0, 1]$.

The next simple characterisation of $m-$convex dominated functions holds.

Lemma 1. *Let $g : [0, b] \to \mathbb{R}$ be an $m-$convex function on the interval $[0, b]$ and the function $f : [0, b] \to \mathbb{R}$. The following statements are equivalent:*

(1) f is $(g, m)-$convex dominated on $[0, b]$.
(2) The mappings $g - f$ and $g + f$ are $m-$convex functions on $[0, b]$.
(3) There exist two $m-$convex mappings h, k defined on $[0, b]$ such that

$$f = \tfrac{1}{2}(h - k) \quad \text{and} \quad g = \tfrac{1}{2}(h + k)\ .$$

Proof. 1$\Longleftrightarrow$2 The condition (2.1) is equivalent to

$$\begin{aligned} &g(\lambda x + m(1-\lambda)y) - \lambda g(x) - m(1-\lambda)g(y) \\ \le\ &\lambda f(x) + m(1-\lambda)f(y) - f(\lambda x + m(1-\lambda)y) \\ \le\ &\lambda g(x) + m(1-\lambda)g(y) - g(\lambda x + m(1-\lambda)y) \end{aligned}$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. The two inequalities may be rearranged as

$$(g + f)(\lambda x + m(1-\lambda)y) \le \lambda(g + f)(x) + m(1-\lambda)(g + f)(y)$$

and

$$(g - f)(\lambda x + m(1-\lambda)y) \le \lambda(g - f)(x) + m(1-\lambda)(g - f)(y).$$

which are eqivalent to the $m-$convexity of $g + f$ and $g - f$, respectively.

2$\Longleftrightarrow$3 We define the mappings f, g as $f = \frac{1}{2}(h - k)$ and $g = \frac{1}{2}(h + k)$. Then, if we sum and subtract f, g, respectively, we have $g + f = h$ and $g - f = k$. By the condition 2, the mappings $g - f$ and $g + f$ are $m-$convex on $[0, b]$, so h, k are $m-$convex mappings too. □

Theorem 5. *Let $g:[0,\infty)\to\mathbb{R}$ be an $m-$convex function with $m\in(0,1]$. $f:[0,\infty)\to\mathbb{R}$ is $(g,m)-$convex dominated mapping and $0\leq a<b$. If $f\in L_1[a,b]$, then one has the inequalities:*

$$\left|\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx-f\left(\frac{a+b}{2}\right)\right| \tag{2.2}$$
$$\leq \frac{1}{b-a}\int_a^b\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx-g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{1}{2}\left[\frac{f(a)+mf\left(\dfrac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\dfrac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\right| \tag{2.3}$$
$$\leq \frac{1}{2}\left[\frac{g(a)+mg\left(\dfrac{a}{m}\right)}{2}+m\frac{g\left(\frac{b}{m}\right)+mg\left(\dfrac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b\frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx.$$

Proof. By Definition 5 with $\lambda=\frac{1}{2}$, as the mapping f is $(g,m)-$convex dominated function, we have that

$$\left|\frac{f(x)+mf(y)}{2}-f\left(\frac{x+my}{2}\right)\right|\leq\frac{g(x)+mg(y)}{2}-g\left(\frac{x+my}{2}\right)$$

for all $x,y\in[0,\infty)$ and $m\in(0,1]$. If we choose $x=ta+(1-t)b$, $y=(1-t)\frac{a}{m}+t\frac{b}{m}$ and $t\in[0,1]$, $m\in(0,1]$, then we get

$$\left|\frac{f(ta+(1-t)b)+mf\left(\frac{(1-t)a+tb}{m}\right)}{2}-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{g(ta+(1-t)b)+mg\left(\frac{(1-t)a+tb}{2}\right)}{2}-g\left(\frac{a+b}{2}\right).$$

Integrating over t on $[0,1]$ we deduce that

$$\left|\frac{\int_0^1 f(ta+(1-t)b)\,dt+m\int_0^1 f\left(\frac{(1-t)a+tb}{m}\right)dt}{2}-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{\int_0^1 g(ta+(1-t)b)\,dt+m\int_0^1 g\left(\frac{(1-t)a+tb}{m}\right)dt}{2}-g\left(\frac{a+b}{2}\right)$$

and so the first inequality is proved.

Since f is $(g,m)-$convex dominated function, we have

$$|tf(x)+m(1-t)f(y)-f(tx+m(1-t)y)|$$
$$\leq tg(x)+m(1-t)g(y)-g(tx+m(1-t)y),\ \text{for all } x,y>0$$

which gives for $x=a$ and $y=\frac{b}{m}$

$$\begin{aligned}&\left|tf(a)+m(1-t)f\left(\frac{b}{m}\right)-f\left(ta+m(1-t)\frac{b}{m}\right)\right|\\ \leq\;& tg(a)+m(1-t)g\left(\frac{b}{m}\right)-g\left(ta+m(1-t)\frac{b}{m}\right)\end{aligned}$$

and for $x=\frac{a}{m}$, $y=\frac{b}{m^2}$ and then multiply with m

$$\begin{aligned}&\left|mtf\left(\frac{a}{m}\right)+m^2(1-t)f\left(\frac{b}{m^2}\right)-mf\left(t\frac{a}{m}+(1-t)\frac{b}{m}\right)\right|\\ \leq\;& mtg\left(\frac{a}{m}\right)+m^2(1-t)g\left(\frac{b}{m^2}\right)-mg\left(t\frac{a}{m}+(1-t)\frac{b}{m}\right)\end{aligned}$$

for all $t\in[0,1]$. By properties of modulus, if we add the above inequalities we get

$$\begin{aligned}&\left|t\left[f(a)+mf\left(\frac{a}{m}\right)\right]+m(1-t)\left[f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)\right]\right.\\ &\left.-\left[f\left(ta+m(1-t)\frac{b}{m}\right)+mf\left(t\frac{a}{m}+(1-t)\frac{b}{m}\right)\right]\right|\\ \leq\;& t\left[g(a)+mg\left(\frac{a}{m}\right)\right]+m(1-t)\left[g\left(\frac{b}{m}\right)+mg\left(\frac{b}{m^2}\right)\right]\\ &-\left[g\left(ta+m(1-t)\frac{b}{m}\right)+mg\left(t\frac{a}{m}+(1-t)\frac{b}{m}\right)\right].\end{aligned}$$

Thus, integrating over t on $[0,1]$ we obtain the second inequality. The proof is completed.

Another proof can be done as the following.

Since f is $(g,m)-$convex dominated, we have by Lemma 1 that $g+f$ and $g-f$ are $m-$convex on $[0,\infty]$, and so by the Hermite-Hadamard inequalities for $m-$convex functions in 1.2

$$\begin{aligned}(f+g)\left(\frac{a+b}{2}\right)&\leq\frac{1}{b-a}\int_a^b\frac{(f+g)(x)+m(f+g)\left(\frac{x}{m}\right)}{2}dx\\ &\leq\frac{1}{2}\left[\frac{(f+g)(a)+m(f+g)\left(\frac{a}{m}\right)}{2}+m\frac{(f+g)\left(\frac{b}{m}\right)+m(f+g)\left(\frac{b}{m^2}\right)}{2}\right].\end{aligned}$$

and

$$\begin{aligned}(g-f)\left(\frac{a+b}{2}\right)&\leq\frac{1}{b-a}\int_a^b\frac{(g-f)(x)+m(g-f)\left(\frac{x}{m}\right)}{2}dx\\ &\leq\frac{1}{2}\left[\frac{(g-f)(a)+m(g-f)\left(\frac{a}{m}\right)}{2}+m\frac{(g-f)\left(\frac{b}{m}\right)+m(g-f)\left(\frac{b}{m^2}\right)}{2}\right].\end{aligned}$$

Therefore, we write the following inequalities

(2.4)
$$-\left[\frac{1}{b-a}\int_a^b \frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx-g\left(\frac{a+b}{2}\right)\right]\leq\frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx-f\left(\frac{a+b}{2}\right),$$

(2.5)
$$\begin{aligned}&-\left[\frac{1}{2}\left[\frac{g(a)+mg\left(\frac{a}{m}\right)}{2}+m\frac{g\left(\frac{b}{m}\right)+mg\left(\frac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b \frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx\right]\\ \leq\;&\frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx,\end{aligned}$$

(2.6)
$$\frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx-f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b \frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx-g\left(\frac{a+b}{2}\right),$$

and

(2.7)
$$\begin{aligned}&\frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b \frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx\\ \leq\;&\frac{1}{2}\left[\frac{g(a)+mg\left(\frac{a}{m}\right)}{2}+m\frac{g\left(\frac{b}{m}\right)+mg\left(\frac{b}{m^2}\right)}{2}\right]-\frac{1}{b-a}\int_a^b \frac{g(x)+mg\left(\frac{x}{m}\right)}{2}dx.\end{aligned}$$

The inequalities (2.4) and (2.6) are equivalent to the inequality (2.2) and the inequalities (2.5) and (2.7) are equivalent to the inequality (2.3). This completes the proof. □

Corollary 1. *If we choose $m=1$ in Theorem 5, we get two inequalities of Hermite-Hadamard type for functions that are convex dominated in Theorem 1.*

Example 1. *Let $f,g:[0,\infty)\to\mathbb{R}$, $g(x)=bx^2$ is an $m-$convex function and $f(x)=ax^2$. If $b>0$ and $|a|\leq b$, then f is $(g,m)-$convex dominated function.*

3. $(g,s)-$CONVEX DOMINATED FUNCTIONS

Definition 6. *Let $g:\mathbb{R}_+\to\mathbb{R}$ be given $s-$convex function, the real function $f:\mathbb{R}_+\to\mathbb{R}$ is called $(g,s)-$convex dominated on $\mathbb{R}_+$, if the following condition is satisfied*

(3.1)
$$|\lambda^s f(x)+(1-\lambda)^s f(y)-f(\lambda x+(1-\lambda)y)|\leq\lambda^s g(x)+(1-\lambda)^s g(y)-g(\lambda x+(1-\lambda)y)$$

for all $x,y\in\mathbb{R}_+,\lambda\in[0,1]$ and $s\in(0,1]$.

The next simple characterisation of $s-$convex dominated functions holds.

Lemma 2. *Let $g:\mathbb{R}_+\to\mathbb{R}$ be an $s-$convex function and the real function $f:\mathbb{R}_+\to\mathbb{R}$. The following statements are equivalent:*

(1) f is $(g,s)-$convex dominated on $\mathbb{R}_+$.
(2) The mappings $g-f$ and $g+f$ are $s-$convex in the second sense on $\mathbb{R}_+$.
(3) There exist two $s-$convex mappings h,k defined on $\mathbb{R}_+$ such that

$$f=\frac{1}{2}(h-k)\quad\text{and}\quad g=\frac{1}{2}(h+k)\ .$$

Proof. 1$\Longleftrightarrow$2 The condition (3.1) is equivalent with

$$\begin{aligned}g(\lambda x+(1-\lambda)y)-\lambda^s g(x)-(1-\lambda)^s g(y) &\leq \lambda^s f(x)+(1-\lambda)^s f(y)-f(\lambda x+(1-\lambda)y)\\ &\leq \lambda^s g(x)+(1-\lambda)^s g(y)-g(\lambda x+(1-\lambda)y)\,.\end{aligned}$$

Then using the above inequality we can write

$$(g+f)(\lambda x+(1-\lambda)y)\leq\lambda^s(g+f)(x)+(1-\lambda)^s(g+f)(y)$$

and

$$(g-f)(\lambda x+(1-\lambda)y)\leq\lambda^s(g-f)(x)+(1-\lambda)^s(g-f)(y).$$

So, we obtained that the mappings $g-f$ and $g+f$ are $s-$convex in the second sense on $\mathbb{R}_+$.

2$\Longleftrightarrow$3 We define the mappings f,g as $f=\frac{1}{2}(h-k)$ and $g=\frac{1}{2}(h+k)$. Then if we sum and subtract f,g, respectively, we have $g+f=h$ and $g-f=k$. By the condition 2, the mappings $g-f$ and $g+f$ are $s-$convex in the second sense on $\mathbb{R}_+$, so h,k are $s-$convex mappings in the second sense too. □

Theorem 6. *Let $g:\mathbb{R}_+\to\mathbb{R}$ be an $s-$convex function in the second sense and the real function $f:\mathbb{R}_+\to\mathbb{R}$ be $(g,s)-$convex dominated $\mathbb{R}_+$. Then for all $a,b\in\mathbb{R}_+$, one has the inequalities:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-2^{s-1}f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{b-a}\int_a^b g(x)\,dx-2^{s-1}g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{s+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{g(a)+g(b)}{s+1}-\frac{1}{b-a}\int_a^b g(x)\,dx$$

for all $s\in(0,1]$.

Proof. By Definition 6, as the mapping f is $(g,s)-$convex dominated function, we can write 3.1 for $\lambda=\frac{1}{2}$

$$\left|\frac{f(x)+f(y)}{2^s}-f\left(\frac{x+y}{2}\right)\right|\leq\frac{g(x)+g(y)}{2^s}-g\left(\frac{x+y}{2}\right)$$

for all $x,y\in[a,b]\subseteq\mathbb{R}_+$ and $s\in(0,1]$. If we choose $x=ta+(1-t)b$, $y=(1-t)a+tb$ and $t\in[0,1]$, then we get

$$\begin{aligned}&\left|\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2^s}-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{g(ta+(1-t)b)+g((1-t)a+tb)}{2^s}-g\left(\frac{a+b}{2}\right).\end{aligned}$$

Integrating the above inequality over t on $[0,1]$ and then we obtained the first inequality in Theorem 6.

Since f is $(g,s)-$convex dominated function, we have

$$|t^s f(a)+(1-t)^s f(b)-f(ta+(1-t)b)| \leq t^s g(a)+(1-t)^s g(b)-g(ta+(1-t)b).$$

Then, we integrate the above inequality over t on $[0,1]$ we get

$$\left|\frac{f(a)+f(b)}{s+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{g(a)+g(b)}{s+1}-\frac{1}{b-a}\int_a^b g(x)\,dx$$

which is the second inequality in Theorem 6. The proof is completed.

Another proof can be done as the following.

Since f is $(g,s)-$convex dominated, we have by Lemma 2 that $g+f$ and $g-f$ are $s-$convex on $\mathbb{R}_+$, and so by the Hermite-Hadamard inequalities for $s-$convex functions in (1.3)

$$2^{s-1}(f+g)\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b (f+g)(x)\,dx \leq \frac{(f+g)(a)+(f+g)(b)}{s+1}$$

and

$$2^{s-1}(g-f)\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b (g-f)(x)\,dx \leq \frac{(g-f)(a)+(g-f)(b)}{s+1}$$

These inequalities are equivalent to those in the enunciation. □

Corollary 2. *If we choose $s=1$ in Theorem 6, we get two inequalities of Hermite-Hadamard type for functions that are convex dominated in Theorem 1.*

Example 2. *Let $f,g:\mathbb{R}_+\to\mathbb{R}$, $f(x)=x^k$ and $g(x)=x^l$. If $k,l,s\in(0,1]$, $k\leq l$ and g is $s-$convex function in the second sense, then f is $(g,s)-$convex dominated function.*

Example 3. *Let $f,g:\mathbb{R}_+\to\mathbb{R}$, $f(x)=\frac{1}{x^k}$ and $g(x)=x^l$. If $k,l,s\in(0,1]$, $k\leq l$ and g is $s-$convex function in the second sense, then f is $(g,s)-$convex dominated function.*

4. $(g,h)-$CONVEX DOMINATED FUNCTIONS

Definition 7. *Let $h:J\to\mathbb{R}$ be a non-negative function, $h\not\equiv 0$, $g:I\to\mathbb{R}$ be an $h-$convex function. The real function $f:I\to\mathbb{R}$ is called $(g,h)-$convex dominated on I if the following condition is satisfied*

$$\begin{aligned} &|h(\lambda)f(x)+h(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)| \\ \leq\ & h(\lambda)g(x)+h(1-\lambda)g(y)-g(\lambda x+(1-\lambda)y) \end{aligned} \tag{4.1}$$

for all $x,y\in I$ and $\lambda\in(0,1]$.

The next simple characterisation of $h-$convex dominated functions holds.

Lemma 3. *Let $h:J\to\mathbb{R}$ be a non-negative function, $h\neq 0$, $g:I\to\mathbb{R}$ be an $h-$convex function and $f:I\to\mathbb{R}$ be a real function. The following statements are equivalent:*

1. f is $(g,h)-$convex dominated on I.
2. The mappings $g-f$ and $g+f$ are $h-$ convex on I.
3. There exist two $h-$convex mappings l,k defined on I such that

$$f=\tfrac{1}{2}(l-k) \quad \text{and} \quad g=\tfrac{1}{2}(l+k)\ .$$

Proof. 1$\Longleftrightarrow$2 The condition (4.1) is equivalent to

$$\begin{aligned} & g\left(\lambda x+(1-\lambda)y\right)-h\left(\lambda\right)g(x)-h(1-\lambda)g(y) \\ \leq\ & h\left(\lambda\right)f(x)+h(1-\lambda)f(y)-f\left(\lambda x+(1-\lambda)y\right) \\ \leq\ & h\left(\lambda\right)g(x)+h(1-\lambda)g(y)-g\left(\lambda x+(1-\lambda)y\right)\end{aligned}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. The two inequalities may be rearranged as

$$(g+f)\left(\lambda x+(1-\lambda)y\right)\leq h\left(\lambda\right)(g+f)(x)+h(1-\lambda)(g+f)(y)$$

and

$$(g-f)\left(\lambda x+(1-\lambda)y\right)\leq h\left(\lambda\right)(g-f)(x)+h(1-\lambda)(g-f)(y).$$

which are eqivalent to the $h-$convexity of $g+f$ and $g-f$, respectively.

2$\Longleftrightarrow$3 Let we define the mappings f,g as $f=\frac{1}{2}(l-k)$ and $g=\frac{1}{2}(l+k)$. Then if we sum and subtract f, g, respectively, we have $g+f=l$ and $g-f=k$. By the condition 2, the mappings $g-f$ and $g+f$ are $m-$convex on $[0,b]$, so l,k are $h-$convex mappings on I too. □

Theorem 7. *Let $h:J\to\mathbb{R}$ be a non-negative function, $h\not\equiv 0$, $g:I\to\mathbb{R}$ be an $h-$convex function and the real function $f:I\to\mathbb{R}$ be $(g,h)-$convex dominated on I. Then one has the inequalities:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{1}{2h\left(\frac{1}{2}\right)}g\left(\frac{a+b}{2}\right)$$

and

$$\left|[f(a)+f(b)]\int_0^1 h(\lambda)\,d\lambda-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq[g(a)+g(b)]\int_0^1 h(\lambda)\,d\lambda-\frac{1}{b-a}\int_a^b g(x)\,dx$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Proof. By the Definition 7 with $\lambda=\frac{1}{2}$, $x=ta+(1-t)b$, $y=(1-t)a+tb$ and $t\in[0,1]$, as the mapping f is $(g,h)-$convex dominated function, we have that

$$\begin{aligned} & \left|h\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)]-f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & h\left(\frac{1}{2}\right)[g(ta+(1-t)b)+g((1-t)a+tb)]-g\left(\frac{a+b}{2}\right).\end{aligned}$$

Integrating the above inequality over t on $[0,1]$, the first inequality is proved.

Since f is $(g,h)-$convex dominated we can write the inequality in (4.1) for $x=a$ and $y=b$,

$$\begin{aligned} & \left|h(t)f(a)+h(1-t)f(b)-f(ta+(1-t)b)\right| \\ \leq\ & h(t)g(a)+h(1-t)g(b)-g(ta+(1-t)b).\end{aligned}$$

Then, we integrate the above inequality over t on $[0,1]$ we get

$$\begin{aligned} & \left|f(a)\int_0^1 h(t)\,dt+f(b)\int_0^1 h(1-t)\,dt-\int_0^1 f(ta+(1-t)b)\,dt\right| \\ \leq\ & g(a)\int_0^1 h(t)\,dt+g(b)\int_0^1 h(1-t)\,dt-\int_0^1 g(ta+(1-t)b)\,dt\end{aligned}$$

using the fact that $\int_0^1 h(t)\,dt = \int_0^1 h(1-t)\,dt$, we get

$$\left|[f(a)+f(b)]\int_0^1 h(t)\,dt - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq [g(a)+g(b)]\int_0^1 h(t)\,dt - \frac{1}{b-a}\int_a^b g(x)\,dx$$

and the second inequality is proved.

Another proof can be done as the following.

Since f is $(g,h)-$convex dominated, we have by Lemma 3 that $g+f$ and $g-f$ are $h-$convex mapping on I, and so by the Hermite-Hadamard inequalities for $h-$convex functions in (1.5)

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}(f+g)\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b (f+g)(x)\,dx \\ &\leq [(f+g)(a)+(f+g)(b)]\int_0^1 h(\alpha)\,d\alpha\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}(g-f)\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b (g-f)(x)\,dx \\ &\leq [(g-f)(a)+(g-f)(b)]\int_0^1 h(\alpha)\,d\alpha\end{aligned}$$

These inequalities are equivalent to those in the enunciation. □

Corollary 3. *If we choose $h(\alpha)=\alpha$ in Theorem 7, we get two inequalities of Hermite-Hadamard type for functions that are convex dominated in Theorem 1.*

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