Received 04/01/13

JENSEN TYPE WEIGHTED INEQUALITIES FOR FUNCTIONS OF SELFADJOINT AND UNITARY OPERATORS

S.S. DRAGOMIR[1,2]

Abstract. On making use of the spectral representations in terms of the Riemann-Stieltjes integral for the selfadjoint and unitary operators in Hilbert spaces we establish here some weighted inequalities of Jensen's type for convex, square-convex and Arg-square-convex functions. Some applications for simple functions of operators that belong to those classes are also provided.

1. Introduction

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [19, p. 257]):

$$\langle f(A)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{1.1}$$

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2 \, d\|E_\lambda x\|^2, \tag{1.2}$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [23] (see also [18, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [23])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$h(\langle Ax, x \rangle) \le \langle h(A)x, x \rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 47A63.

Key words and phrases. Jensen's inequality, Hölder's inequality, Measurable functions, Lebesgue integral, Selfadjoint operators, Unitary operators, Spectral family.

Theorem 2 (Hölder-McCarthy, 1967, [21])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r > 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all* $0 < r < 1$;
(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r < 0$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [18, p. 57]:

Theorem 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$\text{(LR)} \qquad \langle h(A)x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot h(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot h(M)$$

for each $x \in H$ with $\|x\| = 1$.

We recall that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [19, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{1.3}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{1.4}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle, \tag{1.5}$$

for any $x, y \in H$.

From the above properties it follows that the function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[0, 2\pi]$ for any $x \in H$.

For $z \in \mathbb{C}\setminus\{0\}$ we call the *principal value* of $log(z)$ the complex number

$$Log(z) := \ln|z| + iArg(z)$$

where $0 \leq Arg(z) < 2\pi$.

We observe that for $t \in [0, 2\pi)$ we have

$$Log\left(e^{it}\right) = it.$$

If we extend this equality by continuity in the point $t = 2\pi$, then we can define the operator $Log(U) : H \to H$ as

$$Log(U)x = \int_0^{2\pi} Log\left(e^{i\lambda}\right) dE_\lambda x = \int_0^{2\pi} (i\lambda)\, dE_\lambda x,\ x \in H. \tag{1.6}$$

Utilizing these spectral representations in terms of the Riemann-Stieltjes integral for the selfadjoint and unitary operators we establish here some weighted inequalities of Jensen's type for three classes of functions: convex, square-convex and Arg-square-convex functions. Some applications for simple functions of operators that belong to those classes are also provided.

For classical and recent result concerning inequalities for continuos functions of selfadjoint operators, see [23], [24], [25], [20], [18], [6], [9], [10], [12], [11], [16], [15], [14], [13], [7], and [8].

2. Weighted Inequalities for the Riemann-Stieltjes Integral

We can state the following result concerning the weighted Riemann-Stieltjes integral of monotonic nondecreasing integrators:

Theorem 4. *Let $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \to \mathbb{R}$ be a continuous convex function on the interval $[\gamma, \Gamma]$, $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function on the interval $[a, b]$ and with the property that*

$$\gamma \leq f(t) \leq \Gamma \text{ for any } t \in [a, b] \tag{2.1}$$

and $w : [a, b] \to [0, \infty)$ be continuos on $[a, b]$. Then for each monotonic nondecreasing function $u : [a, b] \to \mathbb{R}$ such that $\int_a^b w(t)\, du(t) > 0$ we have the inequalities

$$\begin{aligned} &\Phi\left(\frac{\int_a^b w(t) f(t)\, du(t)}{\int_a^b w(t)\, du(t)}\right) \\ &\leq \frac{\int_a^b w(t) (\Phi \circ f)(t)\, du(t)}{\int_a^b w(t)\, du(t)} \\ &\leq \frac{\Phi(\gamma)\left(\Gamma - \frac{\int_a^b w(t)f(t)du(t)}{\int_a^b w(t)du(t)}\right) + \Phi(\Gamma)\left(\frac{\int_a^b w(t)f(t)du(t)}{\int_a^b w(t)du(t)} - \gamma\right)}{\Gamma - \gamma}. \end{aligned} \tag{2.2}$$

Proof. Utilising the gradient inequality for the convex function Φ, namely

$$\Phi(\varsigma) - \Phi(\tau) \geq \delta_\Phi(\tau)(\varsigma - \tau)$$

for any $\varsigma, \tau \in [\gamma, \Gamma]$ where $\delta_\Phi(\tau) \in \left[\Phi'_-(\tau), \Phi'_+(\tau)\right]$, (for $\tau = \gamma$ we take $\delta_\Phi(\tau) = \Phi'_+(\gamma)$ and for $\tau = \Gamma$ we take $\delta_\Phi(\tau) = \Phi'_-(\Gamma)$) then we get

$$\begin{aligned} &\Phi(\varsigma) - \Phi\left(\frac{\int_a^b w(t) f(t)\, du(t)}{\int_a^b w(t)\, du(t)}\right) \\ &\geq \delta_\Phi\left(\frac{\int_a^b w(t) f(t)\, du(t)}{\int_a^b w(t)\, du(t)}\right)\left(\varsigma - \frac{\int_a^b w(t) f(t)\, du(t)}{\int_a^b w(t)\, du(t)}\right) \end{aligned} \tag{2.3}$$

for any $\varsigma \in [\gamma, \Gamma]$, since obviously, by (2.1)

$$\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)} \in [\gamma, \Gamma].$$

Since f satisfies (2.1), then by (2.3) we get

$$(\Phi \circ f)(s) - \Phi\left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)}\right) \geq \delta_\Phi\left(\frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)}\right)\left(f(s) - \frac{\int_a^b w(t) f(t) du(t)}{\int_a^b w(t) du(t)}\right) \tag{2.4}$$

for any $s \in [a, b]$.

Now, if we multiply (2.4) by $w(s) \geq 0$ and integrate the result over the monotonic nondecreasing integrator u on the interval $[a, b]$ we obtain the first inequality in (2.2).

By the convexity of Φ we also have the inequality

$$\Phi(\tau) \leq \frac{(\Gamma - \tau)\Phi(\gamma) + (\tau - \gamma)\Phi(\Gamma)}{\Gamma - \gamma}$$

for any $\tau \in [\gamma, \Gamma]$, which, by (2.3) implies that

$$(\Phi \circ f)(s) \leq \frac{(\Gamma - f(s))\Phi(\gamma) + (f(s) - \gamma)\Phi(\Gamma)}{\Gamma - \gamma} \tag{2.5}$$

for any $s \in [a, b]$.

Now, if we multiply (2.5) by $w(s) \geq 0$ and integrate the result over the monotonic nondecreasing integrator u on the interval $[a, b]$ we obtain the second inequality in (2.5).

The proof is complete. □

Remark 1. *The above inequality provides a generalization for the unweighted case, namely $w(t) = 1$, $t \in [a, b]$, which can be stated as*

$$\begin{aligned} \Phi\left(\frac{\int_a^b f(t) du(t)}{u(b) - u(a)}\right) &\leq \frac{\int_a^b (\Phi \circ f)(t) du(t)}{u(b) - u(a)} \\ &\leq \frac{\Phi(\gamma)\left(\Gamma - \frac{\int_a^b f(t)du(t)}{u(b)-u(a)}\right) + \Phi(\Gamma)\left(\frac{\int_a^b f(t)du(t)}{u(b)-u(a)} - \gamma\right)}{\Gamma - \gamma}. \end{aligned} \tag{2.6}$$

For inequalities related to the Jensen's result, see [1], [2], [3], [17], [4], [26] and [27].

Corollary 1. *Let $h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function on the interval $[a, b]$ and with the property that*

$$0 \leq \gamma \leq h(t) \leq \Gamma \text{ for any } t \in [a, b] \tag{2.7}$$

and $w : [a, b] \to [0, \infty)$ be continuos on $[a, b]$. Assume also that $u : [a, b] \to \mathbb{R}$ is a monotonic nondecreasing function such that $\int_a^b w(t) du(t) > 0$.

(i) *If* $p \geq 1$, *then*

$$\begin{aligned}(2.8)\qquad & \left(\int_a^b w(t)\,h(t)\,du(t)\right)^p \\ & \leq \left[\int_a^b w(t)\,du(t)\right]^{p-1} \int_a^b w(t)\,h^p(t)\,du(t) \\ & \leq \frac{1}{\Gamma-\gamma}\left[\int_a^b w(t)\,du(t)\right]^p \\ & \times \left[\gamma^p\left(\Gamma - \frac{\int_a^b w(t)\,h(t)\,du(t)}{\int_a^b w(t)\,du(t)}\right) + \Phi^p\left(\frac{\int_a^b w(t)\,h(t)\,du(t)}{\int_a^b w(t)\,du(t)} - \gamma\right)\right].\end{aligned}$$

(ii) *If* $p \in (0,1)$, *then the inequalities reverse in (2.8).*
(iii) *If* $p < 0$ *and* $\gamma > 0$, *then the inequality (2.8) also holds.*

The proof follows by Theorem 4 applied for the convex (concave) function $f(t) = t^p$, $p \in (-\infty, 0) \cup [1, \infty)$ $(p \in (0,1))$.

The following result is the well known version of the Hölder inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators $u : [a,b] \to \mathbb{R}$:

$$(2.9)\qquad \int_a^b |f(t)\,g(t)|\,du(t) \leq \left[\int_a^b |f(t)|^p\,du(t)\right]^{1/p}\left[\int_a^b |g(t)|^q\,du(t)\right]^{1/q},$$

where $f, g : [a,b] \subset \mathbb{R} \to \mathbb{C}$ are continuous and $p, q > 1$ with $1/p + 1/q = 1$.

Proposition 1. *Let* $f, g : [a,b] \subset \mathbb{R} \to \mathbb{C} \setminus \{0\}$ *be continuous on* $[a,b]$ *and* $u : [a,b] \to \mathbb{R}$ *monotonic nondecreasing on* $[a,b]$. *Let* $p, q \in \mathbb{R} \setminus \{0\}$ *with* $1/p + 1/q = 1$ *and assume that*

$$(2.10)\qquad 0 \leq \gamma \leq \frac{|f(t)|}{|g(t)|^{q-1}} \leq \Gamma \text{ for any } t \in [a,b].$$

(i) *If* $p > 1$, *then*

$$\begin{aligned}(2.11)\qquad & \int_a^b |f(t)\,g(t)|\,du(t) \\ & \leq \left[\int_a^b |g(t)|^q\,du(t)\right]^{1/q}\left[\int_a^b |f(t)|^p\,du(t)\right]^{1/p} \\ & \leq \frac{1}{(\Gamma-\gamma)^{1/p}}\int_a^b |g(t)|^q\,du(t) \\ & \times \left[\gamma^p\left(\Gamma - \frac{\int_a^b |f(t)\,g(t)|\,du(t)}{\int_a^b |g(t)|^q\,du(t)}\right) + \Phi^p\left(\frac{\int_a^b |f(t)\,g(t)|\,du(t)}{\int_a^b |g(t)|^q\,du(t)} - \gamma\right)\right]^{1/p}.\end{aligned}$$

(ii) *If* $p \in (0,1)$, *then the inequalities in (2.11) reverse.*
(iii) *If* $p < 0$ *and* $\gamma > 0$ *then the inequalities in (2.11) also reverse.*

Proof. Follows by Corollary 1 on choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q$$

and performing some simple calculation.

The details are omitted. □

Corollary 2. *Let $h : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function on the interval $[a,b]$ and with the property that*

$$0 < \gamma \leq h(t) \leq \Gamma \text{ for any } t \in [a,b] \tag{2.12}$$

and $w : [a,b] \to [0,\infty)$ be continuos on $[a,b]$. Assume also that $u : [a,b] \to \mathbb{R}$ is a monotonic nondecreasing function such that $\int_a^b w(t)\,du(t) > 0$. Then

$$\begin{aligned} &\frac{\int_a^b w(t)\,h(t)\,du(t)}{\int_a^b w(t)\,du(t)} \\ &\geq \exp\left[\frac{\int_a^b w(t)\,(\ln\circ h)(t)\,du(t)}{\int_a^b w(t)\,du(t)}\right] \\ &\geq \gamma^{\frac{1}{\Gamma-\gamma}\left(\Gamma-\frac{\int_a^b w(t)h(t)du(t)}{\int_a^b w(t)du(t)}\right)}\Gamma^{\frac{1}{\Gamma-\gamma}\left(\frac{\int_a^b w(t)h(t)du(t)}{\int_a^b w(t)du(t)}-\gamma\right)}. \end{aligned} \tag{2.13}$$

The proof follows by Theorem 4 applied for the convex function $\Phi(t) = -\ln t, t > 0$.

3. Weighted Inequalities for Convex Functions of Selfadjoint Operators

We can state the following result concerning the weighted Jensen's inequality for continuous functions of selfadjoint operators:

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m,M with $m < M$. If $\Phi : [\gamma,\Gamma] \subset \mathbb{R} \to \mathbb{R}$ is a continuous convex function on the interval $[\gamma,\Gamma]$, $f : [m,M] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function on the interval $[m,M]$ and with the property that*

$$\gamma \leq f(t) \leq \Gamma \text{ for any } t \in [m,M] \tag{3.1}$$

and $w : [m,M] \to [0,\infty)$ is continuos on $[m,M]$, then

$$\begin{aligned} &\Phi\left(\frac{\langle w(A)\,f(A)\,x,x\rangle}{\langle w(A)\,x,x\rangle}\right) \\ &\leq \frac{\langle w(A)\,(\Phi\circ f)(A)\,x,x\rangle}{\langle w(A)\,x,x\rangle} \\ &\leq \frac{\Phi(\gamma)\left(\Gamma-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)+\Phi(\Gamma)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-\gamma\right)}{\Gamma-\gamma}, \end{aligned} \tag{3.2}$$

for any $x \in H$ with $\|x\| = 1$ and $\langle w(A)\,x,x\rangle \neq 0$.

Proof. Let $\{E_\lambda\}_\lambda$ be the spectral family of the operator A. Let $\varepsilon > 0$ and write the inequality (2.2) on the interval $[a,b] = [m-\varepsilon, M]$ and for the monotonic nondecreasing function $g(t) = \langle E_t x, x\rangle$, $x \in H$ with $\|x\| = 1$, to get

$$\begin{aligned}
&\Phi\left(\frac{\int_{m-\varepsilon}^{M} w(t) f(t)\, d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^{M} w(t)\, d\langle E_t x, x\rangle}\right) \\
&\le \frac{\int_{m-\varepsilon}^{M} w(t) (\Phi\circ f)(t)\, d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^{M} w(t)\, d\langle E_t x, x\rangle} \\
&\le \frac{\left(\Gamma - \frac{\int_{m-\varepsilon}^{M} w(t) f(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^{M} w(t) d\langle E_t x, x\rangle}\right)\Phi(\gamma) + \left(\frac{\int_{m-\varepsilon}^{M} w(t) f(t) d\langle E_t x, x\rangle}{\int_{m-\varepsilon}^{M} w(t) d\langle E_t x, x\rangle} - \gamma\right)\Phi(\Gamma)}{\Gamma - \gamma}.
\end{aligned} \tag{3.3}$$

Letting $\varepsilon \to 0+$ and utilizing the spectral representation (1.1) we deduce from (3.3) the desired result (3.2). $\square$

Remark 2. *If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M]$ then we get from (3.2) the inequalities (MP) and (LR).*

We have the following generalization and reverse for the Hölder-McCarthy inequality:

Corollary 3. *Let A be a selfadjoint positive operator on a Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If the functions $f, w : [m, M] \to [0, \infty)$ are continuous and f satisfies the condition (3.1) with $\gamma \ge 0$, then for any $p \ge 1$ we have*

$$\begin{aligned}
&\langle w(A) f(A) x, x\rangle^p \\
&\le \langle w(A) f^p(A) x, x\rangle \langle w(A) x, x\rangle^{p-1} \\
&\le \frac{1}{\Gamma - \gamma} \langle w(A) x, x\rangle^{p-1} \\
&\times \left[\gamma^p \left(\langle w(A) [\Gamma 1_H - f(A)] x, x\rangle\right) + \Gamma^p \left(\langle w(A) [f(A) - \gamma 1_H] x, x\rangle\right)\right]
\end{aligned} \tag{3.4}$$

where $x \in H$ with $\|x\| = 1$.

If $p \in (0,1)$ then the inequalities reverse in (3.4).

If $\gamma > 0$ and $p < 0$ the inequalities in (3.4) also hold.

Remark 3. *If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from (3.4)*

$$\begin{aligned}
\langle Ax, x\rangle^p &\le \langle A^p x, x\rangle \\
&\le \frac{1}{M - m} \left[m^p \left(\langle (M 1_H - A) x, x\rangle\right) + M^p \left(\langle (A - m 1_H) x, x\rangle\right)\right]
\end{aligned} \tag{3.5}$$

for any $p \ge 1$, where $x \in H$ with $\|x\| = 1$.

If $p \in (0,1)$, then the inequalities reverse in (3.5).

If $m > 0$ and $p < 0$ then the inequalities in (3.5) also hold.

Remark 4. *If we choose $w(t) = f(t) = t$ with $t \in [m, M] \subset [0, \infty)$ then we get from (3.4)*

$$\left\langle A^2 x, x\right\rangle^p \leq \langle A^p x, x\rangle \langle Ax, x\rangle^{p-1} \leq \frac{1}{M-m} \langle Ax, x\rangle^{p-1} \times [m^p (\langle A(M1_H - A)x, x\rangle) + M^p (\langle A(A - m1_H)x, x\rangle)] \tag{3.6}$$

for any $p \geq 1$, where $x \in H$ with $\|x\| = 1$.

If $p \in (0, 1)$, then the inequalities reverse in (3.6).

If $m > 0$ and $p < 0$ then the inequalities in (3.6) also hold.

Corollary 4. *Let A be a selfadjoint positive operator on a Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If the functions $f, w : [m, M] \to [0, \infty)$ are continuous and f satisfies the condition (3.1) with $\gamma > 0$ then*

$$\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} \geq \exp\left[\frac{\langle w(A)(\ln \circ f)(A) x, x\rangle}{\langle w(A) x, x\rangle}\right] \geq \gamma^{\frac{1}{\Gamma-\gamma}\left(\Gamma - \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)} \Gamma^{\frac{1}{\Gamma-\gamma}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle} - \gamma\right)} \tag{3.7}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 5. *If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [m, M] \subset (0, \infty)$ then we get from (3.7)*

$$\langle Ax, x\rangle \geq \exp[\langle \ln Ax, x\rangle] \geq m^{\frac{1}{M-m}\langle (M1_H - A)x, x\rangle} M^{\frac{1}{M-m}\langle (A - 1_H m)x, x\rangle} \tag{3.8}$$

for any $x \in H$ with $\|x\| = 1$.

Also, if we choose $w(t) = f(t) = t$ with $t \in [m, M] \subset (0, \infty)$ then we get from (3.7) that

$$\frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle} \geq \exp\left[\frac{\langle A \ln Ax, x\rangle}{\langle Ax, x\rangle}\right] \geq m^{\frac{1}{M-m}\left(M - \frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle}\right)} M^{\frac{1}{M-m}\left(\frac{\left\langle A^2 x, x\right\rangle}{\langle Ax, x\rangle} - m\right)} \tag{3.9}$$

for any $x \in H$ with $\|x\| = 1$.

Remark 6. *If we choose $w(t) = t^r$ and $f(t) = t^q$ with $t \in [m, M] \subset (0, \infty)$ where $r, q > 0$, then we get from (3.2) that*

$$\Phi\left(\frac{\langle A^{r+q} x, x\rangle}{\langle A^r x, x\rangle}\right) \leq \frac{\langle A^r \Phi(A^q) x, x\rangle}{\langle A^r x, x\rangle} \leq \frac{\Phi(\gamma^q)\left(\Gamma^q - \frac{\langle A^{r+q}x,x\rangle}{\langle A^r x,x\rangle}\right) + \Phi(\Gamma^q)\left(\frac{\langle A^{r+q}x,x\rangle}{\langle A^r x,x\rangle} - \gamma^q\right)}{\Gamma^q - \gamma^q}, \tag{3.10}$$

for a continuous convex function $\Phi : [m^q, M^q] \to \mathbb{R}$ *and for any* $x \in H$ *with* $\|x\| = 1$.

We have the following Hölder type inequality for continuous functions of selfadjoint operators:

Proposition 2. *Let* A *be a selfadjoint positive operator on a Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* $f, g : [a, b] \subset \mathbb{R} \to \mathbb{C} \setminus \{0\}$ *are continuous on* $[a, b]$ *and* $p, q \in \mathbb{R} \setminus \{0\}$ *with* $1/p + 1/q = 1$ *are such that*

$$0 \le \gamma \le \frac{|f(t)|}{|g(t)|^{q-1}} \le \Gamma \text{ for any } t \in [a, b], \tag{3.11}$$

then we have the inequalities

$$\begin{aligned} &\langle |f(A) g(A)| x, x\rangle \\ &\le [\langle |g(A)|^q x, x\rangle]^{1/q} [\langle |f(A)|^p x, x\rangle]^{1/p} \\ &\le \frac{1}{(\Gamma - \gamma)^{1/p}} \langle |g(A)|^q x, x\rangle \\ &\times \left[\gamma^p \left(\Gamma - \frac{\langle |f(A) g(A)| x, x\rangle}{\langle |g(A)|^q x, x\rangle}\right) + \Gamma^p \left(\frac{\langle |f(A) g(A)| x, x\rangle}{\langle |g(A)|^q x, x\rangle} - \gamma\right)\right]^{1/p}, \end{aligned} \tag{3.12}$$

for $p > 1$ *and for any* $x \in H$ *with* $\|x\| = 1$ *and* $\langle |g(A)|^q x, x\rangle \ne 0$.

If $p \in (0, 1)$, *then the inequalities in (3.12) reverse;*

If $p < 0$ *and* $\gamma > 0$ *then the inequalities in (3.12) also reverse.*

4. Weighted Inequalities for Square-convex Functions

We introduce the following class of complex valued functions:

Definition 1. *A function* $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \to \mathbb{C}$ *is called square-convex on* $[\gamma, \Gamma]$ *if the associated function* $\varphi : [\gamma, \Gamma] \to [0, \infty)$, $\varphi(t) = |\Phi(t)|^2$ *is convex on* $[\gamma, \Gamma]$.

A simple example of such a function is the concave power function $\Phi : [\gamma, \Gamma] \subset [0, \infty) \to [0, \infty)$, $\Phi(t) = t^r$ with $r \in \left[\frac{1}{2}, 1\right]$. Also, if $h : [\gamma, \Gamma] \to [0, \infty)$ is convex then the complex valued function $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \to \mathbb{C}$ given by $\Phi(t) = h^{1/2}(t) e^{it}$ is square-convex on $[\gamma, \Gamma]$.

Consider the function $f(t) = \ln(t + 1)$. We observe that it is concave and positive on $(0, \infty)$ and if we define $\varphi(t) = [\ln(t + 1)]^2$, then we have that

$$\varphi''(t) = \frac{2}{(t+1)^2} [1 - \ln(t + 1)], \ t > -1,$$

showing that f is square-convex on the interval $[0, e - 1]$.

Another example for trigonometric functions is for instance $f(t) = \cos t, t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. The function $\varphi(t) = \cos^2 t$ has the second derivative $\varphi''(t) = -2\cos(2t)$ which is positive for $t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. Therefore f is square-convex on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Theorem 6. *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* $\Phi : [\gamma, \Gamma] \subset \mathbb{R} \to \mathbb{C}$ *is a continuous square-convex function on the interval* $[\gamma, \Gamma]$, $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function on the interval* $[m, M]$ *and with the property that*

$$\gamma \le f(t) \le \Gamma \text{ for any } t \in [m, M] \tag{4.1}$$

and $w:[m,M]\to[0,\infty)$ *is continuos on* $[m,M]$*, then*

$$\begin{aligned}(4.2)\qquad &\left|\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\right| \\ &\leq\left[\frac{\left\langle w(A)\left(|\Phi|^2\circ f\right)(A)x,x\right\rangle}{\langle w(A)x,x\rangle}\right]^{1/2} \\ &\leq\left[\frac{|\Phi(\gamma)|^2\left(\Gamma-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)+|\Phi(\Gamma)|^2\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-\gamma\right)}{\Gamma-\gamma}\right]^{1/2},\end{aligned}$$

for any $x\in H$ *with* $\|x\|=1$ *and* $\langle w(A)x,x\rangle\neq0$.

The proof follows from Theorem 4 applied for the function $\varphi:[\gamma,\Gamma]\to[0,\infty)$, $\varphi(t)=|\Phi(t)|^2$ that is continuous convex on $[\gamma,\Gamma]$. The details are omitted.

Remark 7. *If* $w(t)=1$*, then we get from (4.2) the following simpler result*

$$\begin{aligned}(4.3)\qquad &|\Phi(\langle f(A)x,x\rangle)| \\ &\leq\|(\Phi\circ f)(A)x\| \\ &\leq\left[\frac{|\Phi(\gamma)|^2\langle(\Gamma 1_H-f(A))x,x\rangle+|\Phi(\Gamma)|^2\langle(f(A)-1_H\gamma)x,x\rangle}{\Gamma-\gamma}\right]^{1/2},\end{aligned}$$

for any $x\in H$ *with* $\|x\|=1$.

This is true since

$$\begin{aligned}\left\langle\left(|\Phi|^2\circ f\right)(A)x,x\right\rangle &= \int_{m-0}^{M}|\Phi(f(t))|^2\,d\langle E_t x,x\rangle \\ &= \|\Phi(f(A))x\|^2\end{aligned}$$

for any $x\in H$ *with* $\|x\|=1$ *(for the second equality see for instance* [19, p. 257]*).*

Corollary 5. *With the assumptions of Theorem 6 for* A,f,w *and if* $\gamma>0$*, then we have*

$$\begin{aligned}(4.4)\qquad &\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^q \\ &\leq\left[\frac{\left\langle w(A)f^{2q}(A)x,x\right\rangle}{\langle w(A)x,x\rangle}\right]^{\frac12} \\ &\leq\left[\frac{\gamma^{2q}\left(\Gamma-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)+\Gamma^{2q}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-\gamma\right)}{\Gamma-\gamma}\right]^{\frac12},\end{aligned}$$

for any $q\in\left[\frac12,1\right]$ *and any* $x\in H$ *with* $\|x\|=1$ *and* $\langle w(A)x,x\rangle\neq0$.

Remark 8. *If we choose* $w(t)=1$ *and* $f(t)=t$ *with* $t\in[m,M]\subset(0,\infty)$ *then we get from (4.4)*

$$\begin{aligned}(4.5)\qquad \langle Ax,x\rangle^q &\leq\|A^q x\| \\ &\leq\left[\frac{m^{2q}\langle(M1_H-A)x,x\rangle+M^{2q}\langle(A-1_H m)x,x\rangle}{M-m}\right]^{1/2},\end{aligned}$$

for any $q \in \left[\frac{1}{2}, 1\right]$ *and any* $x \in H$ *with* $\|x\| = 1$.

Also, if we choose $w(t) = f(t) = t$ *with* $t \in [m, M] \subset (0, \infty)$ *then we get from (4.4)*

$$\left(\frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle}\right)^q \le \left[\frac{\langle A^{2q+1} x, x\rangle}{\langle Ax, x\rangle}\right]^{1/2} \le \left[\frac{m^{2q}\left(M - \frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle}\right) + M^{2q}\left(\frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle} - m\right)}{M - m}\right]^{1/2}, \tag{4.6}$$

for any $q \in \left[\frac{1}{2}, 1\right]$ *and any* $x \in H$ *with* $\|x\| = 1$.

Remark 9. *If we choose* $w(t) = t^r$ *and* $f(t) = t^s$ *with* $t \in [m, M] \subset (0, \infty)$ *where* $r, s > 0$, *then we get from (4.4) that*

$$\left(\frac{\langle A^{r+s} x, x\rangle}{\langle A^r x, x\rangle}\right)^q \le \left[\frac{\langle A^{r+2qs} x, x\rangle}{\langle A^r x, x\rangle}\right]^{\frac{1}{2}} \le \left[\frac{m^{2qs}\left(M^s - \frac{\langle A^{r+s} x, x\rangle}{\langle A^r x, x\rangle}\right) + M^{2qs}\left(\frac{\langle A^{r+s} x, x\rangle}{\langle A^r x, x\rangle} - m^s\right)}{M^s - m^s}\right]^{\frac{1}{2}}, \tag{4.7}$$

for any $q \in \left[\frac{1}{2}, 1\right]$ *and any* $x \in H$ *with* $\|x\| = 1$.

5. Weighted Inequalities for Arg-square-convex Functions

The function $\Phi : \mathcal{C}(0,1) \to \mathbb{C}$ will be called *Arg-square-convex* if the composite function $\varphi : [0, 2\pi] \to [0, \infty)$,

$$\varphi(t) := \begin{cases} \left|\Phi\left(e^{it}\right)\right|^2 & t \in [0, 2\pi) \\ \lim_{s \to 2\pi-} \left|\Phi\left(e^{is}\right)\right|^2 & t = 2\pi \end{cases}$$

is *continuous and convex* on $[0, 2\pi]$.

To make the distinction between the value $\varphi(0) = \left|\Phi\left(e^{i0}\right)\right|^2 = |\Phi(1)|^2$ and the value $\varphi(2\pi) = \lim_{s \to 2\pi-} \left|\Phi\left(e^{is}\right)\right|^2$ we denote by $\Phi_c(1) := \lim_{s \to 2\pi-} \Phi\left(e^{is}\right)$. With this notation we have $\varphi(2\pi) = |\Phi_c(1)|^2$.

The function $\Phi_n : \mathcal{C}(0,1) \to \mathbb{C}$, $\Phi_n(z) = [Log(z)]^n$, where n is a positive integer, is Arg-square-convex. We have

$$\varphi_n(t) = \left|\Phi_n\left(e^{it}\right)\right|^2 = \left|\left[Log\left(e^{it}\right)\right]^n\right|^2 = |it|^{2n} = t^{2n}, t \in [0, 2\pi),$$

and

$$\varphi_n(2\pi) = \lim_{s \to 2\pi-} \left|\Phi_n\left(e^{is}\right)\right|^2 = |\Phi_{n,c}(1)|^2 = (2\pi)^{2n}.$$

For $q \geq \frac{1}{2}$ define the function $\Phi_q : \mathcal{C}(0,1) \to [0,\infty)$ by $\Phi_q(z) = |Log(z)|^q$. We have

$$\varphi_q(t) = \left|\Phi_q\left(e^{it}\right)\right|^2 = \left|Log\left(e^{it}\right)\right|^{2q} = |it|^{2q} = t^{2q}, t \in [0,2\pi)$$

and

$$\varphi_q(2\pi) = \lim_{s \to 2\pi-} \left|\Phi_q\left(e^{is}\right)\right|^2 = \left|\Phi_{q,c}(1)\right|^2 = (2\pi)^{2q}.$$

The function Φ_q for $q \geq \frac{1}{2}$ is an Arg-square-convex function.

If $g : [0,2\pi] \to [0,\infty)$ is continuous and convex on $[0,2\pi]$, then the composite function $\Phi : \mathcal{C}(0,1) \to [0,\infty)$ defined by

$$\Phi(z) := [g(|Log(z)|)]^{1/2}$$

is an Arg-square-convex function on $\mathcal{C}(0,1)$.

Theorem 7. *Let $U \in B(H)$ be a unitary operator on the Hilbert space H and $\Phi : \mathcal{C}(0,1) \to \mathbb{C}$ a continuous and Arg-square-convex function on $\mathcal{C}(0,1)$. If $w : \mathcal{C}(0,1) \to [0,\infty)$ is a continuous function, then we have*

$$\left|\Phi\left(\exp\left[\frac{\langle w(U)\,Log(U)x,x\rangle}{\langle w(U)\,x,x\rangle}\right]\right)\right| \tag{5.1}$$
$$\leq \left[\frac{\left\langle w(U)\,|\Phi(U)|^2\,x,x\right\rangle}{\langle w(U)\,x,x\rangle}\right]^{1/2}$$
$$\leq \left[\frac{\left(2\pi - \frac{\langle w(U)|Log(U)|x,x\rangle}{\langle w(U)x,x\rangle}\right)|\Phi(1)|^2 + \frac{\langle w(U)|Log(U)|x,x\rangle}{\langle w(U)x,x\rangle}|\Phi_c(1)|^2}{2\pi}\right]^{1/2}$$

for any $x \in H, \|x\| = 1$, where $\Phi_c(1) := \lim_{s\to 2\pi-} \Phi\left(e^{is}\right)$.

Proof. We apply Theorem 4 to the function $\varphi : [0,2\pi] \to [0,\infty)$,

$$\varphi(t) = \begin{cases} \left|\Phi\left(e^{it}\right)\right|^2 & t \in [0,2\pi) \\ \\ \lim_{s\to 2\pi-}\left|\Phi\left(e^{is}\right)\right|^2 & t = 2\pi \end{cases}$$

that is continuous and convex on $[0,2\pi]$.

If $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of the operator U, then we can write the inequality (2.2) on the interval $[a,b] = [0,2\pi]$ for the monotonic nondecreasing integrator $u(t) = \langle E_t x, x\rangle$ and for the identity function $f(t) = t, t \in [0,2\pi]$ to get

$$\left|\Phi\left(\exp\left[\frac{i\int_0^{2\pi} w\left(e^{it}\right)t d\langle E_t x,x\rangle}{\int_0^{2\pi} w\left(e^{it}\right)d\langle E_t x,x\rangle}\right]\right)\right|^2 \tag{5.2}$$
$$\leq \frac{\int_0^{2\pi} w\left(e^{it}\right)\left|\Phi\left(e^{it}\right)\right|^2 d\langle E_t x,x\rangle}{\int_0^{2\pi} w\left(e^{it}\right)d\langle E_t x,x\rangle}$$
$$\leq \frac{\left(2\pi - \frac{\int_0^{2\pi} w\left(e^{it}\right)td\langle E_t x,x\rangle}{\int_0^{2\pi} w\left(e^{it}\right)d\langle E_t x,x\rangle}\right)|\Phi(1)|^2 + \left(\frac{\int_0^{2\pi} w\left(e^{it}\right)\Phi(t)d\langle E_t x,x\rangle}{\int_0^{2\pi} w(t)d\langle E_t x,x\rangle}\right)|\Phi_c(1)|^2}{2\pi},$$

for any $x \in H, \|x\| = 1$.

Since, by the spectral representation of functions of unitary operators (1.3) we have

$$\begin{aligned} i\int_0^{2\pi} w\left(e^{it}\right) td\left\langle E_t x,x\right\rangle &= \int_0^{2\pi} w\left(e^{it}\right) Log\left(e^{it}\right) d\left\langle E_t x,x\right\rangle \\ &= \left\langle w\left(U\right) Log(U)x,x\right\rangle \end{aligned}$$

$$\int_0^{2\pi} w\left(e^{it}\right) d\left\langle E_t x,x\right\rangle = \left\langle w\left(U\right)x,x\right\rangle,$$

$$\int_0^{2\pi} w\left(e^{it}\right)\left|\Phi\left(e^{it}\right)\right|^2 d\left\langle E_t x,x\right\rangle = \left\langle w\left(U\right)\left|\Phi\left(U\right)\right|^2 x,x\right\rangle \text{ and}$$

$$\int_0^{2\pi} w\left(e^{it}\right) td\left\langle E_t x,x\right\rangle = \left\langle w\left(U\right)\left|Log(U)\right|x,x\right\rangle$$

for any $x\in H, \|x\|=1$, then the inequality (5.2) produces the desired result (5.1). □

Remark 10. *If $w(t)=1$, then we get from (5.1) the following simpler result*

$$\begin{aligned} &\left|\Phi\left(\exp\left[\left\langle Log(U)x,x\right\rangle\right]\right)\right| \\ &\leq \left\|\Phi\left(U\right)x\right\| \\ &\leq \left[\frac{\left\langle\left(2\pi 1_H-\left|Log(U)\right|\right)x,x\right\rangle\left|\Phi\left(1\right)\right|^2+\left\langle\left|Log(U)\right|x,x\right\rangle\left|\Phi_c\left(1\right)\right|^2}{2\pi}\right]^{1/2} \end{aligned} \tag{5.3}$$

for any $x\in H$ with $\|x\|=1$.

This is true since

$$\left\langle\left|\Phi\left(U\right)\right|^2 x,x\right\rangle = \int_0^{2\pi}\left|\Phi\left(e^{it}\right)\right|^2 d\left\langle E_t x,x\right\rangle = \left\|\Phi\left(U\right)x\right\|^2$$

for any $x\in H$ with $\|x\|=1$ (for the second equality see (1.5)).

The interested reader may apply the inequality (5.1) for different examples of Arg-square-convex functions. We give here only one example, for instance if we choose the function $\Phi_q(z)=\left|Log(z)\right|^q, q\geq 1/2$ as introduced above, then we get from (5.1)

$$\begin{aligned} \left|Log\left(\exp\left[\frac{\left\langle w\left(U\right)Log(U)x,x\right\rangle}{\left\langle w\left(U\right)x,x\right\rangle}\right]\right)\right|^q &\leq \left[\frac{\left\langle w\left(U\right)\left|Log\left(U\right)\right|^{2q}x,x\right\rangle}{\left\langle w\left(U\right)x,x\right\rangle}\right]^{1/2} \\ &\leq \frac{\left\langle w\left(U\right)\left|Log(U)\right|x,x\right\rangle^{1/2}}{\left\langle w\left(U\right)x,x\right\rangle^{1/2}}\left(2\pi\right)^{q-1/2} \end{aligned} \tag{5.4}$$

for any $x\in H$ with $\|x\|=1$ and $w:\mathcal{C}(0,1)\to[0,\infty)$ a continuous function.

In particular we have

$$\begin{aligned} \left|Log\left(\exp\left[\left\langle Log(U)x,x\right\rangle\right]\right)\right|^q &\leq \left\|\left|Log\left(U\right)\right|^q x\right\| \\ &\leq \left(2\pi\right)^{q-1/2}\left\langle\left|Log(U)\right|x,x\right\rangle^{1/2} \end{aligned} \tag{5.5}$$

for any $x\in H$ with $\|x\|=1$.

Finally, we notice that the following result providing Hölder's type inequalities for continuous functions of unitary operators can be stated:

Proposition 3. *Let $U \in B(H)$ be a unitary operator on the Hilbert space H and. If $f, g : \mathcal{C}(0,1) \to \mathbb{C} \setminus \{0\}$ are continuous on $\mathcal{C}(0,1)$ and $p, q \in \mathbb{R} \setminus \{0\}$ with $1/p + 1/q = 1$ are such that*

$$0 \leq \gamma \leq \frac{\left|f\left(e^{it}\right)\right|}{\left|g\left(e^{it}\right)\right|^{q-1}} \leq \Gamma \text{ for any } t \in [0, 2\pi] \tag{5.6}$$

then we have the inequalities

$$\begin{aligned} &\langle |f(U) g(U)| x, x\rangle \\ &\leq \left[\langle |g(U)|^q x, x\rangle\right]^{1/q} \left[\langle |f(U)|^p x, x\rangle\right]^{1/p} \\ &\leq \frac{1}{(\Gamma - \gamma)^{1/p}} \langle |g(U)|^q x, x\rangle \\ &\times \left[\gamma^p \left(\Gamma - \frac{\langle |f(U) g(U)| x, x\rangle}{\langle |g(U)|^q x, x\rangle}\right) + \Gamma^p \left(\frac{\langle |f(U) g(U)| x, x\rangle}{\langle |g(U)|^q x, x\rangle} - \gamma\right)\right]^{1/p}, \end{aligned} \tag{5.7}$$

for $p > 1$ and for any $x \in H$ with $\|x\| = 1$ and $\langle |g(U)|^q x, x\rangle \neq 0$.

If $p \in (0,1)$, then the inequalities in (5.7) reverse;

If $p < 0$ and $\gamma > 0$ then the inequalities in (5.7) also reverse.

The proof follows by Proposition 1 and the spectral representation for continuous functions of unitary operators.

If $g : [0, 2\pi] \to [0, \infty)$ is continuous and convex on $[0, 2\pi]$, then the composite function $f : \mathcal{C}(0,1) \to [0, \infty)$ defined by

$$f(z) := \left[g\left(|Log(z)|\right)\right]^{1/2}$$

is an Arg-square-convex function on $\mathcal{C}(0,1)$.

As examples of such functions we have

$$f_\alpha(z) := \exp\left(\alpha \left|Log(z)\right|\right)$$

which are Arg-square-convex functions on $\mathcal{C}(0,1)$ for any real number $\alpha \neq 0$.

We also notice that the family of functions $f_{m,n} : \mathcal{C}(0,1) \to \mathbb{C}$, $f_{m,n}(z) = z^m \left[Log(z)\right]^n$, where $m \neq 0$ is an integer and n is a positive integer, are Arg-square-convex functions.

The reader may apply the above inequalities for these functions as well. However, the details are omitted.

References

[1] S.S. Dragomir, A converse result for Jensen's discrete inequality via Gruss' inequality and applications in information theory. *An. Univ. Oradea Fasc. Mat.* **7** (1999/2000), 178–189.

[2] S.S. Dragomir, On a reverse of Jessen's inequality for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **2**(2001), No. 3, Article 36.

[3] S.S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), no. 3, 551–562. Preprint RGMIA *Res. Rep. Coll.* **5**(2002), Suplement, Art. 12. [ONLINE:http://rgmia.org/v5(E).php].

[4] S.S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 471-476.

[5] S.S. Dragomir, Bounds for the deviation of a function from the chord generated by its extremities. *Bull. Aust. Math. Soc.* **78** (2008), no. 2, 225–248.

[6] S.S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: http://rgmia.org/v11(E).php].

[7] S.S. Dragomir, Some inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Filomat* **23**(2009), No. 3, 81–92. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 10.

[8] S.S. Dragomir, Some Jensen's type inequalities for twice differentiable functions of selfadjoint operators in Hilbert spaces, *Filomat* **23**(2009), No. 3, 211-222. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 13.

[9] S.S. Dragomir, Some new Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, *Sarajevo J. Math.* **6(18)**, (2010), No. 1, 89-107. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 12. [ONLINE: http://rgmia.org/v11(E).php].

[10] S.S. Dragomir, New bounds for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, *Filomat* **24**(2010), No. 2, 27-39.

[11] S.S. Dragomir, Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces, *Bull. Malays. Math. Sci. Soc.* **34**(2011), No. 3. Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), Sup. Art. 2.

[12] S.S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *J. Ineq. & Appl.,* Vol. **2010**, Article ID 496821. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 15. [ONLINE: http://rgmia.org/v11(E).php].

[13] S.S. Dragomir, Some Slater's type inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Rev. Un. Mat. Argentina*, **52**(2011), No.1, 109-120. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 7.

[14] S.S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions, *Appl. Math. Comp.* **218**(2011), 766-772. Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), No. 1, Art. 7.

[15] S.S. Dragomir, Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), No. 2, Art 1.

[16] S.S. Dragomir, New Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces, *Sarajevo J. Math.* **19**(2011), No. 1, 67-80. Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), Sup. Art. 2.

[17] S.S. Dragomir and N.M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78. MR:1325895 (96c:26012).

[18] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[19] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space,* John Wiley, New York, 1969.

[20] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), no. 2-3, 551–564.

[21] C.A. McCarthy, c_p, *Israel J. Math.*, **5**(1967), 249-271.

[22] J. Mićić, Y. Seo, S.-E. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Ineq. Appl.*, **2**(1999), 83-111.

[23] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420.

[24] B. Mond and J. Pečarić, On some operator inequalities, *Indian J. Math.*, **35**(1993), 221-232.

[25] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.

[26] C.P. Niculescu, An extension of Chebyshev's inequality and its connection with Jensen's inequality. *J. Inequal. Appl.* **6** (2001), no. 4, 451–462.

[27] S. Simic, On a global upper bound for Jensen's inequality, *J. Math. Anal. Appl.* **343**(2008), 414-419.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa