COMPARING TWO INTEGRAL MEANS FOR MAPPING OF BOUNDED VARIATION AND APPLICATIONS

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Abstract. Some new estimates for the difference between the integral mean of a function with bounded variation and its mean over a subinterval are established and new applications for probability density functions are also given.

1. Introduction

The classical Ostrowski type integral inequality [1] stipulates a bound for the difference between a function evaluated at an interior point and the average of the function over an interval. More precisely,

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \le \left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$, where $f' \in L_\infty(a,b)$, that is,

$$\|f'\|_\infty = ess \sup_{t\in[a,b]} |f'(t)| < \infty,$$

and $f : [a,b] \to R$ is a differentiable function on (a,b). Here, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [2], Dragomir pointed out the following generalization of (1.1) for mapppings $f : [a,b] \to R$ of bounded variation on $[a,b]$ and provided some applications in Numerical Analysis and for Euler's Beta function:

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(x)\right| \le \left[\frac{1}{2}(b-a)+(x-\frac{a+b}{2})\right]\bigvee_a^b(f), \tag{1.2}$$

for any $x \in [a,b]$, where $\vee_a^b(f)$ denotes the total variation of f on [a, b]. The constant $\frac{1}{2}$ is sharp.

For various results and generalizations concerning Ostrowski's inequality, see [3]-[14] and the references therein.

In [15], Barnett et al. compared the difference of two integral means as in the following Theorem 1 in which the function has the first derivative bounded where is defined. The obtained results are also generalizations of (1.1) and have been applied to probability density functions, special means, Jeffreys divergence in Information Theory and the sampling of continuous streams in Statistics. For recent related results, see [16], [17] and [18].

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Theorem 1. *Let $f : [a,b] \to R$ be an absolutely continuous function with the property that $f' \in L_\infty[a,b]$. Then, for $a \le x < y \le b$, we have the inequality*

$$\begin{aligned}(1.3)\qquad &\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{y-x}\int_x^y f(u)du\right| \\ &\le \left\{\frac{1}{4} + \left[\frac{\frac{a+b}{2}-\frac{x+y}{2}}{b-a-y+x}\right]^2\right\}(b-a-y+x)\|f'\|_\infty \\ &\le \frac{1}{2}(b-a-y+x)\|f'\|_\infty.\end{aligned}$$

The constant $\frac{1}{4}$ is best possible in the first inequality and $\frac{1}{2}$ is best in the second one.

The main purpose of this article is to establish some new results related to the inequality (1.3) for the functions with bounded variation. As applications, some new inequalities for the probability density functions will be also given.

2. The main results

Theorem 2. *Let $f : [a,b] \to R$ be a mapping with bounded variation on $[a,b]$. Then, for $a \le x < y \le b$, we have the inequality*

$$\begin{aligned}(2.1)\qquad &\left|\frac{1}{b-a}\int_a^b f(s)ds - \frac{1}{y-x}\int_x^y f(s)ds\right| \\ &\le \frac{x-a}{b-a}\bigvee_a^x(f) + \max\left\{\frac{x-a}{b-a},\frac{b-y}{b-a}\right\}\bigvee_x^y(f) + \frac{b-y}{b-a}\bigvee_y^b(f).\end{aligned}$$

The inequality (2.1) *is sharp.*

Proof. Integrating by parts in Riemann-Stieltjes integral, we obtain

$$\int_a^x \frac{a-s}{b-a}df(s) = \frac{a-x}{b-a}f(x) + \frac{1}{b-a}\int_a^x f(s)ds,$$

$$\begin{aligned}&\int_x^y \left(\frac{s-x}{y-x}+\frac{a-s}{b-a}\right)df(s) \\ &= f(y) - \frac{1}{y-x}\int_x^y f(s)ds + \frac{a-y}{b-a}f(y) - \frac{a-x}{b-a}f(x) + \frac{1}{b-a}\int_x^y f(s)ds,\end{aligned}$$

and

$$\int_y^b \frac{b-s}{b-a}df(s) = -\frac{b-y}{b-a}f(y) + \frac{1}{b-a}\int_y^b f(s)ds,$$

Adding the above three identity, we have

$$(2.2)\qquad \frac{1}{b-a}\int_a^b f(s)ds - \frac{1}{y-x}\int_x^y f(s)ds = \int_a^b K_{x,y}(s)df(s)$$

where $K_{x,y} : [a,b] \to R$, given by

$$K_{x,y}(s) = \begin{cases} \frac{a-s}{b-a}, & \text{if } s \in [a,x], \\ \frac{s-x}{y-x}+\frac{a-s}{b-a}, & \text{if } s \in (x,y), \\ \frac{b-s}{b-a}, & \text{if } s \in [y,b]. \end{cases}$$

It is known that, see [19], if $u, v : [a, b] \to R$ are such that u is continuous on $[a, b]$ and v is of bounded variation on $[a, b]$, then

$$\left| \int_a^b u(t)dv(t) \right| \le \sup_{t\in[a,b]} |u(t)| \bigvee_a^b (v).$$

Therefore,

$$\begin{aligned} &\left| \int_a^b K_{x,y}(s)df(s) \right| \\ &\le \left| \int_a^x \frac{a-s}{b-a} df(s) \right| + \left| \int_x^y \left(\frac{s-x}{y-x} + \frac{a-s}{b-a}\right) df(s) \right| + \left| \int_y^b \frac{b-s}{b-a} df(s) \right| \\ &\le \frac{x-a}{b-a} \bigvee_a^x (f) + \max\left\{ \frac{x-a}{b-a}, \frac{b-y}{b-a} \right\} \bigvee_x^y (f) + \frac{b-y}{b-a} \bigvee_y^b (f) \end{aligned} \tag{2.3}$$

for $a \le x < y \le b$. By (2.2) and (2.3), the (2.1) holds immediately.

To prove the shapness, consider the mapping $f : [a, b] \to R$ given by

$$f(s) = \begin{cases} 0, & \text{if } s \in [a, x), \\ 1, & \text{if } s \in [x, y], \\ 0, & \text{if } s \in (y, b]. \end{cases}$$

Then f is bounded variation on $[a, b]$, and $\bigvee_a^x (f) = 1$, $\bigvee_x^y (f) = 0$, $\bigvee_y^b (f) = 1$, $\int_a^b f(s)ds = y - x$, $\int_x^y f(s)ds = y - x$. Obviously, the identity holds in (2.1). This completes the proofs. ∎

The following corollary holds.

Corollary 1. *Let $f : [a, b] \to R$ be a monotonic mapping on $[a, b]$. Then we have the inequality*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(s)ds - \frac{1}{y-x} \int_x^y f(s)ds \right| \\ &\le \frac{x-a}{b-a} \left| f(x) - f(a) \right| + \max\left\{ \frac{x-a}{b-a}, \frac{b-y}{b-a} \right\} \left| f(y) - f(x) \right| \\ &+ \frac{b-y}{b-a} \left| f(b) - f(y) \right| \end{aligned} \tag{2.4}$$

for $a \le x < y \le b$.

The case of lipschitzian mapping is embodied in the following corollary.

Corollary 2. *Let $f : [a, b] \to R$ be a L-lipschitzian mapping on $[a, b]$, that is,*

$$\left| f(x) - f(y) \right| \le L \left| x - y \right|$$

for positive number L and all $x, y \in [a, b]$. Then we have the inequality

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(s)ds - \frac{1}{y-x} \int_x^y f(s)ds \right| \\ &\le \frac{(x-a)^2 L}{b-a} + \max\left\{ \frac{x-a}{b-a}, \frac{b-y}{b-a} \right\} L(y-x) + \frac{(b-y)^2 L}{b-a} \end{aligned} \tag{2.5}$$

for $a \le x < y \le b$.

Donote $\|f'\|_{1,[s,t]} = \int_s^t |f'(u)|du$, for $s,t \in [a,b]$. Using Theorem 2, we have the following corollary immediately.

Corollary 3. *Let $f:[a,b]\to R$ be a continuous differentiable on (a,b), and f' is integrable on (a,b). Then we have the inequality*

$$\begin{aligned}(2.6)\qquad &\left|\frac{1}{b-a}\int_a^b f(s)ds - \frac{1}{y-x}\int_x^y f(s)ds\right| \\ &\le \frac{x-a}{b-a}\|f'\|_{1,[a,x]} + \max\left\{\frac{x-a}{b-a},\frac{b-y}{b-a}\right\}\|f'\|_{1,[x,y]} + \frac{b-y}{b-a}\|f'\|_{1,[y,b]}\end{aligned}$$

for $a\le x<y\le b$.

Remark 1. *If we set $y=x+h$ with $x+h\in(a,b)$, then by (2.1) we get*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(s)ds - \frac{1}{h}\int_x^{x+h} f(s)ds\right| \\ &\le \frac{x-a}{b-a}\|f'\|_{1,[a,x]} + \max\left\{\frac{x-a}{b-a},\frac{b-x-h}{b-a}\right\}\|f'\|_{1,[x,x+h]} + \frac{b-x-h}{b-a}\|f'\|_{1,[x+h,b]}.\end{aligned}$$

Now, letting $h\to 0^+$, we have

$$(2.7)\qquad \left|\frac{1}{b-a}\int_a^b f(u)du - f(x)\right| \le \frac{x-a}{b-a}\|f'\|_{1,[a,x]} + \frac{b-x}{b-a}\|f'\|_{1,[x,b]}.$$

We note that the inequality (2.7) is a new inequality of Ostrowski in L_1 norm and better than the one in [20] given by Dragomir and Wang.

Now, for any $x\in(a,b)$ and some $\delta>0$, let the function $F(x,\cdot):[-\delta,\delta]\to R$ be defined by

$$F(x,t) = \frac{1}{t}\int_{x-t/2}^{x+t/2} f(s)ds.$$

We have the following corollary.

Corollary 4. *Assume that the function $f:[a,b]\to R$ is L-lipschitzian mapping on $[a,b]$. Then the function $F(x,\cdot)$ is locally lipschitzian and the lipschitzian constant is $\frac{L}{2}$ and is independent of x.*

Proof. Assume that $x\in(a,b), t_1,t_2\in[-\delta,\delta]$, with $t_2>t_1$. For $a<x-\frac{t_2}{2}<x-\frac{t_1}{2}<x+\frac{t_1}{2}<x+\frac{t_2}{2}<b$, by (2.5), we obtain

$$\left|\frac{1}{t_2}\int_{x-t_2/2}^{x+t_2/2} f(u)du - \frac{1}{t_1}\int_{x-t_1/2}^{x+t_1/2} f(u)du\right| \le \frac{L}{2}(t_2-t_1)$$

which shows that

$$\Big|F(x,t_2)-F(x,t_1)\Big| \le \frac{L}{2}(t_2-t_1),$$

Similarly, for $t_1>t_2$, we get

$$\Big|F(x,t_2)-F(x,t_1)\Big| \le \frac{L}{2}(t_1-t_2),$$

and then, we have

$$\Big|F(x,t_2)-F(x,t_1)\Big| \le \frac{L}{2}|t_2-t_1|$$

which proves the corollary. ∎

3. Applications for Probability Density Functions

In the following, assume that $f : [a,b] \to R^+$ is a probability density function of a certain random variable X and $F : [a,b] \to R^+$, $F(t) = \int_a^t f(x)dx$ is its cumulative distribution function.

Proposition 1. *Let f and F be as above. Then we have*

$$\left|F(t) - \frac{t-a}{b-a}\right| \leq \frac{(b-t)(t-a)}{b-a} \bigvee_a^b (f) \tag{3.1}$$

provided that f is bounded variation on $[a,b]$.

Proof. Taking $x = a$ and $y = t$ in (2.1), we have the desired inequality. ∎

Proposition 2. *Let f and F be as above. Then we have*

$$\left|F(t) - \frac{t-a}{b-a}\right| \leq \frac{(b-t)(t-a)}{b-a}\Big(\big|f(t)-f(a)\big| + \big|f(b)-f(t)\big|\Big). \tag{3.2}$$

provided that f is a monotonous mapping on $[a,b]$.

Proof. Taking $x = a$ and $y = t$ in (2.4), we have the desired inequality. ∎

Proposition 3. *Let f and F be as above. Then we have*

$$\begin{aligned} &\left|F(t) - \frac{t-a}{b-a}\right| \\ &\leq \frac{L}{b-a}\left[(t-a)^2 + (b-t)(t-a) + (b-t)^2\right] \end{aligned} \tag{3.3}$$

provided that f be a L-lipschitzian mapping on $[a,b]$.

Proof. Taking $x = a$ and $y = t$ in (2.5), we have the desired inequality. ∎

Proposition 4. *Let f and F be as above. Then we have*

$$\left|F(t) - \frac{t-a}{b-a}\right| \leq \frac{(b-t)(t-a)}{b-a}\|f'\|_{1,[a,b]} \tag{3.4}$$

provided that f be a continuous differentiable on (a,b) and f' is integrable on (a,b).

Proof. Taking $x = a$ and $y = t$ in (2.6), we have the desired inequality. ∎

Proposition 5. *Let f and F be as above and let*

$$E_t(X) = \int_a^t uf(u)du, t \in [a,b].$$

Then, for $t \in [a,b]$, we have

$$\left|\frac{(b-E(X))(t-a)}{b-a} + E_t(X) - tF(t)\right| \leq \frac{(b-t)(t-a)}{b-a}\|f\|_{1,[a,b]} \tag{3.5}$$

provided that F is a continuous differentiable on (a,b) and F' is integrable on (a,b).

Proof. Taking $F = f, x = a$ and $y = t$ in (2.6), we get

$$\left|\frac{1}{b-a}\int_a^b F(x)dx - \frac{1}{t-a}\int_a^t F(u)du\right| \leq \frac{(b-t)}{b-a}\|F'\|_{1,[a,b]}. \tag{3.6}$$

Since

$$\int_a^b F(x)dx = b - E(X)$$

and

$$\int_a^t F(u)du = tF(t) - \int_a^t uf(u)du = tF(t) - E_t(X),$$

thus, by (3.6), we have the desired inequality. ∎

Remark 2. *We note that the result from Proposition 5 is an extension of the result from Proposition 3.2 in* [15] *for the class of the mapping f.*

Let us consider the *Beta function*

$$B(p,q) := \int_a^b t^{p-1}(1-t)^{q-1}dt,\ \ p,q > -1$$

and the *incomplete Beta function*

$$B(x;p,q) := \int_a^x t^{p-1}(1-t)^{q-1}dt.$$

we have the following proposition.

Proposition 6. *Let X be a Beta random variable with the parameters $(p,q), p,q > 0$. Then we have the inequality*

$$\left| B(x;p+1,q) - xB(x;p,q) + \frac{qx}{p+q}B(p,q) \right| \leq (1-x)xB(p,q), \tag{3.7}$$

for all $x \in [0,1]$.

Proof. Define $f(t) = t^{p-1}(1-t)^{q-1}, p,q > 0$ and consider the random X having the p.d.f. $\sigma(t) = \frac{f(t)}{B(p,q)}, t \in (0,1)$. The following identities hold.

$$E(X) = \frac{1}{B(p,q)}\int_0^1 t^p(1-t)^{q-1}dt = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q}, \tag{3.8}$$

$$E_x(X) = \frac{1}{B(p,q)}\int_0^x t^p(1-t)^{q-1}dt = \frac{B(x;p+1,q)}{B(p,q)}, \tag{3.9}$$

and,

$$F(x) = \frac{1}{B(p,q)}\int_0^x t^{p-1}(1-t)^{q-1}dt = \frac{B(x;p,q)}{B(p,q)}. \tag{3.10}$$

Using (3.5), we have

$$\left| E_x(X) - xF(x) + (1-E(X))x \right| \leq (1-x)x\int_0^1 \left|\sigma(t)\right|dt,$$

and, then, by the identities (3.8), (3.9) and (3.10), we have the desired inequality (3.7). ∎

Remark 3. *We note that the range of paramaters (p,q) in Proposition 6 is larger than the one in Proposition 3.3 in* [15].

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