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LOWER AND UPPER BOUNDS FOR POSITIVE LINEAR FUNCTIONALS

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Abstract. This paper deals with the problem of finding lower and upper-bounds in a set of convex functions to a given positive linear functional; that is, bounds which estimate always below (or above) a functional over a family of convex functions. A new set of upper and lower bounds are provided and their extremal properties are established. Moreover, we show how such bounds can be combined to produce better error estimates. In addition, we also extends many results from [7], which hold true for simplices, to results for any convex polytopes. Particularly, we use our result to obtain multivariate versions of some inequalities first given, respectively, by Favard in [3] and Hammer in [14], over any convex polytope. For smooth (nonconvex) twice continuously differentiable functions, we will also show how both the lower and upper bounds could be improved. Finally, we establish a general result concerning error estimates. This seems to suggest a more unified and effective approach for problems of this sort.

1. Introduction, notations and preliminary results

Let Ω be a *convex* polytope in $\mathbb{R}^d$ with (finite) set of $(n+1)$ distinct vertices $V_n = \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\}$. Throughout this paper d and n denote positive integers such that $n \geq d$. Let $C(\Omega)$ denote the class of all real-valued continuous functions on Ω. The set of all continuous convex functions defined on Ω will be denoted by $K(\Omega)$. A linear functional T which is defined on $C(\Omega)$ is called positive, if it takes non-negative values when applied to each nonnegative function in the set $C(\Omega)$. We say that T is a normalized linear functional if $T[1] = 1$. We next denote by $N^*(\Omega)$ the set of all normalized positive linear functionals on $C(\Omega)$. We shall always assume that the elements of $N^*(\Omega)$ are strictly positive on $C(\Omega)$, i.e. if $T \in N^*(\Omega)$ then $T[f] > 0$ whenever $f(\boldsymbol{x}) \geq 0$, and f is non-identically null continuous function on Ω.
Throughout the paper, the symbol T will be reserved to denote a fixed normalized positive linear functional belonging to $N^*(\Omega)$.

The problem of lower and upper bounds of functionals in its most general form can be described in the following way: Given $T \in N^*(\Omega)$, a common problem in numerical analysis is that of estimating $T[f]$ for some f in $C(\Omega)$. One popular numerical approach is to replace $T[f]$ by an other simple approximating functional

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$A[f]$, which can relatively easy to evaluate numerically. The key idea, to quantify the quality of the numerical approximation $A[f]$ is to use two different functionals belonging to $N^*(\Omega)$, say L and R, to estimate the absolute value of the error $|T[f]-A[f]|$ by $|R[f]-L[f]|$. If no other information is available, we are forced to accept this (or some scaling of it) as the error estimate of $|T[f]-A[f]|$. However, to get a better estimate of $T[f]$, we need some a priori information about T in a given subset G of $C(\Omega)$ (not necessarily containing f). A natural approach is to construct lower and upper bounding functionals L and R, such that

$$L[g] \leq T[g] \leq R[g] \tag{1.1}$$

for any $g \in G$. We will sometimes use the terminology that T dominates L and T is dominated by R on G.
Hence, the following problem arises: We must select the functionals A, L, R, and also decide how the deviation of $A[f]$ from $T[f]$ should measured. Obviously, if we know that f belongs to the set G, we can sometimes better evaluate and estimate $T[f]$. Indeed, if that is the case, we may take the approximation functional $A := (1-\lambda)L + \lambda R$, any convex combination of L and R then, since inequalities (1.1) are satisfied by f, the error estimate can always be controlled as follows:

$$|T[f]-A[f]| \leq R[f]-L[f]. \tag{1.2}$$

Equation (1.2), clearly, shows that when $R[f]-L[f]$ is small, we are confident that $|T[f]-A[f]|$ is also small. So we can know how closer our approximation is to the exact value. This procedure can still be used even if we do not know the exact value of $T[f]$.
Therefore, we are interested in solving the following lower and upper bounds problem:

- For a given subset G of $C(\Omega)$, a linear functional T and a function $f \in C(\Omega)$ determine two functionals L, R, in such a way that $L \leq T \leq R$ on G.

The above problem is obviously too general to be dealt with under a unifying aspect. From now on, we restrict ourselves to the case where the set $G = K(\Omega)$. Formally, the problem to be solved can be formulated as follows: Given a linear functional T in $N^*(\Omega)$.

- How to obtain upper and lower bounds $L, R \in N^*(\Omega)$ for T on $K(\Omega)$, namely,

$$L[g] \leq T[g] \leq R[f], \forall g \in K(\Omega)? \tag{1.3}$$

- Assume given lower and upper bounds. How to properly derive a procedure to be able to generate tighter lower and upper bounds?

The problem of finding lower and upper-bounds in a set of convex functions to a given positive linear functional is a classical topic, there was a flourishing activity on this field even in the last ten years. For instance, it played a crucial role in our research on the determination of the 'best' (or 'optimal') cubature formulae, see [7, 9, 10, 11, 13], where the latter problem has been extensively reviewed both from the theoretical study as well as the numerical point of view. Our aim in this paper is to merge and generalize the recent results from [7], which hold true for simplices, to results for any convex polytopes.

A brief outline of the paper is : After establishing some preliminary definitions and results about generalized barycentric coordinates and Jensen type inequalities, Section 2 provides lower and upper bounds in a set of convex functions to a given positive linear functional. In Section 3 we will discuss some extremal properties of our upper and lower bounds. We will also obtain tighter lower and upper bounds by just using a simple averaging technique and combining those that are derived in Section 2. We will also extends many results from [7], which hold true for simplices, to results for any convex polytopes. Some of the arguments originate in [7], while others are our own. Consequently, we find an extension of a quadrature formula over a finite interval first given by Hammer in [14], over any convex polytope. As an application, in Section 4, we prove refinements, extensions and counterparts of the well-known Favard inequality. We also justify why we have limited our analysis to the case of polytopes. For smooth (nonconvex) twice continuously differentiable functions, Section 5 shows how both the lower and upper bounds given in Section 2 could be improved. This section also establishes a general result concerning error estimates.

The starting-point of this paper is the use of a Delaunay triangulation and therefore do not resemble the proofs in [7], where the early techniques used were mathematically rather simple. Furthermore, it is the power of Delaunay triangulation that allows us to extend our results to all convex polytopes.

2. Lower and upper bounds of positive linear functionals

After establishing some notational conventions which will be used throughout the rest of the paper, this section focuses on the theoretical framework requisite for generating a new class of modified lower and upper bounds of a given normalized positive linear functional on $K(\Omega)$. We first recall some well known results on generalized barycentric coordinates and Jensen type inequalities. For a given $T \in N^*(\Omega)$, let $\boldsymbol{cg}_T(\Omega) := (T[e_1], \ldots, T[e_d])$, which we will call the center of gravity with respect to the functional T in the polytope Ω. Here, we let $e_1, \ldots, e_d$ denote the projections $e_i : \boldsymbol{x} = (x_1, \ldots, x_d) \to x_i$. The so-called Jensen's inequality for the expectation of a convex real-valued function [17, p. 288] can be stated in the following way: Let $I \subseteq R$ be a (real) interval, and let $f : I \to \mathbb{R}$ be a continuous convex function on I. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and let $g : \Omega \to I$ be a μ-integrable function over Ω. Then $E_\mu[g] \in I$, $E_\mu[f(g)]$ exists, and it holds that

$$f(E_\mu[g]) \leq E_\mu[f(g)], \tag{2.1}$$

where E_μ denotes mathematical expectation with respect to the probability measure μ on Ω.

The above inequality (2.1) was later generalized by McShane, see [16], as follows: Let $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ be a continuous convex function on the polytope Ω. Let $g_i \in C(\Omega), i = 1, \ldots, d$, such that $\boldsymbol{g}(\boldsymbol{z}) := (g_1(\boldsymbol{z}), \ldots, g_d(\boldsymbol{z})) \in \Omega$, for all $\boldsymbol{z} \in \Omega$, and $f(\boldsymbol{g}) \in L$. Let us denote by $T[\boldsymbol{g}] := (T[g_1], \ldots, T[g_d])$. Then $T[\boldsymbol{g}]$ is in $\Omega, f(T[\boldsymbol{g}])$ is defined and

$$f(T[\boldsymbol{g}]) \leq T[f(\boldsymbol{g})]. \tag{2.2}$$

Thus, if $\boldsymbol{g}$ is chosen to be the identity function on Ω, in the sense that $Id(\boldsymbol{x}) = \boldsymbol{x}$, then it follows from the above considerations that for any $T \in N^*(\Omega)$, we have always $\boldsymbol{c}g_T(\Omega) \in int(\Omega)$, and for any convex function $f \in K(\Omega)$ we have

$$f\left(\boldsymbol{c}g_T(\Omega)\right) \leq T[f]. \tag{2.3}$$

Before we present our lower and upper bounds, we first prove some Lemmas of general usefulness. We begin by defining barycentric coordinates for an arbitrary simplex.
Given any linearly independent set $V_n = \{\boldsymbol{v}_0, \dots, \boldsymbol{v}_n\}$ of $n+1$ points in $\mathbb{R}^d$, $(d \geq n)$, the simplex with the set of vertices V_n is the convex hull of V_n. If Ω is a simplex, and $n = d$, (e.g., a triangle in $2D$ or a tetrahedron in $3D$), with vertices $\boldsymbol{v}_0, \dots, \boldsymbol{v}_d$ $\in \mathbb{R}^d$, then each point $\boldsymbol{x}$ of their convex hull Ω has a (*unique*) representation, that is there exist unique nonnegative real numbers $\{\lambda_i(\boldsymbol{x}), i = 0, \dots, d\}$ so that $\sum_{i=0}^d \lambda_i(\boldsymbol{x}) = 1$, and $\boldsymbol{x} = \sum_{i=0}^d \lambda_i(\boldsymbol{x})\boldsymbol{v}_i$. The barycentric coordinates $\lambda_0, \dots, \lambda_d$ are nonnegative affine functions on Ω, see [4, p. 288]. Note that a simplex is a special polytope given as the convex hull of $n+1$ vertices, each pair of which is joined by an edge.
Barycentric coordinates also exist for more general types of polytopes, see Kalman [15, Theorem 2].

The next lemma without the property of affine independence is due essentially to Kalman [15]. Our statements are stronger than the ones provided in [15], but the proof proceeds along the same lines as the proof in [15, Theorem 2], so we omit it. The most important fact of our extension is that one prescribed coordinate can be chosen convex (or concave) on Ω. We claim no novelty for this result in the convex case, which is proved in [15]. In fact, we have the following:

Lemma 2.1. *Let Ω be a polytope in $\mathbb{R}^d$, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ its vertices. Then there are affinely independent real continuous functions on Ω, $\{\psi_0, \dots, \psi_n\}$, such that*

$$\sum_{i=0}^n \psi_i(\boldsymbol{x}) = 1, \qquad \psi_i(\boldsymbol{x}) \geq 0, \tag{2.4}$$

and

$$\boldsymbol{x} = \sum_{i=0}^n \psi_i(\boldsymbol{x})\boldsymbol{v}_i. \tag{2.5}$$

Moreover, we can choose the barycentric coordinates $\{\psi_0, \dots, \psi_n\}$ in such a way that one of them is convex or concave.

Warren [19] showed that $\{\psi_0, \dots, \psi_n\}$ can be chosen as rational functions, which are *uniquely* determined if we require that each ψ_i have minimal degree. Later, we will characterize the polytopes that have unique barycentric coordinates.

The next result gives lower and upper bounds for any functional T belonging to $N^*(\Omega)$.

Lemma 2.2. *Let $\psi_i, i = 0, \dots, n$ be defined as in Lemma 2.1. Let $T \in N^*(\Omega)$ and let $L, R \in N^*(\Omega)$ such that*

$$L[f] \leq T[f] \leq R[f], \forall f \in K(\Omega), \tag{2.6}$$

then for any $f \in K(\Omega)$ we have

$$f(\boldsymbol{cg}_T(\Omega)) \leq L[f] \leq T[f]. \tag{2.7}$$

Moreover, if Ω is a simplex, then for any $f \in K(\Omega)$ we have

$$T[f] \leq R[f] \leq \sum_{i=0}^{n} \psi_i(\boldsymbol{cg}_T(\Omega)) f(\boldsymbol{v}_i). \tag{2.8}$$

Proof. Since the projection functions e_i and $-e_i$ are both convex, then it follows from inequalities (2.6) that the centers of gravity $\boldsymbol{cg}_L(\Omega), \boldsymbol{cg}_T(\Omega), \boldsymbol{cg}_R(\Omega)$ coincide. Thus, we get from McShane inequality (2.3) that for all $f \in K(\Omega)$

$$f(\boldsymbol{cg}_T(\Omega)) = f(\boldsymbol{cg}_L(\Omega)) \leq L[f]. \tag{2.9}$$

This shows the left-hand-side of the inequality in (2.7). Next we apply f on both sides of linear precision (2.5) and make use of the convexity, we arrive at:

$$f(\boldsymbol{x}) \leq \sum_{i=0}^{n} \psi_i(\boldsymbol{x}) f(\boldsymbol{v}_i),$$

and so apply R of both sides gives for all convex function $f \in K(\Omega)$

$$R[f] \leq \sum_{i=0}^{n} R\left[\psi_i\right] f(\boldsymbol{v}_i), \tag{2.10}$$

where in the last inequality, we have used linearity and positivity of R. The above inequality implies, once again, in view of the fact that e_i and $-e_i$ are both convex:

$$\boldsymbol{cg}_R(\Omega) = \sum_{i=0}^{n} R\left[\psi_i\right] \boldsymbol{v}_i. \tag{2.11}$$

Let us observe that $\boldsymbol{cg}_T(\Omega)$ has also the following two representations

$$\begin{aligned} \boldsymbol{cg}_T(\Omega) &= \sum_{i=0}^{n} T\left[\psi_i\right] \boldsymbol{v}_i & (2.12)\\ &= \sum_{i=0}^{n} \psi_i(\boldsymbol{cg}_T(\Omega)) \boldsymbol{v}_i, & (2.13) \end{aligned}$$

where in the last equality we have used the identity (2.5) and the fact that $cg_T(\Omega)$ is an interior point of Ω. But, the centers of gravity $\boldsymbol{cg}_T(\Omega)$ and $\boldsymbol{cg}_R(\Omega)$ coincide, then the uniqueness of barycentric coordinates for simplices guarantees that we can write

$$R\left[\psi_i\right] = T\left[\psi_i\right] = \psi_i(\boldsymbol{cg}_T(\Omega)), \forall i = 0, \ldots, n. \tag{2.14}$$

This, combined with (2.10) shows the left-hand inequality in (2.8) and completes the proof Lemma 2.2. □

Let us note that trivial upper and lower worst-case bounds are easiest to find if the functions is continuous only, since for any $T \in N^*(\Omega)$ and $f \in C(\Omega)$, we have

$$\inf_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x}) \leq T[f] \leq \max_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x}). \tag{2.15}$$

In order to generate 'good' lower and upper bounds, than those given by Lemma 2.2 and in inequalities above, we need to subdivide the polytope Ω. From a numerical

FIGURE 1. The left figure is a Voronoi diagram with C_i is the Voronoi region associated to vertex $\boldsymbol{v}_i$. $C(\boldsymbol{y})$ is the Voronoi region associated to $\boldsymbol{y}$. The right figure shows the associated Delaunay triangulation. $T_i, 1, \ldots, 4$, are the simplices of the triangulation.

point of view, the precise effective lower and upper bounds in which we shall be chiefly interested here are those in which we chose a_i, b_i and $\boldsymbol{x}_i$ so that:

$$\sum_{i=0}^{m} a_i f(\boldsymbol{x}_i) \leq T[f] \leq \sum_{i=0}^{m} b_i f(\boldsymbol{x}_i), \forall f \in K(\Omega),$$

where a_i, b_i are positive numbers and $\boldsymbol{x}_i$ are some points in the polytope Ω.

Let $\boldsymbol{y}$ be an arbitrary but fixed point in $int(\Omega)$. With a slight abuse of notation, we will sometimes denote by $\boldsymbol{v}_{n+1}$ the point $\boldsymbol{y}$. We shall then define, when V_n is the set of all vertices of the polytope Ω,

$$Y := V \cup \{\boldsymbol{y}\} = \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n, \boldsymbol{v}_{n+1}\}.$$

For any $\boldsymbol{z} \in Y$, we denote by $V_Y(\boldsymbol{z})$ the (Euclidean) Voronoi region of $\boldsymbol{z}$ with respect to the point set Y, that is the region consisting of all points of Ω that are closer to $\boldsymbol{z}$ than to any other point in $Y \setminus \boldsymbol{z}$,

$$V_Y(\boldsymbol{z}) = \{\boldsymbol{x} \in \Omega : \|\boldsymbol{x} - \boldsymbol{z}\| \leq \|\boldsymbol{x} - \boldsymbol{p}\| \text{ for all } \boldsymbol{p} \in Y \setminus \boldsymbol{z}\},$$

where $\|.\|$ is the (usual) Euclidean norm in $\mathbb{R}^d$. The set of all Voronoi regions for all $V_Y(\boldsymbol{z}), \boldsymbol{z} \in Y$ forms the Dirichlet-Voronoi diagram of Y. Dirichlet-Voronoi diagrams are also called Voronoi diagrams, Voronoi tessellations, or Thiessen polytopes. Following common usage, we will use the terminology Voronoi diagrams. It is easy to see that all of the Voronoi regions $V_Y(\boldsymbol{z}), \boldsymbol{z} \in Y$ are (non-empty) convex polytopes, and their union is Ω. Furthermore, the interiors of $V_Y(\boldsymbol{z}), \boldsymbol{z} \in Y$ are disjoint convex sets. A triangulation of Ω with respect to Y is a partition of Ω into a finite number of d-dimensional simplices such that the vertices of which are points of Y and any two of them are either disjoint or meet in a common lower dimensional simplex. There is a very natural triangulation $DT(\Omega)$ of Ω, that uses only the points of Y as triangulation vertices and such that any simplex in $DT(\Omega)$ has $\boldsymbol{y}$ as a vertex. Such a triangulation exists and it is called a Delaunay triangulation of Ω with respect to Y and it can be obtained as the geometric dual of the Voronoi diagram of Y, see, e. g., [6]. It is a triangulation of Ω with respect to Y such that no point of Y is inside the hyper circumsphere of any simplex of the triangulation, see Figure 1 in the case when the domain Ω is a rhombus. Alternatively, such a triangulation can also be characterized by an approximation problem; see [2, Theorem 2.3]. Uniqueness of Delaunay triangulations is guaranteed if no $d+1$ points of Y lie on a common hyperplane in $\mathbb{R}^d$ and no $d+2$ points lie on a common hypersphere.

Let $\boldsymbol{y} \in int(\Omega)$ and let $\mathcal{S}_1^{\boldsymbol{y}}, \ldots \mathcal{S}_m^{\boldsymbol{y}}$ be the simplices of $DT(\Omega)$ with respect the set of points Y and let N_i be the set of all integers j such that $\boldsymbol{v}_i$ is a vertex of $\mathcal{S}_j^{\boldsymbol{y}}$. If $\boldsymbol{x} \in \mathcal{S}_j^{\boldsymbol{y}}$ and $j \in N_i$, then we denote by $\lambda_{ij}^{\boldsymbol{y}}(\boldsymbol{x})$ the barycentric co-ordinate of $\boldsymbol{x}$

with respect to $\boldsymbol{v}_i$ for the simplex $\mathcal{S}_j^{\boldsymbol{y}}$. It is easily verified that if $\boldsymbol{x} \in \mathcal{S}_j^{\boldsymbol{y}} \cap \mathcal{S}_k^{\boldsymbol{y}}$, then $\lambda_{ij}^{\boldsymbol{y}}(\boldsymbol{x}) = \lambda_{ik}^{\boldsymbol{y}}(\boldsymbol{x})$ if $j, k \in N_i$ and $\lambda_{ij}(\boldsymbol{x}) = 0$ if $j \in N_i$, $k \notin N_i$. Therefore, setting

$$\phi_i^{\boldsymbol{y}}(\boldsymbol{x}) := \begin{cases} \lambda_{ij}^{\boldsymbol{y}}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \mathcal{S}_j^{\boldsymbol{y}} \text{ and } j \in N_i \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 0, \ldots, n+1$, we trivially obtain a set of well-defined functions, which are a generalization of barycentric coordinates when Ω is a simplex. We list some basic properties of the functions $\phi_0^{\boldsymbol{y}}, \ldots, \phi_{n+1}^{\boldsymbol{y}}$, of which the following are particularly relevant to us:

(1) They are well-defined, piecewise linear and nonnegative real-valued continuous functions.
(2) They send Ω to the unit interval $[0, 1]$.
(3) They form a partition of unity, so $\sum_{i=0}^{n+1} \phi_i^{\boldsymbol{y}} \equiv 1$.
(4) The function $\phi_i^{\boldsymbol{y}}$ has to equal 1 at $\boldsymbol{v}_i$ and 0 at all other points in $Y \setminus \boldsymbol{v}_i$, i.e., $\phi_i^{\boldsymbol{y}}(\boldsymbol{v}_j) = \delta_{ij}$ (δ is the Kronecker delta).
(5) They satisfy the linear precision property, that is,

$$\boldsymbol{x} = \sum_{i=0}^{n+1} \phi_i^{\boldsymbol{y}}(\boldsymbol{x}) \boldsymbol{v}_i. \tag{2.16}$$

Since any simplex in $DT(\Omega)$ contains $\boldsymbol{y}$ as a vertex, to simplify notation from here forward, we shall write $\phi_{n+1}^{\boldsymbol{y}}$ simply as

$$\phi_{n+1}^{\boldsymbol{y}}(\boldsymbol{x}) := \begin{cases} \lambda_j^{\boldsymbol{y}}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \mathcal{S}_j^{\boldsymbol{y}} \text{ and } j \in \{1, \ldots m\} \\ 0 & \text{otherwise,} \end{cases} \tag{2.17}$$

here m is the number of simplices in $DT(\Omega)$ and $\lambda_j^{\boldsymbol{y}}(\boldsymbol{x})$ is the barycentric coordinate of $\boldsymbol{x}$ with respect to $\boldsymbol{y}$ for the simplex $\mathcal{S}_j^{\boldsymbol{y}}$, (see Figure 1).

Kalman showed in [15, Theorem 2] that, for any given finite set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l\}$ of points of $\mathbb{R}^d$ we can assign barycentric coordinates $\{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ to their convex hull X in such a way that each coordinate is continuous on X and that one prescribed coordinate is convex on X. He does not identified exactly which barycentric coordinate function has this property. Kalman also asked if whether it is always possible to make all the coordinates convex simultaneously (see [15, Example 3]). Fuglede [5] answered this question by showing that if all the barycentric coordinates are convex (or if they are all concave), then they are all affine, and consequently, X must be a simplex.

With these important general facts in mind, we now complement the Kalman result. The next Lemma shows that in our situation, for any fixed $\boldsymbol{y} \in int\,(\Omega)$, the function $1 - \phi_{n+1}^{\boldsymbol{y}}$ (assigned to the point $\boldsymbol{y}$) is convex: This convexity property will be an important key to establish some extremal properties of our lower and upper bounds. Thus, the first main point of this paper is the following result, which we will have several occasions to use.

Lemma 2.3. *For any $\boldsymbol{y} \in int(\Omega)$, the function $1-\phi_{n+1}^{\boldsymbol{y}}$ is a convex continuous piecewise-linear function on Ω and for $i=1,\dots,m$ satisfies the identity:*

$$\frac{1}{vol(S_i^{\boldsymbol{y}})}\int_{S_i^{\boldsymbol{y}}} \phi_{n+1}^{\boldsymbol{y}}(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{d+1} = \frac{1}{vol(\Omega)}\int_{\Omega} \phi_{n+1}^{\boldsymbol{y}}(\boldsymbol{x})d\boldsymbol{x}. \tag{2.18}$$

Proof. First we show that $-\phi_{n+1}^{\boldsymbol{y}}$ can be expressed as the maximum of a finite set of affine functions. To this end let $\mathcal{S}_j^{\boldsymbol{y}}$ be any fixed simplex in $DT(\Omega)$. We begin by observing that for any $i=1,\dots,m$, $\lambda_i^{\boldsymbol{y}}$ is nonnegative (affine function) on Ω. To see that, it suffices to note that, for any fixed i, $\lambda_i^{\boldsymbol{y}}(\boldsymbol{y})=1$ and $\lambda_i^{\boldsymbol{y}}(\boldsymbol{x})=0$ is a supporting hyperplane for Ω. Recall that an affine function is uniquely determined by its values at the vertices of the simplex. For any fixed $i=1,\dots,m$, define the functions $d_j, j=1,\dots m$, by $d_j=\lambda_i^{\boldsymbol{y}}-\lambda_j^{\boldsymbol{y}}$, then a simple inspection shows that d_j is nonnegative on $\mathcal{S}_j^{\boldsymbol{y}}$. Hence $h(\boldsymbol{x}) := \max\left\{-\lambda_1^{\boldsymbol{y}}(\boldsymbol{x}),\dots,-\lambda_m^{\boldsymbol{y}}(\boldsymbol{x})\right\} = -\lambda_i^{\boldsymbol{y}}(\boldsymbol{x})$ for all $\boldsymbol{x}\in S_i^{\boldsymbol{y}}$. Thus, it follows that for all i the restriction of h to $S_i^{\boldsymbol{y}}$ coincides with the function $-\phi_{n+1}^{\boldsymbol{y}}$. But now, h and $-\phi_{d+1}^{\boldsymbol{y}}$ are identically equal in Ω, we then get that $-\phi_{n+1}^{\boldsymbol{y}}$ is convex, because affine functions are convex and the maximum of convex functions is a convex function. This means that, $1-\phi_{n+1}^{\boldsymbol{y}}$ is convex, since it is the sum of two convex functions.
To show the left-hand inequality in (2.18), we use the identity (2.17), for the simplex $S_i^{\boldsymbol{y}}$, to find that

$$\int_{S_i^{\boldsymbol{y}}} \phi_{n+1}^{\boldsymbol{y}}(\boldsymbol{x})\,d\boldsymbol{x} = \int_{S_i^{\boldsymbol{y}}} \lambda_i^{\boldsymbol{y}}(\boldsymbol{x})\,d\boldsymbol{x} = \frac{vol(S_i^{\boldsymbol{y}})}{d+1}\sum_{\boldsymbol{v}} \lambda_i^{\boldsymbol{y}}(\boldsymbol{v}), \tag{2.19}$$

where the sum is taken over all vertices of the simplex $S_i^{\boldsymbol{y}}$. The last identity can be deduced using [13, Theorem 2.2] applied to the simplex $S_i^{\boldsymbol{y}}$. Therefore, the Kronecker delta property implies the desired result. Finally, a simple computation now shows that adding inequalities (2.19) for $i=1,\dots,m$, and dividing by $vol(S)$, yields the second desired equality in (2.18). □

It should be noted a remarkable fact that integral identities (2.18) are independent of $\boldsymbol{y}$ and i. This property will be used later in some proofs.

Now we can state and prove our improved lower and upper bounds. We continue to denote $\boldsymbol{v}_{n+1}$ the point $\boldsymbol{y}$. The next result is a main result in this section.

Theorem 2.4. *Let $\boldsymbol{y}\in int(\Omega)$ be fixed. Then, for any $T\in N^*(\Omega)$ and $f\in K(\Omega)$ the following inequalities hold*

$$f\left(cg_T(\Omega)\right) \le L^{\boldsymbol{y}}[f] := \sum_{i=0}^{n+1} A_i^{\boldsymbol{y}} f\left(\boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)\right) \le T[f] \le R^{\boldsymbol{y}}[f] := \sum_{i=0}^{n+1} A_i^{\boldsymbol{y}} f(\boldsymbol{v}_i), \tag{2.20}$$

with for any $i=0,\dots,n+1$, $A_i^{\boldsymbol{y}} = T\left[\phi_i^{\boldsymbol{y}}\right]$ and

$$\boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right) = \left(\frac{T\left[\phi_i^{\boldsymbol{y}} e_1\right]}{T\left[\phi_i^{\boldsymbol{y}}\right]},\dots,\frac{T\left[\phi_i^{\boldsymbol{y}} e_d\right]}{T\left[\phi_i^{\boldsymbol{y}}\right]}\right), (i=0,\dots,n). \tag{2.21}$$

Moreover, if Ω is a simplex, then for any $i = 0, \ldots, n$,

$$A_i^{\boldsymbol{y}} = \left(1 - \frac{\psi_i(\boldsymbol{y})}{\psi_i(\boldsymbol{c}g_T(\Omega))} A_{n+1}^{\boldsymbol{y}}\right) \psi_i(\boldsymbol{c}g_T(\Omega)). \tag{2.22}$$

Proof. In order to obtain the estimation from below (2.20), let us first observe that the center of gravity $cg_T(\Omega)$ can be expressed as a convex combination of the points $\boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)$. Indeed, it follows from the definition of $cg_T(\Omega)$ and the partition of unity functions $\phi_i^{\boldsymbol{y}}, i = 0, \ldots, n+1$ that $cg_T(\Omega) = \sum_{i=0}^{n+1} T\left[\phi_i^{\boldsymbol{y}}\right] \boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)$. Let $f \in K(\Omega)$, then, the convexity of f implies that

$$f\left(cg_T(\Omega)\right) \leq \sum_{i=0}^{n+1} T\left[\phi_i^{\boldsymbol{y}}\right] f\left(\boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)\right).$$

This shows the left-hand inequality of (2.20).
We now prove the second inequality in (2.20). Let us define first the functionals on $C(\Omega)$ by

$$T_i[g] = \frac{T\left[\phi_i^{\boldsymbol{y}} g\right]}{T\left[\phi_i^{\boldsymbol{y}}\right]}, \ (i = 0, \ldots, n+1). \tag{2.23}$$

Now it is easily seen that for all i, the functional T_i belongs to $N^*(\Omega)$, and its center of gravity is $\boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)$. Thus, according to Jensen's inequality (2.7) applied to the functional T_i instead of T, it follows from the definition of $cg_{T_i}(\Omega)$ that $cg_{T_i}(\Omega) := \boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)$. Moreover, we have for all $f \in K(\Omega)$

$$T\left[\phi_i^{\boldsymbol{y}}\right] f\left(\boldsymbol{x}\left(\phi_i^{\boldsymbol{y}}\right)\right) \leq T\left[\phi_i^{\boldsymbol{y}} f\right]. \tag{2.24}$$

Summing up for $i = 0, 1, \ldots, n+1$, and taking into account that the functions $\phi_i^{\boldsymbol{y}}, i = 0, \ldots, n+1$, form a partition of unity, we get the desired result. Let us mention that the left-hand inequality of (2.20) can also be proved by using Lemma 2.2 and the inequality we just proved.
To obtain the estimate from above, we first observe that equation (2.16) tells us that any point $\boldsymbol{x} \in \Omega$ can be written as convex combination of the extreme points of Ω and the point $\boldsymbol{y} \in \Omega$, then if we apply f on both sides of (2.16) and make use of the convexity, we get

$$f(\boldsymbol{x}) \leq \sum_{i=0}^{n} \phi_i^{\boldsymbol{y}}(\boldsymbol{x}) f(\boldsymbol{v}_i) + \phi_{n+1}^{\boldsymbol{y}}(\boldsymbol{x}) f(\boldsymbol{y}).$$

So by applying T of both sides of the above equation and using the linearity and positivity properties of T, we get the desired result. Again, linear precision (2.16) implies that the center of gravity $\boldsymbol{c}g_T(\Omega)$ can also be expressed as follows:

$$\boldsymbol{c}g_T(\Omega) = \sum_{i=0}^{n} T\left[\phi_i^{\boldsymbol{y}}\right] \boldsymbol{v}_i + T\left[\phi_{d+1}^{\boldsymbol{y}}\right] \boldsymbol{y}.$$

But also by Lemma 2.1 we may write $\boldsymbol{y} = \sum_{i=0}^{n} \psi_i(\boldsymbol{y}) \boldsymbol{v}_i$, then we have immediately

$$\boldsymbol{c}g_T(\Omega) = \sum_{i=0}^{n} \left(T\left[\phi_i^{\boldsymbol{y}}\right] + T\left[\phi_{d+1}^{\boldsymbol{y}}\right] \psi_i(\boldsymbol{y})\right) \boldsymbol{v}_i.$$

Thus, using the uniqueness of barycentric coordinates for a simplex, we conclude that

$$\psi_i(\boldsymbol{c}g_T(\Omega)) = T\left[\phi_i^{\boldsymbol{y}}\right] + T\left[\phi_{n+1}^{\boldsymbol{y}}\right]\psi_i(\boldsymbol{y}),\ (i = 0,\ldots,n). \tag{2.25}$$

This prepares us for the final part of the proof of the theorem. Indeed, the above equation immediately yields

$$T\left[\phi_i^{\boldsymbol{y}}\right] = \left(1 - \frac{\psi_i(\boldsymbol{y}))}{\psi_i(\boldsymbol{c}g_T(\Omega))}A_{n+1}^{\boldsymbol{y}}\right)\psi_i(\boldsymbol{c}g_T(\Omega)), \quad (i = 0,\ldots,n). \tag{2.26}$$

This shows identities (2.22), and the proof is complete. □

Remark 2.5. It should be noted that for any point $\boldsymbol{y}$ in the interior of Ω, Lemma 2.2 tells us that if Ω is a simplex, then the upper bound given in Theorem 2.4 must be better on $K(\Omega)$ than that those given by (2.8) of Lemma 2.2. This can also be easily verified directly.

Before we proceed further, we would like to remark that under uniqueness of barycentric coordinates, the reader should note that inequality (2.8) and the right inequality (2.20) with coefficients as given in (2.22) remain valid. This result motivates a question about polytopes with a related property. Let us raise the following:

Problem 2.6. *For which polytopes Ω in $\mathbb{R}^d$, the barycentric coordinates given in lemma 2.1 are uniquely defined?*

The following Lemma shows that unfortunately the only polytopes with this property are the simplices. More precisely, we have the following characterization:

Lemma 2.7. *Let Ω be a polytope in $\mathbb{R}^d$, with $n+1$ vertices $\{\boldsymbol{v}_0,\ldots,\boldsymbol{v}_n\}$, $(n \geq 1)$. Then, the following properties of Ω are equivalent*

1. *Ω is a simplex;*
2. *The barycentric coordinates with respect to the vertices of Ω are unique.*

Proof. The proof sketch is as follows: Given a simplex Ω with vertices $\{\boldsymbol{v}_0,\ldots,\boldsymbol{v}_n\}$. By Lemma 2.1, there must exist at least one barycentric coordinate system on Ω, $\{\psi_0,\ldots,\psi_n\}$, such that for all $\boldsymbol{x} \in \Omega$ we have

$$\sum_{i=0}^{n}\psi_i(\boldsymbol{x}) \,=\, 1, \qquad \psi_i(\boldsymbol{x}) \geq 0, \tag{2.27}$$

and

$$\boldsymbol{x} \,=\, \sum_{i=0}^{n}\psi_i(\boldsymbol{x})\boldsymbol{v}_i. \tag{2.28}$$

For any $\boldsymbol{x} \in \Omega$, we can calculate $\psi_i(\boldsymbol{x})$ by defining M to be the $d \times n$ matrix $\{\boldsymbol{v}_1 - \boldsymbol{v}_0,\ldots,\boldsymbol{v}_n - \boldsymbol{v}_0\}$, and solving the equation

$$\boldsymbol{x} - \boldsymbol{v}_0 = M\left(\psi_1(\boldsymbol{x}),\ldots,\psi_n(\boldsymbol{x})\right)^T,$$

where $\left(\psi_1(\boldsymbol{x}),\ldots,\psi_n(\boldsymbol{x})\right)^T$ is the transpose of the row vector $\left(\psi_1(\boldsymbol{x}),\ldots,\psi_n(\boldsymbol{x})\right)$. Recall that our definition of a simplex stated that its vertices are linearly independent points, then the matrix M is assumed to have full column rank, and

FIGURE 2. This figure gives an example of a triangulation using only the vertices of the polytope.

consequently the following system of linear equations has a unique solution given by

$$(\psi_1(\boldsymbol{x}), \ldots, \psi_n(\boldsymbol{x}))^T = (M^T M)^{-1} M(\boldsymbol{x} - \boldsymbol{v}_0),$$

with $\psi_0(\boldsymbol{x}) = 1 - \sum_{i=1}^n \psi_i(\boldsymbol{x})$. This shows that the barycentric coordinates of a point are uniquely determined by the vertices of the simplex in question. In order to prove the converse assume the contrary, namely that Ω is not a simplex. We first observe that, if n is less than or equal to 2 then we have nothing to prove, since Ω is already a simplex. Here we used the fact that Ω is assumed to be described by $n+1$ vertices. Thus without loss of generality we may assume that n is greater than or equal to 3. The polytope Ω admits at least one triangulation $\mathcal{T}$ in which every vertex is an extreme point of Ω, see [5, Theorem 2]. Note that since Ω is not a simplex, and n is greater than 3, then $\mathcal{T}$ has at least two adjacent simplices, say S and S'. Denote by F the common facet of S and S', and $\boldsymbol{v}$ (respectively $\boldsymbol{v}'$) the vertex of S (respectively S'), lying in opposite the facet F, see Figure 2. Since, $\mathcal{T}$ does not use any interior points, then $S \cup S'$ is a convex polytope with vertices necessarily among those of Ω. Consequently the line segment joining the two points $\boldsymbol{v}$ and $\boldsymbol{v}'$ meets F in at least one point, say $\boldsymbol{y}$. Thus, there is a common point $\boldsymbol{y}$ in the interiors of the two different simplices F and $[\boldsymbol{v}, \boldsymbol{v}']$, and then $\boldsymbol{y}$ must have two different sets of barycentric coordinates. This contradicts the uniqueness property for the barycentric coordinates and shows that Ω must be a simplex. □

In order to simplify the notation, in what follows we denote by $A_{n+1}^{cg} = T\left[\phi_{n+1}^{cg_T(\Omega)}\right]$, and for any $i = 0, \ldots, n, \phi_i^{cg} = \phi_i^{cg_T(\Omega)}$. By using these notations, if the domain is a simplex Theorem 2.4 tells us the following:

Corollary 2.8. *Let Ω be a simplex and let*

$$\begin{aligned} L^{cg}[f] &:= (1 - A_{n+1}^{cg}) \sum_{i=0}^n \psi_i(\boldsymbol{c}g_T(\Omega)) f\left(\boldsymbol{x}\left(\phi_i^{cg}\right)\right) + A_{n+1}^{cg} f\left(\boldsymbol{x}\left(\phi_{n+1}^{cg}\right)\right) \\ R^{cg}[f] &:= (1 - A_{n+1}^{cg}) \sum_{i=0}^n \psi_i(\boldsymbol{c}g_T(\Omega)) f(\boldsymbol{v}_i) + A_{n+1}^{cg} f(\boldsymbol{c}g_T(\Omega)). \end{aligned}$$

Then the following inequality is valid for any $f \in K(\Omega)$

$$L^{cg_T(\Omega)}[f] \le T[f] \le R^{cg_T(\Omega)}[f].$$

Remark 2.9. Let us denote by $K^+(\Omega)$ the set of nonnegative convex functions belonging to $K(\Omega)$. It should be noted that under the hypotheses of Corollary 2.8, the functional T is bounded from below on $K^+(\Omega)$ by

$$\tilde{L}^{cg}[f] := (1 - A_{n+1}^{cg}) \sum_{i=0}^n \psi_i(\boldsymbol{c}g_T(\Omega)) f\left(\boldsymbol{x}\left(\phi_i^{cg}\right)\right). \tag{2.29}$$

Indeed, recalling inequalities (2.24), we can see that, for all $i = 0, \ldots n$,

$$(1 - A_{n+1}^{cg}) \psi_i(\boldsymbol{c}g_T(\Omega)) f\left(\boldsymbol{x}\left(\phi_i^{cg}\right)\right) \le T[\phi_i^{cg} f],$$

and then summing up the above inequalities, we have immediately for all function from $K^+(\Omega)$

$$\begin{aligned}\tilde{L}^{cg}[f] &\leq T[(1-\phi_{n+1}^{cg})f] \\ &= T[f] - T[\phi_{n+1}^{cg} f],\end{aligned}$$

and consequently it follows that $\tilde{L}^{cg}T[f] \leq T[f]$. But obviously we have $\tilde{L}^{cg}[f] \leq L^{cg}[f]$, for all $f \in K^+(\Omega)$, then the functional L^{cg} is an improvement of the lower bound $\tilde{L}^{cg}$ on $K^+(\Omega)$.
By analogy, we can write the expression for R^{cg} as follows:

$$R^{cg}[f] = \tilde{R}^{cg} - A_{n+1}^{cg}\left(\sum_{i=0}^{n} \psi_i(\boldsymbol{c}g_T(\Omega)) f(\boldsymbol{v}_i) - f(\boldsymbol{c}g_T(\Omega))\right),$$

where $\tilde{R}^{cg}[f] := \sum_{i=0}^{n} \psi_i(\boldsymbol{c}g_T(\Omega)) f(\boldsymbol{v}_i)$. Then, Lemma 2.2 informs us that the functional $\tilde{R}^{cg}$ is an upper bound for T on $K(\Omega)$. Since we have subtracted a nonnegative term, then R^{cg} appears here as an improvement for the upper bound $\tilde{R}^{cg}$. But this time, the improvement is valid for any continuous convex function.

In particular, if T is the functional-integral over a simplex, Corollary 2.8 allows us to extend a quadrature formula over an interval, first given by Hammer in [14], to a multivariate setting. This result was first proved in [7]. Indeed, as a consequence of the above result combined with Lemma 2.3, we immediately obtain the following:

Corollary 2.10. *Let $\Omega \subset \mathbb{R}^d$ be a simplex with vertices $\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_d\}$ and center of gravity $\boldsymbol{v}^* = \frac{\boldsymbol{v}_0 + \ldots + \boldsymbol{v}_d}{d+1}$. Set*

$$\begin{aligned}L^{cg}[f] &:= \frac{1}{d+1}\left(\frac{d}{d+1}\sum_{i=0}^{d} f\left(\left(\frac{\boldsymbol{v}^* - \boldsymbol{v}_i}{d+1}\right) + \boldsymbol{v}^*\right) + f(\boldsymbol{v}^*)\right) \\ R^{cg}[f] &:= \frac{1}{d+1}\left(\frac{d}{d+1}\sum_{i=0}^{d} f(\boldsymbol{v}_i) + f(\boldsymbol{v}^*)\right).\end{aligned}$$

Then for every convex function from $K(\Omega)$, we have

$$L^{cg}[f] \leq \frac{1}{vol(\Omega)} \int_{\Omega} f(\boldsymbol{x})\, d\boldsymbol{x} \leq R^{cg}[f]. \tag{2.30}$$

Proof. Just apply the integral identity (2.18) proved in Lemma 2.3. □

3. Refinement and comparison of our upper and lower bounds

The aim of this section is to establish some extremal properties of the upper and lower bounds given by Theorem 2.4. We will also get 'better' lower and upper bounds by just combining those that are derived in Section 2.

The next Theorem describes relevant properties of the upper bound derived in Theorem 2.4. In what follows $R^{\boldsymbol{y}}$ denotes the upper bound derived in Theorem 2.4. This extends to arbitrary polytopes in $\mathbb{R}^d$, a result in [7] in the case of a simplex.

Theorem 3.1. *Let $T \in N^*(\Omega), \boldsymbol{y} \in int(\Omega), A_{n+1}^{\boldsymbol{y}} = T[\phi_{n+1}^{\boldsymbol{y}}]$. Suppose that there are positive real numbers $B_0, \dots, B_{n+1}$ such that their sum is 1, and for every*

$f \in K(\Omega)$, we have

$$T[f] \leq R[f] := \sum_{i=0}^{n} B_i f(\boldsymbol{v}_i) + B_{n+1} f(\boldsymbol{y}). \tag{3.1}$$

Then,

$$B_{n+1} \leq A_{n+1}^{\boldsymbol{y}}. \tag{3.2}$$

Moreover, if Ω is a simplex, then equality in (3.2) is attained if and only if the two functionals R and $R^{\boldsymbol{y}}$ are identically equal. If $B_{n+1} > A_{n+1}^{\boldsymbol{y}}$ then T cannot bounded by R on $K(\Omega)$.

Proof. Fix $\boldsymbol{y} \in int(\Omega)$, then by Lemma 2.3, we already know that the function $1 - \phi_{n+1}^{\boldsymbol{y}} \in K(\Omega)$. Recall that $\phi_{n+1}^{\boldsymbol{y}}$ takes the value 1 at $\boldsymbol{y}$ and vanishes on all vertices of Ω. Consequently, taking into account that the two functionals T and R coincide over the set of constant functions, we get from dominant property (3.1) on $K(\Omega)$

$$-A_{n+1}^{\boldsymbol{y}} = T[-\phi_{n+1}^{\boldsymbol{y}}] \leq R[-\phi_{n+1}^{\boldsymbol{y}}] = -B_{n+1}\phi_{n+1}^{\boldsymbol{y}}(\boldsymbol{y}) = -B_{n+1}.$$

Hence, for every $\boldsymbol{y} \in int(\Omega)$ we have $B_{n+1} \leq A_{n+1}^{\boldsymbol{y}}$. If, moreover, Ω is a simplex, the sufficiency condition is obvious. For this, it suffices to consider the equality condition for the the function $\phi_{n+1}^{\boldsymbol{y}}$. It remains to show that if $B_{n+1} = A_{n+1}^{\boldsymbol{y}}$ then $R = R^{\boldsymbol{y}}$. To this end, since, T is dominated by R and $R^{\boldsymbol{y}}$ on $K(\Omega)$ and affine functions and their opposites belong to $K(\Omega)$, then the functionals R and $R^{\boldsymbol{y}}$ have the same center of gravity. Therefore, we have

$$\boldsymbol{cg}_T(\Omega) = \sum_{i=0}^{n} R^{\boldsymbol{y}}[\psi_i]\, \boldsymbol{v}_i \tag{3.3}$$

$$= \sum_{i=0}^{n} R[\psi_i]\, \boldsymbol{v}_i, \tag{3.4}$$

here $\{\psi_i, i = 0, \dots, n\}$ are the barycentric coordinates defined in Lemma 2.1. Because of the uniqueness of barycentric coordinates we get

$$R[\psi_i] = R^{\boldsymbol{y}}[\psi_i],$$

then clearly $B_i + B_{n+1}\psi_i(\boldsymbol{y}) = A_i^{\boldsymbol{y}} + A_{n+1}^{\boldsymbol{y}}\psi_i(\boldsymbol{y})$. Finally, since $B_{n+1} = A_{n+1}^{\boldsymbol{y}}$, we get $B_i = A_i^{\boldsymbol{y}}$. Thus, we see that $R = R^{\boldsymbol{y}}$.
In order to complete the proof we have to find a function $h \in K(\Omega)$ for which inequality (3.1) is not true. To this end, assume that $B_{n+1} > A_{n+1}^{\boldsymbol{y}}$, then since

$$B_{n+1} = R[\phi_{n+1}^{\boldsymbol{y}}], \tag{3.5}$$

$$A_{n+1}^{\boldsymbol{y}} = T[\phi_{n+1}^{\boldsymbol{y}}], \tag{3.6}$$

it follows that inequality (3.1) does not hold for the convex function $-\phi_{n+1}^{\boldsymbol{y}} \in K(\Omega)$. This contradiction shows that B_{n+1} cannot strictly bigger than $A_{n+1}^{\boldsymbol{y}}$ and completes the proof of the theorem. $\square$

The arguments in Theorem 3.1 generalize easily to yield the following more general result:

Remark 3.2. Assume that $R[f] := \sum_{i=0}^{n} B_i f(\boldsymbol{v}_i) + B_{n+1} f(\boldsymbol{y})$ dominates T on $K(\Omega)$, then the equality $R = R^{\boldsymbol{y}}$ holds true if and only if there exists an index i belonging to $\{0, \dots, n+1\}$ such that $A_i^{\boldsymbol{y}} = B_i$ (here, $R^{\boldsymbol{y}}$ is the functional defined in (2.20)). This extends Theorem 3.1.

Before we go on to our next result, we need yet some more preparation. We recall that a partition of unity $\{p_i\}_{i=0}^{n}$ consists of a finite collection of continuous functions from Ω to the unit interval $[0,1]$ and whose values sum the unity for all $\boldsymbol{x} \in \Omega$. For a partition of unity $\{p_i, i = 0, \dots, n\}$, we simply write $\boldsymbol{p}$ and say that it is a *pu*-system. The collection of all *pu*-systems will be denoted by $\boldsymbol{P}_{n+1}$. If $\boldsymbol{y}_0, \dots, \boldsymbol{y}_n$ denote $(n+1)$-points in Ω and $B_0, \dots, B_n$ be $(n+1)$-nonnegative real numbers such that their sum is 1, let us define

$$R[f] := \sum_{i=0}^{n} B_i f(\boldsymbol{y}_i). \tag{3.7}$$

Let $T \in N^*(\Omega)$, we will assume throughout that for any $\boldsymbol{p} := \{p_i, i = 0, \dots, n\} \in \boldsymbol{P}_{n+1}$, we have $T[p_i] > 0, i = 0, \dots, n$. For a given functional R of the form (3.7), we will say that R is generated by a *pu*-system with respect to T if there exists a *pu*-system $\boldsymbol{p} = \{p_i, = 0, \dots, n\}$, such that

$$B_i = T[p_i] \ \ \textit{and} \ \ \boldsymbol{y}_i = cg_{T_i}(\Omega) := \left(\frac{T\left[p_i e_1\right]}{T\left[p_i\right]}, \dots, \frac{T\left[p_i e_d\right]}{T\left[p_i\right]} \right), (i = 0, \dots, n),$$

where T_i is the functional defined on $C(\Omega)$ by $T_i[f] = \frac{T[p_i f]}{T[p_i]}$.

The next result gives a necessary and sufficient condition under which any functional of the form (3.7) is bounded from above by T on $K(\Omega)$.

Theorem 3.3. *Let $B_0, \dots, B_n$ be $(n+1)$-nonnegative real numbers such that their sum is 1. For any $\boldsymbol{y}_0, \dots, \boldsymbol{y}_n$, $(n+1)$-points in Ω, define*

$$R[f] := \sum_{i=0}^{n} B_i f(\boldsymbol{y}_i).$$

Then R is generated by a pu-system with respect to T if and only if R is bounded above by T on $K(\Omega)$.

Proof. Main idea of the proof. Similar arguments show that Theorem 2.4 also holds when replacing the barycentric coordinates system $\left\{\phi_0^{\boldsymbol{y}}, \dots, \phi_n^{\boldsymbol{y}}\right\}$ by any *pu*-system. Hence, if R is generated by a *pu*-system then R is bounded from above by T on $K(\Omega)$. The difficulty is contained in the converse implication, but it can be proved by an easy adaptation of [8, Theorem 3.8] to the more general setting of positive linear functionals. We leave the details to the reader. □

As a corollary we obtain the following result:

Corollary 3.4. *Let Ω be a polytope in $\mathbb{R}^d$ and $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ its vertices. For any $\boldsymbol{y} \in \Omega$, define*

$$R[f] := \sum_{i=0}^{n} B_i f(\boldsymbol{v}_i) + B_{n+1} f(\boldsymbol{y}).$$

Then the functional T is bounded above by R on $K(\Omega)$ if and only if $B_0 = \ldots = B_n = 0, B_{n+1} = 1$, and $\boldsymbol{y} = cg_T(\Omega)$.

Proof. The proof of this corollary is a standard consequence of Theorem 3.3. The details are as follow. Assume that T is dominated by R on $K(\Omega)$, then, by Theorem 3.3, R is generated by a *pu*-system $\boldsymbol{p} := \{p_i, i = 0, \ldots, n\}$, with

$$\boldsymbol{v}_i = \left(\frac{T\left[p_i e_1\right]}{T\left[p_i\right]}, \ldots, \frac{T\left[p_i e_d\right]}{T\left[p_i\right]}\right), (i = 0, \ldots, n).$$

As the centers of gravity $\left(\frac{T[p_i e_1]}{T[p_i]}, \ldots, \frac{T[p_i e_d]}{T[p_i]}\right), (i = 0, \ldots, n)$, are interior points, this is possible only if $B_0 = \ldots = B_{n+1} = 0, B_{n+2} = 1$, and $\boldsymbol{y} = cg_T(\Omega)$.
The converse implication can be proved by using Lemma 2.2. □

In order to compute better lower and upper bounds, as suggested early in the Introduction, we take a convex combination of nonnegative coefficients of lower and upper bounds, given by Theorem 2.4. The next result gives all admissible convex combinations. For simplicity we continue to denote $A_i^{cg} = T\left[\phi_i^{cg_T(\Omega)}\right], i = 0, \ldots, n+1$.

Theorem 3.5. *Let us define the two linear functionals L^{cg} and R^{cg} by*

$$\begin{aligned} L^{cg}\left[f\right] &:= f(\boldsymbol{c}g_T(\Omega)) \\ R^{cg}\left[f\right] &:= \sum_{i=0}^{n+1} A_i^{cg} f(\boldsymbol{v}_i). \end{aligned}$$

For any $\alpha \in [0,1]$, let us set

$$T_\alpha[f] := (1-\alpha)L^{cg}\left[f\right] + \alpha R^{cg}\left[f\right]. \tag{3.8}$$

Then,

1. *T_α is dominated by T on $K(\Omega)$ if and only if $\alpha = 0$;*
2. *T is dominated by T_α on $K(\Omega)$ if and only if $\alpha = 1$.*

Proof. (1) is an immediate consequence of Corollary 3.4. To prove (2), let us assume that T is dominated by T_α on $K(\Omega)$. Then, since $-\phi_{n+1}^{cg}$ belongs to $K(\Omega)$, it takes the value -1 at $cg_T(\Omega)$ and vanishes on all vertices of Ω, we have

$$-A_{n+1}^{cg} = T[-\phi_{n+1}^{cg}] \leq T_\alpha[-\phi_{n+1}^{cg}] = -(1-\alpha) - \alpha A_{n+1}^{cg},$$

here, for simplicity in notation, we have denoted $\phi_{n+1}^{cg_T(\Omega)}$ by ϕ_{n+1}^{cg}. This shows that

$$(1-\alpha)(1 - A_{n+1}^{cg}) \leq 0.$$

According to the fact, that A_{n+1}^{cg} is strictly less than or equal to 1 and $\alpha \in [0,1]$, it follows that the above inequality is satisfied only if $\alpha = 1$. The converse implication is obvious, since for $\alpha = 1, T_1 := R^{cg}$ and by Theorem 2.4, T is dominated by R^{cg} on $K(\Omega)$. □

Now consider the case when the domain Ω is a simplex. In this case, Lemma 2.2 implies that for all $\alpha \in [0,1]$

$$L^{cg}\left[f\right] \leq T_\alpha[f] := (1-\alpha)L^{cg}\left[f\right] + \alpha R^{cg}\left[f\right] \leq R^{cg}\left[f\right], \forall f \in K(\Omega),$$

where L^{cg} and R^{cg} are defined by

$$\begin{aligned} L^{cg}\left[f\right] &:= f(\boldsymbol{c}g_T(\Omega)) \\ R^{cg}\left[f\right] &:= \sum_{i=0}^{n} \psi_i(\boldsymbol{c}g_T(\Omega)) f(\boldsymbol{v}_i). \end{aligned}$$

The problem we wish to consider is: For which values of $\alpha \in [0,1]$, is T bounded from below or above by T_α on $K(\Omega)$?

The following Theorem gives all admissible values of α for which T_α is bounded below by T on $K(\Omega)$.

Theorem 3.6. *If Ω is a simplex then the following inequalities hold for every $\alpha \in \left[A_{n+1}^{cg}, 1\right]$:*

$$T[f] \leq T_\alpha[f], \forall f \in K(\Omega). \tag{3.9}$$

Moreover, as a function of α, T_α is non-decreasing on $\left[A_{n+1}^{cg}, 1\right]$. In addition, if $\alpha > A_{n+1}^{cg}$, then T_α cannot dominate T on $K(\Omega)$.

Proof. A simple inspection of Theorem 2.4 shows that (3.9) holds for $\alpha = A_{n+1}^{cg}$. Since, by Lemma 2.2, we know already that L^{cg} is dominated by R^{cg}, (note that this inequality is also a very easy consequence of the classical Jensen's discrete inequality for convex functions), it follows that T_α is a non-decreasing function of α and that (3.9) holds for each $\alpha \in \left[A_{n+1}^{cg}, 1\right]$. The rest of proof can be carried out similarly as in Theorem 3.1. □

We can show similarly as in the proof of Theorem 3.6 that T_α is dominated by T on $K(\Omega)$ if and only if $\alpha = 0$.

We naturally asked if a similar result of Theorem 3.6 holds for convex combinations of lower bounds given in Theorem 2.4. To this end, it is also worth emphasizing the following result:

Theorem 3.7. *Let us define the functional U_α on $C(\Omega)$ by*

$$U_\alpha[f] = (1-\alpha) f\left(\boldsymbol{x}\left(\phi_{n+1}^{cg}\right)\right) + \alpha \sum_{i=0}^{n} \psi_i(\boldsymbol{cg}_T(\Omega)) f\left(\boldsymbol{x}\left(\phi_i^{cg}\right)\right).$$

Then if Ω is a simplex the following inequalities hold for every $\alpha \in \left[0, A_{n+1}^{cg}\right]$:

$$U_\alpha[f] \leq T[f], \forall f \in K(\Omega). \tag{3.10}$$

Moreover, as a function of α, U_α is non-decreasing on $\left[0, A_{n+1}^{cg}\right]$. In addition, if $\alpha > A_{n+1}^{cg}$, then U_α cannot dominated by T on $K(\Omega)$.

Proof. The proof is similar to the proof of Theorem 3.6, and it is only necessary to use the second inequality obtained in Theorem 2.4. □

4. A generalization of J. Favard's inequality

Let us now turn to some applications of the previous results. Our intention in this section is to provide a natural generalization of Favard's inequality [3, p. 58]. To this end, we start with recalling the well known Hadamard's inequality in its usual form: which states that for every convex function $f : [a,b] \to \mathbb{R}$, it holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{4.1}$$

Inequalities (4.1) and many of their variants have been extensively studied in the literature, see, e. g., [9, 10, 11, 12]. To prepare for our generalization, we first derive a refinement to the right-hand side of Hadamard's inequality in the following way:

Proposition 4.1. *If $f \in K([a,b]$, then the following refinement of Hadamard's inequality holds.*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{2}\inf_{x\in[a,b]} f(x) + \frac{1}{2}\sup_{x\in[a,b]} f(x). \tag{4.2}$$

Proof. Since we have not been able to find an existing proof for the following result, we include one here. Let us first assume that f is a nonnegative function belonging to $K([a,b]$. We begin the proof of Proposition 4.1 by noting that since f is convex, then, there exists an t_f in $[a,b]$ such that f is non-increasing on $[a,t_f]$ and non-decreasing on $[t_f,b]$. We can assume without loss of generality that t_f belongs to $]a,b[$, since if $t_f = a$ or $t_f = b$ the result is an immediate consequence of Hadamard's inequality. By applying twice Hadamard's inequality, we immediately get

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^{t_f} f(x)dx + \int_{t_f}^b f(x)dx \\
&\le \frac{t_f-a}{2}\left(f(a)+f(t_f)\right) + \frac{b-t_f}{2}\left(f(t_f)+f(b)\right) \\
&= \frac{b-a}{2}f(t_f) + \frac{t_f-a}{2}f(a) + \frac{b-t_f}{2}f(b) \\
&\le \frac{b-a}{2}f(t_f) + \frac{b-a}{2}\max\left\{f(a),f(b)\right\}.
\end{aligned}$$

The convexity of f now implies the simple fact that the maximum $\sup_{x\in[a,b]} f(x)$ is achieved in one of the endpoints, either in a or in b, then after dividing by $b-a$ we get the desired inequality (4.2) immediately.
In general, when f is an arbitrary function fron $K[a,b]$ then, applying the above inequality to the nonnegative convex function

$$f - \inf_{x\in[a,b]} f(x),$$

we obtain the required result, and this completes the proof of the proposition. □

If f is a continuous concave function, then a similar argument shows that the inequality (4.2) is reversed. This inequality is known as Favard's inequality [3, p. 58]. Thus, inequality (4.2) complements the famous Favard's inequality. Recall that Favard has shown the more general result

$$\left(\frac{1}{b-a}\int_a^b f^p(x)dx\right)^{1/p} \le \frac{2}{(p+1)^{1/p}}\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \tag{4.3}$$

for all nonnegative continuous concave functions and all parameters $p > 1$. Let us note that the proof of the Favard's inequality, given in [3], is obtained by letting p tend to infinity in (4.3). Thus we have at hand a very simple proof of the so-called Favard's inequality. We may also observe that inequalities (4.2) are a refinement of inequalities (2.15) established when the function is continuous only.

The next result extends the inequality given in Proposition 4.1 to any polytope Ω and any positive linear functional T from $N^*(\Omega)$.

Theorem 4.2. *Let $f \in K(\Omega)$ such that f attains its minimum over Ω at $\boldsymbol{x}_{min}$. Then the following inequalities are valid:*

$$f\left(cg_T(\Omega)\right) \leq T[f] \leq T[\phi_{n+1}^{\boldsymbol{x}_{min}}] \inf_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}) + \left(1 - T[\phi_{n+1}^{\boldsymbol{x}_{min}}]\right) \sup_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}). \tag{4.4}$$

Proof. Since f is continuous on Ω, there exist points $\boldsymbol{x}_{min}, \boldsymbol{x}_{max} \in \Omega$ such that $f_{min} := f(\boldsymbol{x}_{min}) = \inf_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x})$ and $f_{max} := f(\boldsymbol{x}_{max}) = \sup_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x})$. It can easily be shown that given a convex function over the polytope Ω, then this function achieves its maximum value at some point among the vertices of Ω. We assume first that $\boldsymbol{x}_{min}$ is an interior point of Ω. Next we apply Theorem 2.4 by choosing $\boldsymbol{y} = \boldsymbol{x}_{min}$ then the following inequality holds

$$T\left[h\right] \leq \sum_{i=0}^{n} T[\phi_i^{\boldsymbol{x}_{min}}]h(\boldsymbol{v}_i) + T[\phi_{n+1}^{\boldsymbol{x}_{min}}]h(\boldsymbol{x}_{min}),$$

where $h = f - f_{min}$. Then it is clear that, in order to get an upper bounds for T, we need to maximize $h(\boldsymbol{v}_i)$ for all vertices of Ω. Thus, since h is a nonnegative function on Ω an additional calculation similar to that done above shows that

$$T\left[h\right] \leq (1 - T[\phi_{n+1}^{\boldsymbol{x}_{min}}])h(\boldsymbol{x}_{max}) + T[\phi_{n+1}^{\boldsymbol{x}_{min}}]h(\boldsymbol{x}_{min}).$$

Therefore, it follows from the fact that the above inequality becomes an equality for constants, we deduce the right-hand side in (4.4) for any continuous convex function. Finally, a simple continuity argument shows that equality (4.4) is also valid when $\boldsymbol{x}_{min}$ is a border point of Ω. $\square$

Our next result extends the above theorem to any point $\boldsymbol{y}$ that belongs to the interior of Ω. Indeed, by similar arguments the following more general result can be proved. For convenience, we reformulate the statement of this result here.

Theorem 4.3. *Let $\boldsymbol{y}$ be an arbitrary fixed point in $int(\Omega)$. Then, the following inequalities are valid for any $f \in K(\Omega)$:*

$$f\left(cg_T(\Omega)\right) \leq T[f] \leq T[\phi_{n+1}^{\boldsymbol{y}}]f(\boldsymbol{y}) + \left(1 - T[\phi_{n+1}^{\boldsymbol{y}}]\right) \sup_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}). \tag{4.5}$$

The reader may observe that the bounds derived in Theorem 4.2 and 4.3 are always better than those given in (2.15), where we have assumed only the continuity of the functions involved.

Multidimensional integral version: our general upper bound for the integral functional setting is not different from that in the one dimension. Indeed, we have the following multivariate version of Favard's inequality as a corollary to the above theorem:

Corollary 4.4. *For every convex function belonging to $K(\Omega)$ the following inequalities are valid:*

$$f\left(\frac{\int_\Omega \boldsymbol{x} d\boldsymbol{x}}{vol(\Omega)}\right) \leq \frac{\int_\Omega f(\boldsymbol{x}) d\boldsymbol{x}}{vol(\Omega)} \leq \frac{1}{d+1} \inf_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}) + \frac{d}{d+1} \sup_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x}). \tag{4.6}$$

Proof. Just apply the integral identity (2.18) proved in Lemma 2.3. $\square$

An interesting aspect of this form is that, in the integral situation, the coefficient $T[\phi_{n+1}^{\boldsymbol{x}_{min}}]$ is independent of the point $\boldsymbol{x}_{min}$ and it is a function of d only.

Remark 4.5. Note that if the function f in Theorems 4.2, 4.3 and Corollary 4.4 is concave, then inequalities (4.4), (4.5) and (4.6) hold true, except in these cases each of these inequalities is reversed (with the order of the sup and inf reversed).

We end this section with a question which is concerned with a possible extension of Theorem 4.2 and its Corollary 4.4. We may naturally ask whether inequalities in (4.4) and (4.6) hold for any compact convex set $\Omega \subset \mathbb{R}^d$ with positive measure.

The answer is no. Indeed, this result can be immediately derived from the following general statement:

Theorem 4.6. *Let $T \in N^*(\Omega)$, where Ω is a compact convex subset of $\mathbb{R}^d$ with positive measure. Assume that there are $(m+1)$ points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m \in \Omega$, and real numbers $a_0, \ldots, a_m$, such such that, for every $f \in K(\Omega)$, we have*

$$T[f] \leq R[f] := \sum_{i=0}^{m} a_i f(\boldsymbol{x}_i). \tag{4.7}$$

Then Ω must be a polytope.

Proof. Let Ω^* be the convex hull of $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m$. Let us assume to the contrary that there exists a point, say $\boldsymbol{x}^*$, which is an element of Ω, but does not belong to Ω^*. We will exhibit a function from $K(\Omega)$ for which inequality (4.7) does not hold. Indeed, due to the separation Theorem for closed convex sets (see, e.g., [20, p. 65, Theorem 2.4.1]), there exists an affine function h such that $h(\boldsymbol{x}^*) = 1$ and h is strictly negative on Ω^*. Set

$$\tilde{h}(\boldsymbol{x}) = \max\{h(\boldsymbol{x}), 0\}, \forall \boldsymbol{x} \in \Omega.$$

Note that $\tilde{h}$ is a nonnegative function belonging to $K(\Omega)$, since it is the maximum of two convex functions. It also vanishes on Ω^* and takes the value 1 at $\boldsymbol{x}^*$. We can therefore conclude, in view of the continuity of $\tilde{h}$, that there exists a neighbor U of $\boldsymbol{x}^*$ such that $\tilde{h}$ is positive. Because T is assumed strictly positive, we then have $T[\tilde{h}] > 0$. But, $R[\tilde{h}] = 0$, this contradicts (4.7) and so Ω must be equal the convex hull of the points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m$. Thus we can conclude that Ω is a polytope. □

This justifies why we have limited our analysis to the case of polytope domains. Finally, we should remark that if Ω is a polytope, then Theorem 2.4 tells us that, every $T \in N^*(\Omega)$ is dominated in $K(\Omega)$ by some functional of the form given in (4.7). Thus, domination property (4.7) characterizes the polytopes among the compact convex subsets of $\mathbb{R}^d$.

5. An improvement of the lower and upper bounds and error estimates

Assume given lower and upper bounds in $K(\Omega)$ to a given positive linear functional. In the case when f is smooth, this section answers the following question:

• How to properly derive a procedure to be able to generate 'much better' lower and upper bounds than those derived in Section 2 ?

In this situation, this section also establishes a general result concerning error estimates. More precisely, for smooth (nonconvex) twice continuously differentiable

functions, the first part of this section improves both the lower and upper bounds given in Theorem 2.4 for any $T \in N^*(\Omega)$. Our inequalities can be seen as an extension of Hadamard-Favard inequalities for nonconvex functions.
In order to present the essentials of the technique, we need some additional necessary background and notation.
By S_d we denote the set of all $d \times d$ symmetric matrices in $\mathbb{R}$. Let $A \in S_d$, and $\beta_i[A], i = 1, \ldots, d$, the (real) eigenvalues of A, we define

$$\beta_{min}[A] := \min_{1 \leq i \leq d} \beta_i[A] = \min_{\|\boldsymbol{y}\|=1} \langle A\boldsymbol{y}, \boldsymbol{y} \rangle .$$

We say $A \in S_d$ is positive semidefinite if $\langle A\boldsymbol{y}, \boldsymbol{y} \rangle \geq 0$, for every $\boldsymbol{y} \in \mathbb{R}^d$. The set of positive semidefinite symmetric matrices (all eigenvalues ≥ 0) is denoted by S_d^+ .
By $D^2 f(\boldsymbol{x})$, we mean the $d \times d$ matrix whose entries are the second-order partial derivatives of f at $\boldsymbol{x}$. It is well known that when f is a $C^2(\Omega)$-function, its convexity is characterized by the fact that for all $\boldsymbol{x} \in \Omega, D^2 f(\boldsymbol{x}) \in S_d^+$ (see e.g. [18]). For every $x \in \Omega$, the Hessian matrix $D^2 f(\boldsymbol{x})$, as real-valued and symmetric matrix, has real-valued eigenvalues. Therefore, for every function f in $C^2(\Omega)$, we may define

$$\lambda_{min}[f] := \inf_{\boldsymbol{x} \in \Omega} \beta_{min}[D^2 f(\boldsymbol{x})].$$

Now, let f be any $C^2(\Omega)-$function and set

$$g := f - \frac{\lambda_{min}[f]}{2} \|.\|^2 , \tag{5.1}$$

($\|.\|^2$ denotes the Euclidean norm on $\mathbb{R}^d$.) The Hessian matrix of g is

$$D^2 g(\boldsymbol{x}) = D^2 f(\boldsymbol{x}) - \lambda_{min}[f] I_d,$$

where I_d denotes the $d \times d$ identity matrix. Therefore, for $\boldsymbol{y} \in \mathbb{R}^d$ such that $\|\boldsymbol{y}\| = 1$, we have

$$\left\langle \boldsymbol{y}, D^2 g(\boldsymbol{x})(\boldsymbol{y}) \right\rangle = \left\langle \boldsymbol{y}, D^2 f(\boldsymbol{x})(\boldsymbol{y}) \right\rangle - \lambda_{min}[f].$$

It is clear from the definition of $\lambda_{min}[f]$ that, for every $\boldsymbol{x} \in \Omega$, the right-hand term in the above equation is nonnegative. This means that the Hessian matrix of g is positive semidefinite for all $\boldsymbol{x}$ of the set Ω, and consequently g is convex.
Hence, an arbitrary nonconvex twice continuously differentiable function is made convex after adding to it the quadratic $-\frac{\lambda_{min}[f]}{2} \|.\|^2$.
Note that $\lambda_{min}[f]$ is not necessarily zero if f is convex over Ω. On the other hand, if $\lambda_{min}[f] \geq$ then f is convex over Ω.
Let R (resp. L) be a lower bound (resp. an upper bound) of T on $K(\Omega)$, in the sense

$$L[g] \leq T[g] \leq R[g], \tag{5.2}$$

for all convex functions $g \in C(\Omega)$. Define

$$R^+[\|.\|^2] = R[\|.\|^2] - T[\|.\|^2],$$

and

$$L^-[\|.\|^2] = T[\|.\|^2] - L[\|.\|^2].$$

Note that $R^+[\|.\|^2]$ and $L^-[\|.\|^2]$ are nonnegative.

With this notation, the next result shows that, under some regularity assumptions about the function f, there exist better bounds than those defined in (5.2). More precisely we have:

Theorem 5.1. *Let R and L be an upper bound (resp. a lower bound) of T on $K(\Omega)$. Then the following bounds hold for every function from $C^2(\Omega)$:*

$$L[f] + \frac{\lambda_{min}[f]}{2} L^-[\|.\|^2] \leq T[f] \leq R[f] - \frac{\lambda_{min}[f]}{2} R^+[\|.\|^2]. \tag{5.3}$$

Proof. In fact, under the present assumption about the function f, it is evident that the auxiliary function

$$g := f - \frac{\lambda_{min}[f]}{2} \|.\|^2$$

is convex. Hence we can apply the inequality (5.2) to g and rearranging terms leads to the required inequality. □

We now turn to the error estimates for the approximation of a functional $T \in N^*(\Omega)$ by any lower or upper bounds on $K(\Omega)$. The following result characterizes the error estimates in approximation by functionals of this type.

Proposition 5.2. *Let $A : C^k(\Omega) \to \mathbb{R}$, where $k \in \{0, 1\}$, be a normalized linear functional and let $\sigma \in \{-1, 1\}$. The following statements are equivalent:*

(i) *For every $f \in C^2(\Omega)$, we have*

$$|T[f] - A[f]| \leq \sigma \left(T[\|.\|^2] - A[\|.\|^2] \right) \frac{\lfloor D^2 f \rfloor}{2}.$$

Equality is attained for all functions of the form $f(\boldsymbol{x}) := a(\boldsymbol{x}) + c$, where $c \in \mathbb{R}$ and a is any affine function.

(ii) *For every convex function $f \in C^2(\Omega)$, we have*

$$\sigma \left(T[f] - A[f] \right) \geq 0.$$

Proof. The proof is an immediate consequence of [8, Proposition 2.1]. □

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