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A THREE–POINT QUADRATURE RULE FOR THE RIEMANN–STIELTJES INTEGRAL WITH APPLICATIONS

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ABSTRACT. In this paper, three point quadrature rules for the Riemann–Stieltjes integral are introduced. Applications to numerical integration are provided as well.

1. Introduction

In 2000, Dragomir [17] introduced the following quadrature rule:

$$\int_a^b f(t)\,du(t) \cong f(x)\left[u(b)-u(a)\right], \forall x \in [a,b]$$

For several error bounds for this quadrature under various assumptions to the function involved the reader may refer to [7, 8], [10]–[17], [24, 25], [27]–[29], and the references therein, as well as the recent work [3].

From a different point of view, the authors of [18] considered the problem of approximating the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ via the generalized trapezoid rule $[u(x)-u(a)]\,f(a)+[u(b)-u(x)]\,f(b)$, i.e.,

$$\int_a^b f(t)\,du(t) \cong [u(x)-u(a)]\,f(a)+[u(b)-u(x)]\,f(b), \forall x \in [a,b].$$

For various bounds of the above generalized trapezoid rule the reader may refer to [18]–[22] and the references therein. For new quadrature rules regarding Riemann–Stieltjes integral see [1], [2] and [4].

In order to approximate the Riemann–Stieltjes integral $\int_a^b f(x)\,du(x)$ by the Riemann integral $\int_a^b f(t)\,dt$, Dragomir and Fedotov [23], introduced the following functional:

$$\mathcal{D}(f;u) := \int_a^b f(x)\,du(x) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

In the same paper [23], the authors have proved the following inequality:

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Theorem 1. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K > 0$. Then we have*

$$|\mathcal{D}(f;u)| \leq \frac{1}{2} K (b-a) \bigvee_a^b (u). \tag{1.2}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In [21], Dragomir has obtained the following inequality as well:

Theorem 2. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(y) - u(x)| \leq L|x-y|, \forall x, y \in [a,b], \quad (L>0)$$

and f is Riemann integrable on $[a,b]$.

If $m, M \in \mathbb{R}$, are such that $m \leq f(x) \leq M$, for any $x \in [a,b]$, then

$$|\mathcal{D}(f;u)| \leq \frac{1}{2} L (M-m)(b-a). \tag{1.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In [26], Mercer has introduced new midpoint and trapezoid type rules for the Riemann–Stieltjes integral which engender a natural generalization of Hadamard's integral inequality, as follows:

Theorem 3. *Let g be continuous and increasing on $[a,b]$, let $c \in [a,b]$ which satisfies*

$$\int_a^b g(t)\,dt = (c-a)\,g(a) + (b-c)\,g(b).$$

If $f'' \geq 0$, then we have

$$f(c)[g(b)-g(a)] \leq \int_a^b f dg \leq [G - g(a)] f(a) + [g(b) - G] f(b) \tag{1.4}$$

where, $G := \frac{1}{b-a}\int_a^b g(t)\,dt$.

However, it seems that Mercer didn't notice that the following relation between the right-hand side of (1.4) and $\mathcal{D}(g;f)$, exists

$$\begin{aligned} \int_a^b f(t)\,dg(t) - [G-g(a)]\,f(a) - [g(b)-G]\,f(b) \\ = \frac{f(b)-f(a)}{b-a}\int_a^b g(t)\,dt - \int_a^b g(t)\,df(t) := -\mathcal{D}(g;f). \end{aligned} \tag{1.5}$$

This follows by integration by parts formula

$$\int_a^b f(t)\,dg(t) = f(b)\,g(b) - f(a)\,g(a) - \int_a^b g(t)\,df(t).$$

Motivated by [26], in this paper several new inequalities of Hermite–Hadamard type and new approximations for the Riemann–Stieltjes integral via three point quadrature rules are established. The idea of the results and the analysis of the proofs follows in a similar manner to that one used in [26]. However, the obtained results in this work, are completely different and independent from those established in [26].

2. Introducing quadrature rules for Riemann–Stieltjes integral

To establish a three-point quadrature rule for the Riemann–Stieltjes integral, let us seek numbers A, B, C and D such that

$$\int_a^b f(t)\,dg(t) = \int_a^x f(t)\,dg(t) + \int_x^b f(t)\,dg(t) \tag{2.1}$$

with

$$\int_a^x f(t)\,dg(t) \cong Af(a) + Bf(x) \tag{2.2}$$

and

$$\int_x^b f(t)\,dg(t) \cong Cf(x) + Df(b) \tag{2.3}$$

for all $x \in (a,b)$.

To find the scalars A, B, C and D, let $f(t) = 1$ and then $f(t) = t$ in (2.2) and (2.3); respectively. By simple calculations we get

$$A = \frac{1}{x-a}\int_a^x g(t)\,dt - g(a), \qquad B = g(x) - \frac{1}{x-a}\int_a^x g(t)\,dt$$
$$C = \frac{1}{b-x}\int_x^b g(t)\,dt - g(x), \qquad D = g(b) - \frac{1}{b-x}\int_x^b g(t)\,dt$$

and therefore we have

$$\begin{aligned}\int_a^b f(t)\,dg(t) \cong{} & \left[\frac{1}{b-x}\int_x^b g(t)\,dt - \frac{1}{x-a}\int_a^x g(t)\,dt\right] f(x) \\ & + \left[\frac{1}{x-a}\int_a^x g(t)\,dt - g(a)\right] f(a) \\ & + \left[g(b) - \frac{1}{b-x}\int_x^b g(t)\,dt\right] f(b)\,. \end{aligned} \tag{2.4}$$

for all $a < x < b$.

Theorem 4. *Fix $x \in (a,b)$. Let g be continuous on $[a,b]$ and monotonic nondecreasing on $[a,x]$ and $[x,b]$ (it may not be monotonic nondecreasing on the whole of $[a,b]$). Let $c_1 \in [a,x]$ and $c_2 \in [x,b]$, be such that*

$$c_1 = \frac{xg(x) - ag(a) - \int_a^x g(t)\,dt}{g(x) - g(a)} \quad \text{and} \quad c_2 = \frac{bg(b) - xg(x) - \int_x^b g(t)\,dt}{g(b) - g(x)}.$$

If $f'' \geq 0$, then we have

$$\begin{aligned} & [g(x) - g(a)]\, f(c_1) + [g(b) - g(x)]\, f(c_2) \\ & \leq \int_a^b f(t)\,dg(t) \\ & \leq [G(a,x) - g(a)]\, f(a) + [G(x,b) - G(a,x)]\, f(x) \\ & + [g(b) - G(x,b)]\, f(b)\,, \end{aligned} \tag{2.5}$$

for all $a < x < b$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

Proof. The argument of the proof is similar to the proof of Theorem 3. We first prove the right-hand inequality. Fix $x \in (a,b)$, so that

$$\int_a^b f(t)\,dg(t) = \int_a^x f(t)\,dg(t) + \int_x^b f(t)\,dg(t).$$

Let $h_1(t) = g(t) - G(a,x)$ and $H_1(t) := \int_a^t h(u)\,du$, $t \in [a,x]$. Using integration by parts twice, and the fact that $H_1(a) = H_1(x) = 0$, we get that

$$\begin{aligned}
&\int_a^x f(t)\,dg(t) - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x \\
&= \int_a^x f(t)\,d\left[g(t) - G(a,x)\right] - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x \\
&= -\int_a^x \left[g(t) - G(a,x)\right] df(t) \\
&= -\int_a^x f'(t)\,dH_1(t) \\
&= \int_a^x H_1(t)\,df'(t) - H_1(t)\,f'(t)\Big|_a^x \\
&= \int_a^x H_1(t)\,f''(t)\,dt.
\end{aligned}$$

Now, since $f'' \geq 0$, its enough to show that $H_1 \leq 0$, so that the right-hand inequality is proved, i.e.,

$$\int_a^x f(t)\,dg(t) - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x \leq 0. \tag{2.6}$$

To do this, since g is increasing on $[a,x]$ then by the First Mean Value Theorem for integrals there exists a unique $\tau \in (a,x)$ such that $g(\tau) = G(a,x)$. For $s \in [a,\tau]$, we have $H_1(s) = \int_a^s [g(t) - G(a,x)]\,dt \leq 0$ and for $s \in [\tau, x]$, we also have

$$\begin{aligned}
H_1(s) &= \int_a^\tau \left[g(t) - G(a,x)\right] dt + \int_\tau^s \left[g(t) - G(a,x)\right] dt \\
&= -\int_\tau^x \left[g(t) - G(a,x)\right] dt + \int_\tau^s \left[g(t) - G(a,x)\right] dt \\
&= -\int_s^x \left[g(t) - G(a,x)\right] dt \\
&= -\int_s^x g(t)\,dt + (x-s)\,G(a,x) \\
&= -(x-s)\,g(s) + (x-s)\,G(a,x) \leq 0,
\end{aligned}$$

which proves that $H_1 \leq 0$.

Now, for the integral $\int_x^b f(t)\,dg(t)$, we may define $h_2(t) = g(t) - G(x,b)$ and $H_2(t) := \int_x^t h(u)\,du$, $t \in [x,b]$. By repeating the above argument we get that

$$\int_x^b f(t)\,dg(t) - f(t)\left[g(t) - G(x,b)\right]\Big|_x^b \leq 0 \tag{2.7}$$

and thus by adding (2.6) and (2.7), the right-hand side of (2.5) is proved.

To prove the left-hand side of (2.5), fix $x \in (a,b)$ and define the mapping

$$h_1(t) = \begin{cases} g(t) - g(a), & t \in [a, c_1] \\ g(t) - g(x), & t \in (c_1, x] \end{cases}$$

and therefore we observe that for $H_1(t) = \int_a^t h(u)du$, where $t \in [a,x]$, we have $H_1(a) = 0$ and since

$$\begin{aligned} H_1(x) = \int_a^x h(u)\,du &= \int_a^{c_1} (g(u) - g(a))\,du + \int_{c_1}^x (g(u) - g(x))\,du \\ &= \int_a^x g(u)\,du - (c_1 - a)\,g(a) - (x - c_1)\,g(x) = 0 \end{aligned}$$

by our choice of c_1.

Now, using integration by parts (twice) we may write

$$\int_a^x f(t)\,dg(t) - f(c_1)[g(x) - g(a)] = \int_a^x H_1(t)\,f''(t)\,dt. \tag{2.8}$$

Claiming that $H_1 \geq 0$, then by given hypothesis $f'' \geq 0$ and so

$$\int_a^x f(t)\,dg(t) - f(c_1)[g(x) - g(a)] \geq 0 \tag{2.9}$$

which therefore, prove the left-hand inequality.

To prove our claim, let $y \in [a, c_1]$, since g is monotonic nondecreasing on $[a,x]$, then we have

$$H_1(y) = \int_a^y (g(u) - g(a))\,du.$$

Also, for $y \in (c_1, x]$, we have

$$\begin{aligned} H_1(y) &= \int_a^{c_1} (g(u) - g(a))\,du + \int_{c_1}^y (g(u) - g(x))\,du \\ &= \int_a^y g(u)\,du - (c_1 - a)\,g(a) - (y - c_1)\,g(x) \\ &= \int_a^x g(u)\,du - (c_1 - a)\,g(a) - \int_y^x g(u)\,du - (y - c_1)\,g(y) \\ &= \int_a^x g(u)\,du - (c_1 - a)\,g(a) - \int_y^x g(u)\,du - (y - c_1)\,g(y) \\ &= g(x)(x - c_1) - \int_y^x g(u)\,du - (y - c_1)\,g(y) \\ &= g(x)(x - y) - \int_y^x g(u)\,du \geq 0, \end{aligned}$$

again since g is increasing. So that the claim is proved.

In a similar way, define the mapping

$$h_2(t) = \begin{cases} g(t) - g(x), & t \in [x, c_2] \\ g(t) - g(b), & t \in (c_2, b] \end{cases}$$

and therefore we observe that for $H_2(t) = \int_t^b h_2(u)du$, where $t \in [x,b]$, $H_2(x) = 0$ and $H_2(b) = 0$. Similarly as above we have

$$\int_x^b f(t)\,dg(t) - f(c_2)[g(b) - g(x)] = \int_x^b H_2(t)\,f''(t)\,dt. \tag{2.10}$$

Claiming that $H_2 \geq 0$, then by given hypothesis $f'' \geq 0$,

$$\int_x^b f(t)\,dg(t) - f(c_2)[g(b) - g(x)] \geq 0. \tag{2.11}$$

By repeating the above argument we can prove last claim. So that adding (2.9) and (2.12), we get

$$\int_a^b f(t)\,dg(t) - \{[g(x) - g(a)]\,f(c_1) + [g(b) - g(x)]\,f(c_2)\} \geq 0, \tag{2.12}$$

which therefore, prove the left-hand side of (2.5). □

Corollary 1. *In Theorem 4, choose $g(t) = t$, $t \in [a,b]$, then we have the inequality:*

$$\begin{aligned} &\frac{1}{b-a}\left[(x-a)\,f\left(\frac{a+x}{2}\right) + (b-x)\,f\left(\frac{x+b}{2}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{2}\left[f(x) + \frac{(x-a)\,f(a) + (b-x)\,f(b)}{b-a}\right] \end{aligned} \tag{2.13}$$

for all $a < x < b$. Moreover, if we choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned} \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] &\leq \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]. \end{aligned} \tag{2.14}$$

3. Representation of the Error for Differentiable g

3.1. **A.** Consider the quadrature rule

$$\begin{aligned} \mathcal{R}(f,g;x) := &\int_a^b f(t)\,g'(t)\,dt - [G(a,x) - g(a)]\,f(a) \\ &- [G(x,b) - G(a,x)]\,f(x) - [g(b) - G(x,b)]\,f(b), \end{aligned} \tag{3.1}$$

where, $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

Theorem 5. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic on $[a,x]$ and $[x,b]$. Then there exist $\xi_1,\eta_1 \in (a,x)$ and $\xi_2,\eta_2 \in (x,b)$ such that*

$$\begin{aligned} &\int_a^b f(t)\,g'(t)\,dt - [G(a,x) - g(a)]\,f(a) - [G(x,b) - G(a,x)]\,f(x) \\ &\qquad - [g(b) - G(x,b)]\,f(b) \\ &\qquad = -\frac{1}{12}\left[f''(\xi_1)\,g'(\eta_1)\,(x-a)^3 + f''(\xi_2)\,g'(\eta_2)\,(b-x)^3\right], \end{aligned} \tag{3.2}$$

for all $a < x < b$, where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\,dt$.

Proof. Fix $x \in (a,b)$, so that

$$\int_a^b f(t)\, g'(t)\, dt = \int_a^x f(t)\, g'(t)\, dt + \int_x^b f(t)\, g'(t)\, dt. \tag{3.3}$$

For the first integral on the right hand side of (3.3), consider the functions $h_1(t) = g(t) - G(a,x)$ and $H_1(t) := \int_a^t h(u)\, du$, $t \in [a,x]$. As we pointed out previously,

$$\int_a^x f(t)\, g'(t)\, dt - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x = \int_a^x H_1(t)\, f''(t)\, dt$$

and since g is monotonic on $[a,x]$ then H_1 does not change sign on $[a,x]$. So by the First Mean Value Theorem for integrals, there is $\xi_1 \in (a,x)$ such that

$$\int_a^x f(t)\, g'(t)\, dt - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x = f''(\xi_1)\int_a^x H_1(t)\, dt.$$

Applying the classical Trapezoid Rule to $\int_a^x H_1(t)dt$, on the right side above we have

$$\begin{aligned}
&\int_a^x f(t)\, g'(t)\, dt - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x \\
&= f''(\xi_1)\int_a^x H_1(t)\, dt \\
&= f''(\xi)\left[\frac{H(a) + H(x)}{2}(x-a) - H''(\eta_1)\frac{(x-a)^3}{12}\right]
\end{aligned}$$

for some $\eta_1 \in (a,x)$. Since $H_1'' = g'$ and $H_1(a) = H_1(x) = 0$, then

$$\int_a^x f(t)\, g'(t)\, dt - f(t)\left[g(t) - G(a,x)\right]\Big|_a^x = -f''(\xi_1)\, g'(\eta_1)\frac{(x-a)^3}{12}. \tag{3.4}$$

Now, for the second integral on the right hand side of (3.3), consider the functions $h_2(t) = g(t) - G(x,b)$ and $H_2(t) := \int_x^t h(u)\, du$, $t \in [x,b]$. By repeating the above argument we get that

$$\int_x^b f(t)\, g'(t)\, dt - f(t)\left[g(t) - G(x,b)\right]\Big|_x^b = -f''(\xi_2)\, g'(\eta_2)\frac{(b-x)^3}{12}, \tag{3.5}$$

and thus by adding (3.4) and (3.5), the right-hand side of (3.8) is proved. $\square$

Corollary 2. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. Then there exist $\xi_1, \eta_1 \in \left(a, \frac{a+b}{2}\right)$ and $\xi_2, \eta_2 \in \left(\frac{a+b}{2}, b\right)$ such that*

$$\begin{aligned}
&\int_a^b f(t)\, g'(t)\, dt - \left[G\left(\frac{a+b}{2}, b\right) - G\left(a, \frac{a+b}{2}\right)\right] f\left(\frac{a+b}{2}\right) \\
&\qquad - \left[G\left(a, \frac{a+b}{2}\right) - g(a)\right] f(a) - \left[g(b) - G\left(\frac{a+b}{2}, b\right)\right] f(b) \\
&\qquad\qquad = -\frac{(b-a)^3}{96}\left[f''(\xi_1)\, g'(\eta_1) + f''(\xi_2)\, g'(\eta_2)\right],
\end{aligned} \tag{3.6}$$

where $G(\alpha,\beta) := \frac{1}{\beta-\alpha}\int_\alpha^\beta g(t)\, dt$.

Corollary 3. *In Corollary 2, let $g(t) = t$, for all $t \in [a, b]$, then we have:*

$$\int_a^b f(t)\,dt - \frac{(b-a)}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] = -\frac{(b-a)^3}{96}\left[f''(\xi_1) + f''(\xi_2)\right]. \tag{3.7}$$

Theorem 6. *Suppose that f'' and g'' are continuous on $[a, b]$ and that g is monotonic on $[a, x]$ and $[x, b]$. Then there exist $\xi_1, \eta_1, \tau_1, \sigma_1 \in (a, x)$ and $\xi_2, \eta_2, \tau_2, \sigma_2 \in (x, b)$ such that*

$$\begin{aligned}
&\int_a^b f(t)\,g'(t)\,dt \\
&\quad - \left[\frac{g(x) - g(a)}{2} f(a) + \frac{g(b) - g(a)}{2} f(x) + \frac{g(b) - g(x)}{2} f(b)\right] \\
&= f'(\tau_1)\,g''(\sigma_1)\,\frac{(x-a)^2}{12} + f'(\tau_2)\,g''(\sigma_2)\,\frac{(b-x)^2}{12} \\
&\quad - \frac{1}{12}\left[f''(\xi_1)\,g'(\eta_1)\,(x-a)^3 + f''(\xi_2)\,g'(\eta_2)\,(b-x)^3\right].
\end{aligned} \tag{3.8}$$

Proof. Apply the classical Trapezoid rule to get

$$G(a, x) = \frac{1}{x-a}\int_a^x g(t)\,dt = \frac{g(a) + g(x)}{2} - g''(\sigma_1)\,\frac{(x-a)^2}{12}$$

for some $\sigma_1 \in (a, x)$, and

$$G(x, b) = \frac{1}{b-x}\int_x^b g(t)\,dt = \frac{g(x) + g(b)}{2} - g''(\sigma_2)\,\frac{(b-x)^2}{12}$$

for some $\sigma_2 \in (x, b)$. Therefore, by Theorem 5 we have

$$\begin{aligned}
&\int_a^b f(t)\,g'(t)\,dt \\
&= [G(a, x) - g(a)]\,f(a) + [G(x, b) - G(a, x)]\,f(x) + [g(b) - G(x, b)]\,f(b) \\
&\quad - \frac{1}{12}\left[f''(\xi_1)\,g'(\eta_1)\,(x-a)^3 + f''(\xi_2)\,g'(\eta_2)\,(b-x)^3\right] \\
&= \left[\frac{g(a) + g(x)}{2} - g''(\sigma_1)\,\frac{(x-a)^2}{12} - g(a)\right] f(a) \\
&\quad + \left[\frac{g(x) + g(b)}{2} - g''(\sigma_2)\,\frac{(b-x)^2}{12} - \frac{g(a) + g(x)}{2} + g''(\sigma_1)\,\frac{(x-a)^2}{12}\right] f(x) \\
&\quad + \left[g(b) - \frac{g(x) + g(b)}{2} + g''(\sigma_2)\,\frac{(b-x)^2}{12}\right] f(b) \\
&\quad - \frac{1}{12}\left[f''(\xi_1)\,g'(\eta_1)\,(x-a)^3 + f''(\xi_2)\,g'(\eta_2)\,(b-x)^3\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{g(x)-g(a)}{2} f(a) + \frac{g(b)-g(a)}{2} f(x) + \frac{g(b)-g(x)}{2} f(b) \\
&\quad + g''(\sigma_1)\frac{(x-a)^2}{12}[f(x)-f(a)] + g''(\sigma_2)\frac{(b-x)^2}{12}[f(b)-f(x)] \\
&\quad - \frac{1}{12}\left[f''(\xi_1)g'(\eta_1)(x-a)^3 + f''(\xi_2)g'(\eta_2)(b-x)^3\right] \\
&= \frac{g(x)-g(a)}{2} f(a) + \frac{g(b)-g(a)}{2} f(x) + \frac{g(b)-g(x)}{2} f(b) \\
&\quad + f'(\tau_1)g''(\sigma_1)\frac{(x-a)^2}{12} + f'(\tau_2)g''(\sigma_2)\frac{(b-x)^2}{12} \\
&\quad - \frac{1}{12}\left[f''(\xi_1)g'(\eta_1)(x-a)^3 + f''(\xi_2)g'(\eta_2)(b-x)^3\right]
\end{aligned}
$$

which follows by the Mean Value Theorem, for some $\tau_1 \in (a,x)$ and $\tau_2 \in (x,b)$. □

Corollary 4. *In Theorem 6, choose $g(t)=t$, to get*

$$
\begin{aligned}
\int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] \qquad \\
= -\frac{1}{12}\left[f''(\xi_1)(x-a)^3 + f''(\xi_2)(b-x)^3\right],
\end{aligned} \tag{3.9}
$$

for all $x \in (a,b)$.

3.2. **B.** Consider the quadrature rule

$$
\mathcal{L}(f,g;x) := \int_a^b f(t)g'(t)\,dt - [g(x)-g(a)]f(c_1) - [g(b)-g(x)]f(c_2) \tag{3.10}
$$

where, $c_1 \in [a,x]$ and $c_2 \in [x,b]$, are given by

$$
c_1 = \frac{xg(x)-ag(a)-\int_a^x g(t)\,dt}{g(x)-g(a)} \quad \text{and} \quad c_2 = \frac{bg(b)-xg(x)-\int_x^b g(t)\,dt}{g(b)-g(x)},
$$

for all $a<x<b$.

Theorem 7. *Suppose that f'' and g' are continuous on $[a,b]$ and that g is monotonic on $[a,x]$ and $[x,b]$. Then there exist $\xi_1,\eta_1 \in (a,x)$ and $\xi_2,\eta_2 \in (x,b)$ such that*

$$
\mathcal{L}(f,g;x) = -\frac{1}{12}\left[f''(\xi_1)g'(\eta_1)(x-a)^3 + f''(\xi_2)g'(\eta_2)(b-x)^3\right], \tag{3.11}
$$

for all $a<x<b$.

Proof. Fix $x \in (a,b)$, so that

$$
\int_a^b f(t)g'(t)\,dt = \int_a^x f(t)g'(t)\,dt + \int_x^b f(t)g'(t)\,dt. \tag{3.12}
$$

For the first integral on the right hand side of (3.12), consider the functions

$$
h_1(t) = \begin{cases} g(t)-g(a), & t \in [a,c_1] \\ g(t)-g(x), & t \in (c_1,x] \end{cases}
$$

and therefore we observe that for $H_1(t) = \int_a^t h(u)du$, where $t \in [a,x]$, we have $H_1(a) = H_1(x) = 0$; as we pointed out previously.

$$\int_a^x f(t)\, g'(t)\, dt - f(c_1)[g(x) - g(a)] = \int_a^x H_1(t)\, f''(t)\, dt$$

and since g is monotonic on $[a,x]$ then H_1 does not change sign on $[a,x]$. So by the First Mean Value Theorem for integrals, there is $\xi_1 \in (a,x)$ such that

$$\int_a^x f(t)\, g'(t)\, dt - f(c_1)[g(x) - g(a)] = f''(\xi_1) \int_a^x H_1(t)\, dt.$$

Applying the classical Trapezoid Rule to $\int_a^x H_1(t)dt$, on the right side above

$$\begin{aligned}
&\int_a^x f(t)\, g'(t)\, dt - f(c_1)[g(x) - g(a)] \\
&= f''(\xi_1) \int_a^x H_1(t)\, dt \\
&= f''(\xi_1) \left[\frac{H_1(a) + H_1(x)}{2}(x-a) - H_1''(\eta_1) \frac{(x-a)^3}{12} \right]
\end{aligned}$$

for some $\eta_1 \in (a,x)$. Since $H_1'' = g'$ and $H_1(a) = H_1(x) = 0$, then

$$\int_a^x f(t)\, g'(t)\, dt - f(c_1)[g(x) - g(a)] = -f''(\xi_1)\, g'(\eta_1) \frac{(x-a)^3}{12}. \tag{3.13}$$

Now, for the second integral on the right hand side of (3.12), consider the functions

$$h_2(t) = \begin{cases} g(t) - g(x), & t \in [x, c_2] \\ g(t) - g(b), & t \in (c_2, b] \end{cases}$$

and therefore we observe that for $H_2(t) = \int_a^t h_2(u)du$, where $t \in [x,b]$, we have $H_2(x) = H_2(b) = 0$. By repeating the above argument we get that

$$\int_x^b f(t)\, g'(t)\, dt - f(c_2)[g(b) - g(x)] = -f''(\xi_2)\, g'(\eta_2) \frac{(b-x)^3}{12}, \tag{3.14}$$

and thus by adding (3.13) and (3.14), the right-hand side of (3.11) is proved. □

In particular, for

$$\begin{aligned}
\mathcal{L}\left(f, g; \frac{a+b}{2}\right) &:= \int_a^b f(t)\, g'(t)\, dt \\
&\quad - \left[g\left(\frac{a+b}{2}\right) - g(a) \right] f(c_1) - \left[g(b) - g\left(\frac{a+b}{2}\right) \right] f(c_2),
\end{aligned}$$

so that, we have the following bound:

$$\mathcal{L}\left(f, g; \frac{a+b}{2}\right) = -\frac{(b-a)^3}{96}\left[f''(\xi_1)\, g'(\eta_1) + f''(\xi_2)\, g'(\eta_2)\right]. \tag{3.15}$$

where, $c_1, \xi_1, \eta_1 \in \left[a, \frac{a+b}{2}\right]$ and $c_2, \xi_2, \eta_2 \in \left[\frac{a+b}{2}, b\right]$.

Corollary 5. *In Theorem 7, let $g(t) = t$, for all $t \in [a, b]$, then we have:*

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] = -\frac{(b-a)^3}{96}\left[f''(\xi_1) + f''(\xi_2)\right]. \tag{3.16}$$

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