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NEW INEQUALITIES OF HERMITE-HADAMARD AND FEJÉR TYPE INEQUALITIES VIA PREINVEXITY

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Abstract. Several new weighted inequalities connected with Hermite-Hadamard and Fejér type inequalities are established for functions whose derivatives in absolute value are preinvex. The results presented in this paper provide extensions of those given in earlier works.

1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as (see [23]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

In [7], Fejér gave a weighted generalization of (1.1) as follows:

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx, \tag{1.2}$$

where $f : [a, b] \to \mathbb{R}$ be a convex function and $f : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$.

For several results which generalize, improve and extend the inequalities (1.1) and (1.2) we refer the interested reader [5, 6, 8], [10]-[13], [23, 24], [26]-[31].

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [9], Ben-Israel and Mond [4], Pini [21], M.A.Noor [18, 19], Yang and Li [33] and Weir [32]. Mond [5], Weir [32] and Noor [18, 19], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [9], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4], gave the concept of preinvex function which is special case of invexity. Pini [21], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning invexity and preinvexity

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Let K be a closed set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1. [32] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (1 - t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [32].

In the recent paper, Noor [16] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1. [16] *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.3}$$

For several new results on inequalities for preinvex functions we refer the interested reader to [3, 15, 20, 25] and the references therein.

In the present paper we give new inequalities of Hermite-Hadamard and Fejér for functions whose derivatives are preinvex. Our results generalize those results presented in a very recent paper of M. Z. Sarikaya [25, 28].

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $a < a + \eta(b, a)$ Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $w : [a, a + \eta(b, a)] \to [0, \infty)$ is an integrable mapping, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following equality holds:*

$$\begin{aligned} \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, w(x) dx - \frac{1}{\eta(b, a)} f\left(a + \frac{1}{2}\eta(b, a)\right) \int_a^{a+\eta(b,a)} w(x) dx \\ = \eta(b, a) \int_0^1 k(t) f'(a + t\eta(b, a))\, dt, \end{aligned} \tag{2.1}$$

where

$$k(t) = \begin{cases} \int_0^t w(a + s\eta(b, a))\, ds, & t \in \left[0, \frac{1}{2}\right) \\ -\int_t^1 w(a + s\eta(b, a))\, ds, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Proof. We observe that

$$
\begin{aligned}
I &= \int_0^1 k(t) f^{'}\left(a+t\eta\left(b,a\right)\right) dt \qquad (2.2)\\
&= \int_0^{\frac{1}{2}} \left(\int_0^t w\left(a+s\eta\left(b,a\right)\right) ds\right) f^{'}\left(a+t\eta\left(b,a\right)\right) dt \\
&\quad + \int_{\frac{1}{2}}^1 \left(-\int_t^1 w\left(a+s\eta\left(b,a\right)\right) ds\right) f^{'}\left(a+t\eta\left(b,a\right)\right) dt = I_1 + I_2.
\end{aligned}
$$

By integration by parts, we get

$$
\begin{aligned}
I_1 &= \left(\int_0^t w\left(a+s\eta\left(b,a\right)\right) ds\right) \left.\frac{f\left(a+t\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\right|_0^{\frac{1}{2}} \qquad (2.3)\\
&\quad - \frac{1}{\eta\left(b,a\right)} \int_0^{\frac{1}{2}} w\left(a+t\eta\left(b,a\right)\right) f\left(a+t\eta\left(b,a\right)\right) dt \\
&= \frac{f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)}{\eta\left(b,a\right)} \int_0^{\frac{1}{2}} w\left(a+t\eta\left(b,a\right)\right) dt \\
&\quad - \frac{1}{\eta\left(b,a\right)} \int_0^{\frac{1}{2}} w\left(a+t\eta\left(b,a\right)\right) f\left(a+t\eta\left(b,a\right)\right) dt.
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
I_2 &= \left(-\int_t^1 w\left(a+s\eta\left(b,a\right)\right) ds\right) \left.\frac{f\left(a+t\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\right|_{\frac{1}{2}}^{1} \qquad (2.4)\\
&\quad - \frac{1}{\eta\left(b,a\right)} \int_{\frac{1}{2}}^1 w\left(a+t\eta\left(b,a\right)\right) f\left(a+t\eta\left(b,a\right)\right) dt \\
&= \frac{f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)}{\eta\left(b,a\right)} \int_{\frac{1}{2}}^1 w\left(a+t\eta\left(b,a\right)\right) dt \\
&\quad - \frac{1}{\eta\left(b,a\right)} \int_{\frac{1}{2}}^1 w\left(a+t\eta\left(b,a\right)\right) f\left(a+t\eta\left(b,a\right)\right) dt.
\end{aligned}
$$

From (2.3) and (2.4), we get

$$
\begin{aligned}
I &= \frac{f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)}{\eta\left(b,a\right)} \int_0^1 w\left(a+t\eta\left(b,a\right)\right) dt \\
&\quad - \frac{1}{\eta\left(b,a\right)} \int_0^1 w\left(a+t\eta\left(b,a\right)\right) f\left(a+t\eta\left(b,a\right)\right) dt.
\end{aligned}
$$

Using the change of variable $x = a + t\eta\left(b,a\right)$ for $t \in [0,1]$ and multiplying both sides by $\eta\left(b,a\right)$, we get (2.1). This completes the proof of the lemma. $\square$

Remark 1. *If we take $w(x) = 1$, $x \in [a, a + t\eta(b,a)]$ in Lemma 1, then (2.1) reduces to*

$$\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx + f\left(a + \frac{1}{2}\eta(b,a)\right) = \eta(b,a) \int_0^1 k(t) f'(a + t\eta(b,a))\,dt, \tag{2.5}$$

where

$$k(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right) \\ t-1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Which is one of the results from [25].

Remark 2. *If $\eta(b,a) = b - a$ in Lemma 1, then (2.1) becomes Lemma 2.1 from* [28, page 379].

Now using Lemma 1, we prove our results:

Theorem 2. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$ Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b,a)])$ and $w : [a, a + \eta(b,a)] \to [0, \infty)$ is an integrable mapping and symmetric to $a + \frac{1}{2}\eta(b,a)$. If $\left|f'\right|$ is preinvex on K, then for every $a, b \in K$ with $\eta(b,a) \neq 0$ we have the following inequality:*

$$\left|\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x)dx - \frac{1}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)dx\right| \leq \left(\frac{1}{\eta(b,a)} \int_a^{a+\frac{1}{2}\eta(b,a)} [\eta(b,a) - 2(x-a)] w(x)dx\right) \left(\frac{\left|f'(a)\right| + \left|f'(b)\right|}{2}\right). \tag{2.6}$$

Proof. From Lemma 1 and the preinvexity of $\left|f'\right|$ on K, we have

$$\begin{aligned} &\left|\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) w(x)dx - \frac{1}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)dx\right| \\ &\leq \eta(b,a) \int_0^{\frac{1}{2}} \left(\int_0^t w(a + s\eta(b,a))\,ds\right) \left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right] dt \\ &\quad + \eta(b,a) \int_{\frac{1}{2}}^1 \left(\int_t^1 w(a + s\eta(b,a))\,ds\right) \left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right] dt \end{aligned} \tag{2.7}$$

By the change of the order of integration, we have

$$\int_0^{\frac{1}{2}}\int_0^t w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt \tag{2.8}$$
$$=\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dtds$$
$$=\int_0^{\frac{1}{2}} w(a+s\eta(b,a))\left[\left(\frac{(1-s)^2}{2}-\frac{1}{8}\right)\left|f'(a)\right|+\left(\frac{1}{8}-\frac{s^2}{2}\right)\left|f'(b)\right|\right]ds.$$

Using the change of variable $x=a+s\eta(b,a)$ for $s\in[0,1]$, we have from (2.8) that

$$\int_0^{\frac{1}{2}}\int_0^t w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt \tag{2.9}$$
$$=\frac{1}{\eta(b,a)}\left|f'(a)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{2}\left(1-\frac{x-a}{\eta(b,a)}\right)^2-\frac{1}{8}\right)w(x)\,dx$$
$$+\frac{1}{\eta(b,a)}\left|f'(b)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-a}{\eta(b,a)}\right)^2\right)w(x)\,dx.$$

Similarly by change of order of integration and using the fact that w is symmetric to $a+\frac{1}{2}\eta(b,a)$, we obtain

$$\int_{\frac{1}{2}}^1\int_t^1 w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt \tag{2.10}$$
$$=\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w(a+(1-s)\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dtds$$
$$=\frac{1}{\eta(b,a)}\left|f'(a)\right|\int_{\frac{1}{2}}^1\left(\frac{1}{8}-\frac{1}{2}(1-s)^2\right)w(a+(1-s)\eta(b,a))\,ds$$
$$+\frac{1}{\eta(b,a)}\left|f'(b)\right|\int_{\frac{1}{2}}^1\left(\frac{s^2}{2}-\frac{1}{8}\right)w(a+(1-s)\eta(b,a))\,ds$$

By the change of variable $x=a+(1-s)\eta(b,a)$, we get form (2.10) that

$$\int_{\frac{1}{2}}^1\int_t^1 w(a+s\eta(b,a))\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dsdt \tag{2.11}$$
$$=\frac{1}{\eta(b,a)}\left|f'(a)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-a}{\eta(b,a)}\right)^2\right)w(x)\,dx$$
$$+\frac{1}{\eta(b,a)}\left|f'(b)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{2}\left(1-\frac{x-a}{\eta(b,a)}\right)^2-\frac{1}{8}\right)w(x)\,dx.$$

Substituting (2.9) and (2.11) in (2.7) and simplifying, we get the inequality (2.6). This completes the proof of the theorem. □

Corollary 1. *If we take $w(x) = 1$, for $x \in [a, a + \eta(b, a)]$ in Theorem 2, we get*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right) \right| \leq \frac{\eta(b,a)}{8} \left[\left| f'(a) \right| + \left| f'(b) \right| \right]. \tag{2.12}$$

Which is Theorem 5 from [25].

Remark 3. *If $\left|f'\right|$ is convex on $[a, b]$, then $\eta(b, a) = b - a$. Hence from Theorem 2, and using the symmetricity of w about $\frac{a+b}{2}$, we get Theorem 2.3 from* [28, page 380].

Theorem 3. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$ and $w : [a, a + \eta(b, a)] \to [0, \infty)$ is an integrable mapping and symmetric to $a + \frac{1}{2}\eta(b, a)$. If $\left|f'\right|^q$, $q > 1$, is preinvex on K, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have the following inequality:*

$$\begin{aligned} &\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,w(x)dx - \frac{1}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)dx \right| \\ &\leq \eta(b,a) \left(\frac{1}{(\eta(b,a))^2} \int_a^{a+\frac{1}{2}\eta(b,a)} \left[\frac{\eta(b,a)}{2} - (x-a) \right] w^p(x)\,dx \right)^{\frac{1}{p}} \\ &\quad \times \left[\left(\frac{2\left|f'(a)\right|^q + \left|f'(b)\right|^q}{24} \right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q + 2\left|f'(b)\right|^q}{24} \right)^{\frac{1}{q}} \right], \end{aligned} \tag{2.13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and change of order of integration, we get

$$\begin{aligned} &\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,w(x)dx - \frac{1}{\eta(b,a)} f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)dx \right| \\ &\leq \eta(b,a) \int_0^{\frac{1}{2}} \left(\int_0^t w(a + s\eta(b,a))\,ds \right) \left| f'(a + t\eta(b,a)) \right| dt \\ &+ \eta(b,a) \int_{\frac{1}{2}}^1 \left(\int_t^1 w(a + s\eta(b,a))\,ds \right) \left| f'(a + t\eta(b,a)) \right| dt \\ &= \eta(b,a) \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} w(a + s\eta(b,a)) \left| f'(a + t\eta(b,a)) \right| dtds \\ &\quad + \eta(b,a) \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s w(a + s\eta(b,a)) \left| f'(a + t\eta(b,a)) \right| dtds. \end{aligned} \tag{2.14}$$

By the Hölder's inequality, we have

$$\begin{aligned}(2.15)\quad &\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(a+s\eta(b,a))\left|f^{'}(a+t\eta(b,a))\right|dtds\\ &\leq \eta(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(a+s\eta(b,a))\,dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|f^{'}(a+t\eta(b,a))\right|^q dtds\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|f^{'}\right|^q$, $q>1$, is preinvex on K, for every $a,b\in K$ and $t\in[0,1]$ we have

$$\left|f^{'}(a+t\eta(b,a))\right|^q \leq (1-t)\left|f^{'}(a)\right|^q + t\left|f^{'}(b)\right|^q$$

hence by solving elementary integrals and using the substitution $x=a+s\eta(b,a)$, $s\in[0,1]$, we have from (2.15) that

$$\begin{aligned}(2.16)\quad &\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w(a+s\eta(b,a))\left|f^{'}(a+t\eta(b,a))\right|dtds\\ &\leq \eta(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p(a+s\eta(b,a))\,dtds\right)^{\frac{1}{p}}\\ &\times\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left[(1-t)\left|f^{'}(a)\right|^q + t\left|f^{'}(b)\right|^q\right]dtds\right)^{\frac{1}{q}}\\ &= \eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)\,dx\right)^{\frac{1}{p}}\\ &\times\left(\frac{2\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{24}\right)^{\frac{1}{q}}.\end{aligned}$$

Analogously, using the symmetricity of w about $a+\frac{1}{2}\eta(b,a)$, we also have

$$\begin{aligned}(2.17)\quad &\eta(b,a)\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w(a+s\eta(b,a))\left|f^{'}(a+t\eta(b,a))\right|dtds\\ &\leq \eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)\,dx\right)^{\frac{1}{p}}\\ &\times\left(\frac{\left|f^{'}(a)\right|^q+2\left|f^{'}(b)\right|^q}{24}\right)^{\frac{1}{q}}\end{aligned}$$

Using (2.16) and (2.17) in (2.14), we get the required inequality. This completes the proof of the theorem. $\square$

Corollary 2. *If the conditions of Theorem 3 are satisfied and if $w(x) = 1$, $x \in [a, a + \eta(b, a)]$, then the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx + f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \le \eta(b,a)\left(\frac{1}{8}\right)^{\frac{1}{p}} \times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right], \tag{2.18}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 3. [28, Theorem 2.5, page 381]*Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on I°, a, $b \in I$ with $a < b$. Let $w : [a, b] \to [0,\infty)$ is an integrable mapping and symmetric to $\frac{a+b}{2}$ and $f' \in L([a, b])$. If $\left|f'\right|^q$, $q > 1$, is convex on $[a, b]$, then we have the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,w(x)dx-\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx\right| \le (b-a)\left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)w^p(x)\,dx\right)^{\frac{1}{p}} \times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right], \tag{2.19}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It follows from Theorem 3 by taking $\eta(b, a) = b-a$ and using the symmetry of w about $\frac{a+b}{2}$. □

For our next results we need the following Lemma:

Lemma 2. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $a < a + \eta(b, a)$ Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $w : [a, a + \eta(b, a)] \to [0,\infty)$ is an integrable mapping, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following equality holds:*

$$-\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\int_a^{a+\eta(b,a)} w(x)dx+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx = \frac{\eta(b,a)}{2}\int_0^1 p(t)f'(a+t\eta(b,a))\,dt, \tag{2.20}$$

where

$$p(t)=\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+s\eta(b,a))\,ds,\ \ t\in[0,1].$$

Proof. It suffices to note that

$$J=\int_0^1 p(t)f^{'}\left(a+t\eta\left(b,a\right)\right)dt \tag{2.21}$$
$$=-\int_0^1\left(\int_0^t w\left(a+s\eta\left(b,a\right)\right)ds\right)f^{'}\left(a+t\eta\left(b,a\right)\right)dt$$
$$+\int_0^1\left(\int_t^1 w\left(a+s\eta\left(b,a\right)\right)ds\right)f^{'}\left(a+t\eta\left(b,a\right)\right)dt=J_1+J_2$$

By integration by parts, we get

$$J_1=-\left.\frac{\left(\int_0^t w\left(a+s\eta\left(b,a\right)\right)ds\right)f\left(a+t\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\right|_0^1 \tag{2.22}$$
$$+\frac{1}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)f\left(a+t\eta\left(b,a\right)\right)dt$$
$$=-\frac{f\left(a+\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)dt$$
$$+\frac{1}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)f\left(a+t\eta\left(b,a\right)\right)dt.$$

Similarly. we also have

$$J_2=-\frac{f\left(a\right)}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)dt+\frac{1}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)f\left(a+t\eta\left(b,a\right)\right)dt. \tag{2.23}$$

Using (2.22) and (2.23) in (2.21), we obtain

$$J=-\frac{f(a)+f\left(a+\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)dt \tag{2.24}$$
$$+\frac{2}{\eta\left(b,a\right)}\int_0^1 w\left(a+t\eta\left(b,a\right)\right)f\left(a+t\eta\left(b,a\right)\right)dt$$

By the change of variable $x=a+t\eta\left(b,a\right)$ for $t\in[0,1]$ and by multiplying both sides if (2.6) by $\frac{\eta(b,a)}{2}$, we get (2.20). This completes the proof of the lemma. □

Remark 4. *If we take* $w(x)=1$, $x\in[a,a+\eta\left(b,a\right)]$, *then we get*

$$-\frac{f(a)+f(a+\eta\left(b,a\right)}{2}+\frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)}f\left(x\right)dx \tag{2.25}$$
$$=\frac{\eta\left(b,a\right)}{2}\int_0^1\left(1-2t\right)f^{'}\left(a+t\eta\left(b,a\right)\right)dt,$$

which is Lemma 2.1 from [3, Page 3].

Theorem 4. *Let* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$ *and* a, $b\in K$ *with* $a<a+\eta\left(b,a\right)$ *Suppose* $f:K\to\mathbb{R}$ *is a differentiable mapping on* K *such that* $f^{'}\in L\left([a,a+\eta\left(b,a\right)]\right)$ *and* $w:[a,a+\eta\left(b,a\right)]\to[0,\infty)$ *is an*

integrable mapping and symmetric to $a+\frac{1}{2}\eta(b,a)$*. If* $\left|f^{'}\right|^{q}$*,* $q>1$*, is preinvex on* K*, then for every* $a,b\in K$ *with* $\eta(b,a)\neq 0$ *we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)w(x)dx\right| \le \frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f^{'}(a)\right|^{q}+\left|f^{'}(b)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{2.26}$$

where

$$g(t)=\left|\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx\right|, t\in[0,1] \text{ and } \frac{1}{p}+\frac{1}{q}=1.$$

Proof. From Lemma 2, we get

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)w(x)dx\right| \le \frac{\eta(b,a)}{2}\int_0^1\left|\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+s\eta(b,a))\,ds\right|\left|f^{'}(a+t\eta(b,a))\right|dt. \tag{2.27}$$

Since w is symmetric to $a+\frac{1}{2}\eta(b,a)$, we can write

$$\begin{aligned} &\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+s\eta(b,a))\,ds \\ &=\int_t^1 w(a+s\eta(b,a))\,ds-\int_0^t w(a+(1-s)\eta(b,a))\,ds \\ &=\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+\eta(b,a)}w(x)\,dx+\frac{1}{\eta(b,a)}\int_{a+\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx \\ &=\begin{cases} \frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx, & t\in\left[0,\frac{1}{2}\right] \\ -\frac{1}{\eta(b,a)}\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w(x)\,dx, & t\in\left[\frac{1}{2},1\right]. \end{cases} \end{aligned} \tag{2.28}$$

Using (2.28) in (2.27) we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2\eta(b,a)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)w(x)dx\right| \le \frac{1}{2}\int_0^1 g(x)\left|f^{'}(a+t\eta(b,a))\right|dt, \tag{2.29}$$

where

$$g(t)=\left|\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w(x)\,dx\right|, t\in[0,1].$$

By Hölder's inequality, it follows from (2.29) that

$$\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx\right| \tag{2.30}$$
$$\leq\frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{'}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{'}(a+t\eta(b,a))\right|^q$ is preinvex on K, for every a, $b\in K$ and $t\in[0,1]$, we have

$$\left|f^{'}(a+t\eta(b,a))\right|^q\leq(1-t)\left|f^{'}(a)\right|^q+t\left|f^{'}(b)\right|^q$$

and hence from (2.30), we get that

$$\left|\frac{f(a)+f(a+\eta(b,a)}{2\eta(b,a)}\int_a^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,w(x)dx\right| \tag{2.31}$$
$$\leq\frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-t)\left|f^{'}(a)\right|^q+t\left|f^{'}(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

which completes the proof of the theorem. □

Corollary 4. *If we take $\eta(b,a)=b-a$ in Theorem 4, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)\,w(x)dx\right| \tag{2.32}$$
$$\leq\frac{1}{2}\left(\int_0^1 g^p(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

where

$$g(t)=\left|\int_{tb+(1-t)a}^{ta+(1-t)b}w(x)\,dx\right|,\ t\in[0,1]\ \text{ and }\ \frac{1}{p}+\frac{1}{q}=1.$$

Which is Theorem 2.8 from [28, page 383]*.*

Corollary 5. [3, Theorem 2.2, page 4] *Under the assumptions of Theorem 4, if we take $w(x)=1$, $x\in[a,a+\eta(b,a)]$. Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.33}$$
$$\leq\frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It follows from the fact that

$$\int_0^1 g^p(t)dt = \int_0^1 \left(\left| \int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} dx \right|^p \right) dt$$
$$= (\eta(b,a))^p \int_0^1 |1-2t|^p \, dt = \frac{(\eta(b,a))^p}{p+1}.$$

□

Corollary 6. [5] *If the conditions of Theorem 4 are fulfilled and if $w(x) = 1$, $x \in [a,b]$ and $\eta(b,a) = b-a$, then we have the inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2} \right)^{\frac{1}{q}}, \tag{2.34}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. It follows from Corollary 5. □

3. Applications to Special Means

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 2. [31]*A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, is called a Mean function if it has the following properties:*

(1) *Homogeneity: $M(ax, ay) = aM(x,y)$, for all $a > 0$,*
(2) *Symmetry : $M(x,y) = M(y,x)$,*
(3) *Reflexivity : $M(x,x) = x$,*
(4) *Monotonicity: If $x \le x'$ and $y \le y'$, then $M(x,y) \le M(x',y')$,*
(5) *Internality: $\min\{x,y\} \le M(x,y) \le \max\{x,y\}$.*

We consider some means for arbitrary positive real numbers α, β (see for instance [31]).

(1) The arithmetic mean:
$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}$$
(2) The geometric mean:
$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}$$
(3) The harmonic mean:
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
(4) The power mean:
$$P_r := P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \ r \ge 1$$

(5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right], \ \alpha \neq \beta, \ p \in \mathbb{R}\setminus\{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b, a) = M(b, a)$ in (2.12), (2.18) and (2.33), one can obtain the following interesting inequalities involving means:

$$\left|\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}M(b,a)\right)\right| \leq \frac{M(b,a)}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right], \tag{3.1}$$

$$\left|\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx + f\left(a + \frac{1}{2}M(b,a)\right)\right| \leq M(b,a)\left(\frac{1}{8}\right)^{\frac{1}{p}} \times \left[\left(\frac{2\left|f'(a)\right|^q + \left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q + 2\left|f'(b)\right|^q}{24}\right)^{\frac{1}{q}}\right] \tag{3.2}$$

and

$$\left|\frac{f(a) + f(a + M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{M(b,a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{3.3}$$

Letting $M = A$, G, H, P_r, I, L, L_p in (3.1), (3.2) and (3.3), we can get the required inequalities, and the details are left to the interested reader.

References

[1] T. Antczak, Mean value in invexity analysis, Nonl. Anal., 60 (2005), 1473-1484.

[2] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequsi-invex functions, RGMIA Research Report Collection, 14(2011), Article 48, 7 pp.

[3] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, RGMIA Research Report Collection, 14(2011), Article 64, 11 pp.
[4] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.
[5] S. S. Dragomir, and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5)(1998), 91–95.
[6] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl., 167(1992), 42–56.
[7] L. Fejer, Über die Fourierreihen, II. Math. Naturwiss. Anz Ungar. Akad. Wiss., 24(1906), 369–390. (Hungarian).
[8] D. -Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Appl. Math. Comp., 217(23)(2011), 9598-9605.
[9] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.
[10] U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147(1)(2004), 137-146.
[11] U. S. Kırmacı and M. E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153(2)(2004), 361-368.
[12] K. C. Lee and K. L. Tseng, On a weighted generalization of Hadamard's inequality for G-convex functions, Tamsui-Oxford J. Math. Sci., 16(1)(2000), 91–104.
[13] A. Lupas, A generalization of Hadamard's inequality for convex functions, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz., 544-576(1976), 115–121.
[14] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901–908.
[15] M. Matloka, On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates. (Submitted)
[16] M. Aslam Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, Preprint.
[17] M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323–330.
[18] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463–475.
[19] M. A. Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl.,11(2006),165-171
[20] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-14.
[21] R. Pini, Invexity and generalized convexity, Optimization 22 (1991) 513-525.
[22] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2)(2000), 51–55.
[23] F. Qi, Z. -L.Wei and Q. Yang, Generalizations and refinements of Hermite-Hadamard's inequality, Rocky Mountain J. Math., 35(2005), 235–251.
[24] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial ordering and statistical applications, Academic Press, New York, 1991.
[25] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, arXiv:1203.4759v1.
[26] M. Z. Sarıkaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, 54(9-10)(2011), 2175-2182.
[27] M. Z. Sarikaya, M. Avci and H. Kavurmaci, On some inequalities of Hermite-Hadamard type for convex functions, ICMS International Conference on Mathematical Science, AIP Conference Proceedings 1309, 852(2010).
[28] M. Z. Sarikaya, O new Hermite-Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 377-386.
[29] A. Saglam, M. Z. Sarikaya and H. Yıldırım and, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[30] C. -L. Wang and X. -H. Wang, On an extension of Hadamard inequality for convex functions, Chin. Ann. Math., 3(1982), 567–570.

[31] S. -H. Wu , On the weighted generalization of the Hermite-Hadamard inequality and its applications, The Rocky Mountain J. of Math., 39(2009), no. 5, 1741–1749.
[32] T. Weir, and B. Mond, Preinvex functions in multiple bjective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.
[33] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.

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