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SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE HIGHER ORDER PARTIAL DERIVATIVES ARE CO-ORDINATED s-CONVEX

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Abstract. In this paper we point out some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives of higher order are co-ordinated s-convex in the second sense. Our established results generalize the Hermite-Hadamard type inequalities established for co-ordinated s-convex functions and refine those results established for differentiable functions whose partial derivatives of higher order are co-ordinated convex proved in a recent paper [24].

1. Introduction

The following definition is well known in literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1.1}$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$. The inequality (1.1) holds in reverse direction if f is concave.

The most famous inequality concerning the class of convex functions, is the Hermite-Hadamard's inequality.

This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, a, $b \in I$ with $a < b$. The inequalities in (1.2) are in reversed order if f a concave function.

The inequalities (1.2) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.2), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [8, 14, 19, 29, 32, 33] and the references therein.

In the paper [15], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined follows:

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A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.3}$$

holds for all $x, y \in [0, \infty), \lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [9], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1. [9]*Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$ and $a, b \in [0, \infty)$, $a < b$. If $f \in L^1[a, b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4).

For more about properties and Hermite-Hadamard type inequalities of s-convex functions in the second sense we refer the interested readers to [7, 9, 12, 15, 20].

Let us consider now a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda) f(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A modification for convex functions on Δ, known as co-ordinated convex functions, was introduced by S. S. Dragomir [10] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b], y \in [c, d]$.

A formal definition for co-ordinated convex functions may be stated as follow:

Definition 1. [21] *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality*

$$\begin{aligned} & f(tx + (1-t)y, ru + (1-r)w) \\ & \leq tr f(x, u) + t(1-r) f(x, w) + r(1-t) f(y, u) + (1-t)(1-r) f(y, w) \end{aligned}$$

holds for all $t, r \in [0, 1]$ and $(x, u), (y, w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates but converse may not be true [10].

The following Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ were established in [10]:

Theorem 2. [10] *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, then*

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right]\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \quad (1.5)
\end{aligned}$$

The above inequalities are sharp.

The concept of s-convex functions on the co-ordinates in the second sense was introduced by Alomari and Darus in [3] as a generalization of the usual co-ordinated convexity:

Definition 2. [3] *Consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\to\mathbb{R}$ is s-convex in the second sense on Δ if*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda^s f(x,y)+(1-\lambda)^s f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$, $\lambda\in[0,1]$ with some fixed $s\in(0,1]$.

A function $f:\Delta\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are s-convex in the second sense for all $y\in[c,d]$, $x\in[a,b]$ and $s\in(0,1]$, i.e., the partial mappings f_y and f_x are s-convex in the second sense with some fixed $s\in(0,1]$.

A formal definition of co-ordinated s-convex function in second sense may be stated as follows:

Definition 3. *A function $f:\Delta\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if*

$$\begin{aligned}
&f(tx+(1-t)y,ru+(1-r)w)\\
&\leq t^sr^sf(x,u)+t^s(1-r)^sf(x,w)+r^s(1-t)^sf(y,u)+(1-t)^s(1-r)^sf(y,w) \quad (1.6)
\end{aligned}$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,u),(x,w),(y,w)\in\Delta$, for some fixed $s\in(0,1]$. The mapping f is concave on the co-ordinates on Δ if the inequality (1.6) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$ with some fixed $s\in(0,1]$.

Furthermore, Alomari and Darus [5] introduced a new class of s-convex functions on the co-ordinates on the rectangle from the plane as follows:

Definition 4. [5] *Consider the bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\to\mathbb{R}$ is s-convex in the second sense on Δ if there exist $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$ such that*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda^{s_1}f(x,y)+(1-\lambda)^{s_2}f(z,w) \quad (1.7)$$

holds for all $(x,y),(z,w)\in\Delta$, $\lambda\in[0,1]$. This class of functions is denoted by $MWO^2_{s_1,s_2}$.

A function $f:\Delta\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are s_1-convex and s_2-convex in the second sense for all $y\in[c,d]$, $x\in[a,b]$ and $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$, respectively, i.e., the partial mappings f_y and f_x are s_1-convex and s_2-convex in the second sense, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$.

The definition 3 can be generalized as follows:

Definition 5. *A function $f:\Delta=:[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if*

$$\begin{aligned}&f(tx+(1-t)y,ru+(1-r)w)\\&\le t^{s_1}r^{s_2}f(x,u)+t^{s_1}(1-r)^{s_2}f(x,w)+r^{s_2}(1-t)^{s_1}f(y,u)+(1-t)^{s_1}(1-r)^{s_2}f(y,w)\end{aligned}\tag{1.8}$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,u),(x,w),(y,w)\in\Delta$, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$. The mapping f is concave on the co-ordinates on Δ if the inequality (1.8) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$.

In [5], Alomari et al. also proved a variant of inequalities given above by (1.5) for s-convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$:

Theorem 3. [5] *Suppose $f:\Delta\subseteq[0,\infty)^2\to[0,\infty)$ is s-convex function in the second sense on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned}&\left(\frac{4^{s_1-1}+4^{s_2-1}}{2}\right)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\qquad\le\frac{2^{s_1-2}}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{2^{s_2-2}}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\\&\qquad\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\&\qquad\le\frac{1}{2(s_1+1)(b-a)}\int_a^b\left[f(x,c)+f(x,d)\right]dx\\&\qquad\quad+\frac{1}{2(s_2+1)(d-c)}\int_c^d\left[f(a,y)+f(b,y)\right]dy\\&\le\frac{1}{2}\left(\frac{1}{(s_1+1)^2}+\frac{1}{(s_2+1)^2}\right)\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right].\end{aligned}\tag{1.9}$$

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1, 3, 4, 5, 6], [10], [13], [21]-[24], [25]-[28] and [31]. Alomari et al. [1, 3, 4, 5, 6], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions and co-ordinated log-convex functions. Dragomir [10], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [13], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated to the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al

[12]-[14], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, differentiable functions whose higher order partial derivatives are co-ordinated convex , product of two co-ordinated convex mappings and for co-ordinated h-convex mappings. Özdemir et. al [25]-[28], proved Hadamard's type inequalities for co-ordinated convex functions, co-ordinated s-convex functions and co-ordinated m-convex and (α, m)-convex functions.

The main aim of this paper is to establish some new Hermite-Hadamard type inequalities for differentiable functions whose partial derivatives of higher order are co-ordinated s-convex in the second sense on the rectangle from the plane $\mathbb{R}^2$ which generalize the Hermite-Hadamard type inequalities proved for co-ordinated s-convex functions in the second sense and refine those results established for differentiable functions whose partial derivatives of higher order are co-ordinated convex on the rectangle from the plane $\mathbb{R}^2$ (see [24]).

2. Main Results

In this section we establish new Hermite-Hadamard type inequalities for double integrals of functions whose partial derivatives of higher order are co-ordinated s-convex in the second sense.

To make the presentation easier and compact to understand, we make some symbolic representations as follows:

$$
\begin{aligned}
A^{'} = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f(x,c)+f(x,d)\right]dx + \frac{1}{d-c}\int_c^d \left[f(a,y)+f(b,y)\right]dy\right] \\
+\frac{1}{2}\sum_{l=2}^{m-1}\frac{(l-1)(d-c)^l}{2(l+1)!}\left[\frac{\partial^l f(a,c)}{\partial y^l}+\frac{\partial^l f(b,c)}{\partial y^l}\right] \\
+\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\left[\frac{\partial^k f(a,c)}{\partial x^k}+\frac{\partial^k f(a,d)}{\partial x^k}\right] \\
-\frac{1}{b-a}\sum_{l=2}^{m-1}\frac{(l-1)(d-c)^l}{2(l+1)!}\int_a^b \frac{\partial^l f(x,c)}{\partial y^l}dx \\
-\frac{1}{d-c}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}\int_c^d \frac{\partial^k f(a,y)}{\partial x^k}dy \\
-\sum_{k=2}^{n-1}\sum_{l=2}^{m-1}\frac{(k-1)(l-1)(b-a)^k(d-c)^l}{4(k+1)!(l+1)!}\frac{\partial^{k+l} f(a,c)}{\partial x^k y^l},
\end{aligned}
$$

$$
B_{(n,m)} = \left|\frac{\partial^{n+m} f(a,c)}{\partial t^n \partial r^m}\right|, \quad C_{(n,m)} = \left|\frac{\partial^{n+m} f(a,d)}{\partial t^n \partial r^m}\right|,
$$

$$
D_{(n,m)} = \left|\frac{\partial^{n+m} f(b,c)}{\partial t^n \partial r^m}\right|, \quad E_{(n,m)} = \left|\frac{\partial^{n+m} f(b,d)}{\partial t^n \partial r^m}\right|,
$$

$$
F_{(n,m)} = \left|\frac{\partial^{n+m} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t^n \partial r^m}\right|, G_{(n,m)} = \left|\frac{\partial^{n+m} f\left(a,\frac{c+d}{2}\right)}{\partial t^n \partial r^m}\right|,
$$

$$
H_{(n,m)} = \left|\frac{\partial^{n+m} f\left(\frac{a+b}{2},c\right)}{\partial t^n \partial r^m}\right|, J_{(n,m)} = \left|\frac{\partial^{n+m} f\left(\frac{a+b}{2},d\right)}{\partial t^n \partial r^m}\right|,
$$

$$I_{(n,m)}=\left|\frac{\partial^{n+m}f\left(b,\frac{c+d}{2}\right)}{\partial t^{n}\partial r^{m}}\right|,$$

where the sums above take 0, when $m=n=1$ and $m=n=2$ and hence

$$A^{'}=A=\frac{1}{2}\left[\frac{1}{b-a}\int_{a}^{b}\left[f\left(x,c\right)+f\left(x,d\right)\right]dx+\frac{1}{d-c}\int_{c}^{d}\left[f\left(a,y\right)+f\left(b,y\right)\right]dy\right].$$

.

In what follows Δ° is the interior of $\Delta=[a,b]\times[c,d]$ and $L\left(\Delta\right)$ is the space of integrable functions over Δ.

The following two results will be very useful in the sequel of the paper:

Theorem 4. [18] *Let $f:\Delta\rightarrow\mathbb{R}$ be a continuous mapping such that the partial derivatives $\frac{\partial^{k+l}f(.,.)}{\partial x^{k}\partial y^{l}}$, $k=0,1,\ldots,n-1,$ $l=0,1,\ldots,m-1$ exist on Δ° and are continuous on Δ, then*

$$\begin{aligned}\int_{a}^{b}\int_{c}^{d}f\left(t,r\right)drdt=&\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}X_{k}\left(x\right)Y_{l}\left(y\right)\frac{\partial^{k+l}f\left(x,y\right)}{\partial x^{k}\partial y^{l}}+\\&\left(-1\right)^{m}\sum_{k=0}^{n-1}X_{k}\left(x\right)\int_{c}^{d}S_{m}\left(y,r\right)\frac{\partial^{k+m}f\left(x,r\right)}{\partial x^{k}\partial r^{m}}dr\\&+\left(-1\right)^{n}\sum_{l=0}^{m-1}Y_{l}\left(y\right)\int_{a}^{b}K_{n}\left(x,t\right)\frac{\partial^{n+l}f\left(t,y\right)}{\partial t^{n}\partial y^{l}}dt\\&+\left(-1\right)^{m+n}\int_{a}^{b}\int_{c}^{d}K_{n}\left(x,t\right)S_{m}\left(y,r\right)\frac{\partial^{n+m}f\left(t,r\right)}{\partial t^{n}\partial r^{m}}drdt,\end{aligned}$$

where

$$\left\{\begin{array}{l}K_{n}\left(x,t\right):=\left\{\begin{array}{l}\frac{(t-a)^{n}}{n!},t\in[a,x]\\\frac{(t-b)^{n}}{n!},t\in(x,b]\end{array}\right.\\\\S_{m}\left(y,r\right):=\left\{\begin{array}{l}\frac{(r-c)^{m}}{m!},r\in[c,y]\\\frac{(r-d)^{m}}{m!},r\in(y,d]\end{array}\right.\end{array}\right.\quad\text{and}\quad\left\{\begin{array}{l}X_{k}\left(x\right):=\frac{(b-x)^{k+1}+(-1)^{k}(x-a)^{k+1}}{(k+1)!}\\\\Y_{l}\left(y\right):=\frac{(d-y)^{l+1}+(-1)^{l}(y-c)^{l+1}}{(l+1)!}\end{array}\right.,$$

for $(x,y)\in\Delta$.

Lemma 1. [24] *Let $f:\Delta\rightarrow\mathbb{R}$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial x^{n}\partial y^{m}}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial x^{n}\partial y^{m}}\in L\left(\Delta\right)$ for $m,n\geq1$, then*

$$\begin{aligned}\frac{\left(b-a\right)^{n}\left(d-c\right)^{m}}{4n!m!}&\int_{0}^{1}\int_{0}^{1}t^{n-1}r^{m-1}\left(n-2t\right)\left(m-2r\right)\\&\times\frac{\partial^{n+m}f\left(ta+\left(1-t\right)b,cr+\left(1-r\right)d\right)}{\partial t^{n}\partial r^{m}}dtdr+A^{'}\\&=\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\\&\qquad+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}f\left(x,y\right)dydx.\quad(2.1)\end{aligned}$$

Now we prove our main results.

Theorem 5. *Let $f:\Delta\subseteq[0,\infty)^2\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial r^m}\right|$ is s-convex on the co-ordinates on Δ in the second sense, for m, $n\in\mathbb{N}$, m, $n\geq 2$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\qquad\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|\\&\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\left[LB_{(n,m)}+MC_{(n,m)}+ND_{(n,m)}+RE_{(n,m)}\right],\end{aligned}\tag{2.2}$$

where

$$L=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right]\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right],$$

$$M=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right]\left[mB(m,s_2+1)-2B(m+1,s_2+1)\right],$$

$$N=\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right]\left[nB(n,s_1+1)-2B(n+1,s_1+1)\right],$$

$$R=\left[nB(n,s_1+1)-2B(n+1,s_1+1)\right]\left[mB(m,s_2+1)-2B(m+1,s_2+1)\right],$$

$s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$ and

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

is the Euler Beta function.

Proof. Suppose $m,n\geq 2$. By Lemma 1, we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\qquad\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|\\&\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\\&\qquad\times\left|\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}\right|dtdr.\end{aligned}\tag{2.3}$$

By s-convexity of $\left|\frac{\partial^{m+n}f}{\partial t^n \partial s^m}\right|$ on the co-ordinates on Δ, we get that

$$\begin{aligned}
&\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r) \\
&\qquad\times\left|\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}\right|dtdr \\
&\le B_{(n,m)}\int_0^1\int_0^1 t^{n+s_1-1}r^{m+s_2-1}(n-2t)(m-2r)\,drdt \\
&+C_{(n,m)}\int_0^1\int_0^1 t^{n+s_1-1}r^{m-1}(1-r)^{s_2}(n-2t)(m-2r)\,drdt \\
&+E_{(n,m)}\int_0^1\int_0^1 t^{n-1}(1-t)^{s_1}(n-2t)\,r^{m-1}(1-r)^{s_2}(m-2r)\,drdt \\
&+D_{(n,m)}\int_0^1\int_0^1 t^{n-1}r^{m+s_2-1}(1-t)^{s_1}(n-2t)(m-2r)\,drdt\Big]. \qquad (2.4)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^1\int_0^1 t^{n+s_1-1}r^{m+s_2-1}(n-2t)(m-2r)\,drdt \\
&=\int_0^1 t^{n+s_1-1}(n-2t)\,dt\int_0^1 r^{m+s_2-1}(m-2r)\,dr \\
&=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right]\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right]. \qquad (2.5)
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\int_0^1\int_0^1 t^{n+s_1-1}r^{m-1}(1-r)^{s_2}(n-2t)(m-2r)\,drdt \\
&=\left[\frac{n(n-1)+s_1(n-2)}{(n+s_1)(n+s_1+1)}\right][mB(m,s_2+1)-2B(m+1,s_2+1)], \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1 t^{n-1}r^{m+s_2-1}(1-t)^{s_1}(n-2t)(m-2r)\,drdt \\
&=\left[\frac{m(m-1)+s_2(m-2)}{(m+s_2)(m+s_2+1)}\right][nB(n,s_1+1)-2B(n+1,s_1+1)] \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 t^{n-1}(1-t)^{s_1}(n-2t)\,r^{m-1}(1-r)^{s_2}(m-2r)\,drdt \\
&=[nB(n,s_1+1)-2B(n+1,s_1+1)][mB(m,s_2+1)-2B(m+1,s_2+1)]. \qquad (2.8)
\end{aligned}$$

From (2.4)-(2.8) in (2.3), we get the required inequality. This completes the proof of the theorem. □

Theorem 6. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial r^m}\right|^q$, $q\ge 1$,*

is s-convex on the co-ordinates on Δ, $m,n\in\mathbb{N}$, $m,n\geq 2$, *then*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|$$
$$\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\left(\frac{(n-1)(m-1)}{(n+1)(m+1)}\right)^{1-1/q}$$
$$\times\sqrt[q]{LB^q_{(n,m)}+MD^q_{(n,m)}+NC^q_{(n,m)}+RE^q_{(n,m)}},\quad(2.9)$$

where $s_1,s_2\in(0,1]$ *with* $s=\frac{s_1+s_2}{2}$ *and* L, M, N, R *and* $B(x,y)$ *are as defined in Theorem 5.*

Proof. The case $q=1$ is the Theorem 5. Suppose $q>1$, then by Lemma 1 and the power mean inequality, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|$$
$$\leq\frac{(b-a)^n(d-c)^m}{4n!m!}\left\{\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\,drdt\right\}^{1-1/q}$$
$$\times\left\{\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\right.$$
$$\left.\times\left|\frac{\partial^{n+m}f(ta+(1-t)b,cr+(1-r)d)}{\partial t^n\partial r^m}\right|^q dtdr\right\}^{1/q}.\quad(2.10)$$

By the similar arguments used to obtain (2.2) and the fact

$$\int_0^1\int_0^1 t^{n-1}r^{m-1}(n-2t)(m-2r)\,drdt=\frac{(n-1)(m-1)}{(n+1)(m+1)},$$

we get (2.9). This completes the proof of the theorem. □

Theorem 7. *Let* $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, *be a continuous mapping such that* $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ *exist on* Δ° *and* $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. *If* $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, $q\geq 1$, *is s-convex on the co-ordinates on* Δ, $s_1,s_2\in(0,1]$ *with* $s=\frac{s_1+s_2}{2}$, m, $n\in\mathbb{N}$,

$m,n\geq 1$. Then

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt \\
&\quad-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l} \\
&\quad+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr \\
&\quad+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\Bigg| \\
&\leq\frac{1}{4n!m!}\left(\frac{4}{(n+1)(m+1)}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{2}\right)^n\left(\frac{d-c}{2}\right)^m \\
&\times\Bigg[\left(B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)}\right)B(n+1,s_1+1)\,B(m+1,s_2+1) \\
&\quad+\frac{2\left(G^q_{(n,m)}+I^q_{(n,m)}\right)B(n+1,s_1+1)}{m+s_2+1} \\
&+\frac{2\left(H^q_{(n,m)}+J^q_{(n,m)}\right)B(m+1,s_2+1)}{n+s_1+1}+\frac{4F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}\Bigg]^{\frac{1}{q}},
\end{aligned}\tag{2.11}$$

where

$$P(t):=\begin{cases}(t-a)^n, & t\in\left[a,\frac{a+b}{2}\right]\\(t-b)^n, & t\in\left(\frac{a+b}{2},b\right]\end{cases}\quad\text{and}\quad Q(r):=\begin{cases}(r-c)^m, & r\in\left[c,\frac{c+d}{2}\right]\\(r-d)^m, & r\in\left(\frac{c+d}{2},d\right]\end{cases}.$$

Proof. By letting $x\mapsto\frac{a+b}{2}$ and $y\mapsto\frac{c+d}{2}$ in Theorem 4 and using the properties of the absolute value, we obtain

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt \\
&\quad-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l} \\
&\quad+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr \\
&\quad+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\Bigg| \\
&\leq\frac{1}{(b-a)(d-c)\,m!n!}\int_a^b\int_c^d|P(t)|\,|Q(r)|\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|drdt.
\end{aligned}\tag{2.12}$$

By the power mean inequality for double integrals, we have

$$
\begin{aligned}
&\int_a^b \int_c^d |P(t)|\,|Q(r)| \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right| drdt \\
&\leq \left(\int_a^b \int_c^d |P(t)|\,|Q(r)|\, drdt\right)^{1-\frac{1}{q}} \left(\int_a^b \int_c^d |P(t)|\,|Q(r)| \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt\right)^{\frac{1}{q}} \\
&= \left(\int_a^b \int_c^d |P(t)|\,|Q(r)|\, drdt\right)^{1-\frac{1}{q}} \left[\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^n (r-c)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt\right. \\
&\qquad + \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} (b-t)^n (r-c)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt \\
&\qquad + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d (t-a)^n (d-r)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt \\
&\qquad \left. + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d (b-t)^n (d-r)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt\right]^{\frac{1}{q}}. \qquad (2.13)
\end{aligned}
$$

Now we calculate each integral in (2.13). Since

$$
t = \left(\frac{\frac{a+b}{2} - t}{\frac{a+b}{2} - a}\right) a + \left(\frac{t-a}{\frac{a+b}{2} - a}\right) \frac{a+b}{2}
$$

and

$$
r = \left(\frac{\frac{c+d}{2} - r}{\frac{c+d}{2} - c}\right) c + \left(\frac{r-c}{\frac{c+d}{2} - c}\right) \frac{c+d}{2}.
$$

By the co-ordinated s-convexity of $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|^q$, we have

$$
\begin{aligned}
&\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^n (r-c)^m \left|\frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m}\right|^q drdt \leq \left(\frac{2}{b-a}\right)^{s_1} \left(\frac{2}{d-c}\right)^{s_2} \\
&\times \left[B_{(n,m)}^q \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^n (r-c)^m \left(\frac{a+b}{2} - t\right)^{s_1} \left(\frac{c+d}{2} - r\right)^{s_2} drdt\right. \\
&\qquad + G_{(n,m)}^q \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^n \left(\frac{a+b}{2} - t\right)^{s_1} (r-c)^{s_2+m} drdt \\
&\qquad + H_{(n,m)}^q \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^{s_1+n} \left(\frac{c+d}{2} - r\right)^{s_2} (r-c)^m drdt \\
&\qquad \left. + F_{(n,m)}^q \int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^{s_1+n} (r-c)^{s_2+m} drdt\right]. \qquad (2.14)
\end{aligned}
$$

Now by the change of variables $u=t-a$, $v=r-c$ and then by the change of variables $x=\frac{2u}{b-a}$, $y=\frac{2v}{d-c}$, we get that

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\\
&\qquad\times\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left(\frac{a+b}{2}-t\right)^{s_1}\left(\frac{c+d}{2}-r\right)^{s_2}drdt\\
&=\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_0^{\frac{b-a}{2}}u^n\left(\frac{b-a}{2}-u\right)^{s_1}du\int_0^{\frac{d-c}{2}}v^m\left(\frac{d-c}{2}-v\right)^{s_2}dv\\
&=\int_0^{\frac{b-a}{2}}u^n\left(1-\frac{2u}{b-a}\right)^{s_1}du\int_0^{\frac{d-c}{2}}v^m\left(1-\frac{2v}{d-c}\right)^{s_2}dv\\
&=\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\int_0^1x^n(1-x)^{s_1}dx\int_0^1y^m(1-y)^{s_2}dy\\
&=\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}B(n+1,s_1+1)\,B(m+1,s_2+1)\,. \qquad (2.15)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n\left(\frac{a+b}{2}-t\right)^{s_1}(r-c)^{s_2+m}drdt\\
&\qquad=\frac{\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}B(n+1,s_1+1)}{m+s_2+1}, \qquad (2.16)
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^{s_1+n}\left(\frac{c+d}{2}-r\right)^{s_2}(r-c)^m drdt\\
&\qquad=\frac{\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}B(m+1,s_2+1)}{n+s_1+1} \qquad (2.17)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{2}{b-a}\right)^{s_1}\left(\frac{2}{d-c}\right)^{s_2}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^{s_1+n}(r-c)^{s_2+m}drdt\\
&\qquad=\frac{\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}}{(n+s_1+1)(m+s_2+1)}. \qquad (2.18)
\end{aligned}$$

Using (2.15)-(2.18) in (2.14), we obtain

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(t-a)^n(r-c)^m\left|\frac{\partial^{n+m}f(t,r)}{\partial t^n\partial r^m}\right|^q drdt\le\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\\
&\quad\times\Bigg[B_{(n,m)}^qB(n+1,s_1+1)\,B(m+1,s_2+1)+\frac{G_{(n,m)}^qB(n+1,s_1+1)}{m+s_2+1}\\
&\qquad+\frac{H_{(n,m)}^qB(m+1,s_2+1)}{n+s_1+1}+\frac{F_{(n,m)}^q}{(n+s_1+1)(m+s_2+1)}\Bigg]. \qquad (2.19)
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b}\int_{c}^{\frac{c+d}{2}}(b-t)^{n}(r-c)^{m}\left|\frac{\partial^{n+m}f(t,r)}{\partial t^{n}\partial r^{m}}\right|^{q}drdt \\
&\leq\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\left[\frac{H_{(n,m)}^{q}B(m+1,s_{2}+1)}{n+s_{1}+1}\right. \\
&\quad+D_{(n,m)}^{q}B(n+1,s_{1}+1)B(m+1,s_{2}+1) \\
&\quad\left.+\frac{F_{(n,m)}^{q}}{(n+s_{1}+1)(m+s_{2}+1)}+\frac{I_{(n,m)}^{q}B(n+1,s_{1}+1)}{m+s_{2}+1}\right], \qquad (2.20)
\end{aligned}$$

$$\begin{aligned}
&\int_{a}^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}(t-a)^{n}(d-r)^{m}\left|\frac{\partial^{n+m}f(t,r)}{\partial t^{n}\partial r^{m}}\right|^{q}drdt \\
&\leq\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\left[\frac{G_{(n,m)}^{q}B(n+1,s_{1}+1)}{m+s_{2}+1}\right. \\
&\quad+C_{(n,m)}^{q}B(n+1,s_{1}+1)B(m+1,s_{2}+1) \\
&\quad\left.+\frac{J_{(n,m)}^{q}B(m+1,s_{2}+1)}{n+s_{1}+1}+\frac{F_{(n,m)}^{q}}{(n+s_{1}+1)(m+s_{2}+1)}\right] \qquad (2.21)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}(b-t)^{n}(d-r)^{m}\left|\frac{\partial^{n+m}f(t,r)}{\partial t^{n}\partial r^{m}}\right|^{q}drdt \\
&\leq\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}\left[\frac{F_{(n,m)}^{q}}{(n+s_{1}+1)(m+s_{2}+1)}\right. \\
&\quad+\frac{I_{(n,m)}^{q}B(n+1,s_{1}+1)}{m+s_{2}+1}+\frac{J_{(n,m)}^{q}B(m+1,s_{2}+1)}{n+s_{1}+1} \\
&\quad\left.+E_{(n,m)}^{q}B(n+1,s_{1}+1)B(m+1,s_{2}+1)\right]. \qquad (2.22)
\end{aligned}$$

It is not difficult to observe that

$$\int_{a}^{b}\int_{c}^{d}|P(t)||Q(r)|\,drdt=\frac{4}{(n+1)(m+1)}\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{d-c}{2}\right)^{m+1}. \qquad (2.23)$$

From (2.12)-(2.23), we get the desired inequality. The proof of the Theorem for $q=1$ is the same. This completes the proof. $\square$

Some results can be deduced from the inequalities (2.9) and (2.12) as follows: Letting $s_1=s_2=1$ in Theorem 6 gives the following corollary:

Corollary 1. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^{n}\partial r^{m}}$ exists on Δ° and $\frac{\partial^{m+n}f}{\partial t^{n}\partial r^{m}}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^{n}\partial r^{m}}\right|^{q}$, $q\geq 1$,*

is convex on the co-ordinates on Δ, $m, n \in \mathbb{N}$, $m, n \geq 2$, *then*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|$$
$$\leq \frac{(b-a)^n(d-c)^m(n-1)^{1-1/q}(m-1)^{1-1/q}}{4(n+1)!(m+1)!(n+2)^{1/q}(m+2)^{1/q}}\left[\left(m^2-2\right)\left(n^2-2\right)B^q_{(n,m)}\right.$$
$$\left.+m\left(n^2-2\right)C^q_{(n,m)}+n\left(m^2-2\right)D^q_{(n,m)}+nmE^q_{(n,m)}\right]^{\frac{1}{q}}. \quad (2.24)$$

Corollary 2. *Under the assumptions of Corollary 1 with* $m = n = 2$, *we have*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A'\right|$$
$$\leq \frac{(b-a)^2(d-c)^2}{9\cdot 2^{\frac{2}{q}+4}}\sqrt[q]{\left|\frac{\partial^4 f(a,c)}{\partial t^2\partial r^2}\right|^q+\left|\frac{\partial^4 f(b,c)}{\partial t^2\partial r^2}\right|^q+\left|\frac{\partial^4 f(a,d)}{\partial t^2\partial r^2}\right|^q+\left|\frac{\partial^4 f(b,d)}{\partial t^2\partial r^2}\right|^q}. \quad (2.25)$$

The following corollary is a special case of Theorem 7 for $s_1 = s_2 = 1$:

Corollary 3. *Let* $f : \Delta \subset [0,\infty)\times[0,\infty) \to [0,\infty)$, $a < b$, $c < d$, *be a continuous mapping such that* $\frac{\partial^{m+n} f}{\partial t^n \partial r^m}$ *exist on* Δ° *and* $\frac{\partial^{m+n} f}{\partial t^n \partial r^m} \in L(\Delta)$. *If* $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|^q$, $q \geq 1$, *is convex on the co-ordinates on* Δ, m, $n \in \mathbb{N}$, $m, n \geq 1$. *Then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt\right.$$
$$-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!(l+1)!}\frac{\partial^{k+l} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}$$
$$+\frac{(-1)^{m+1}}{(d-c)m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m} f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr$$
$$\left.+\frac{(-1)^{n+1}}{(b-a)n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l} f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\right|$$
$$\leq \frac{(b-a)^n(d-c)^m}{2^{m+n+\frac{2}{q}}(n+1)!(m+1)!}\left[\frac{B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)}}{(n+2)(m+2)}\right.$$
$$+\frac{2(m+1)\left(G^q_{(n,m)}+I^q_{(n,m)}\right)}{(n+2)(m+2)}+\frac{2(n+1)\left(H^q_{(n,m)}+J^q_{(n,m)}\right)}{(n+2)(m+2)}$$
$$\left.+\frac{4(n+1)(m+1)F^q_{(n,m)}}{(n+2)(m+2)}\right]^{\frac{1}{q}}, \quad (2.26)$$

where $P(t)$ *and* $Q(r)$ *are as defined in Theorem 7.*

The following Corollary is a special case of Theorem 7 for $s_1 = s_2 = 1$ and $m = n = 1$, which gives tighter estimate than those from [23, Theorem 4, page 8]:

Corollary 4. *Under the assumptions of corollary 3 with $m = n = 1$, we have*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt + f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$
$$\left.-\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},r\right)dr - \frac{1}{2(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt\right|$$
$$\leq \frac{(b-a)(d-c)}{2^{4+\frac{2}{q}}}\left[\frac{B^q_{(1,1)}+C^q_{(1,1)}+D^q_{(1,1)}+E^q_{(1,1)}}{9}\right.$$
$$\left.+\frac{4\left(G^q_{(1,1)}+I^q_{(1,1)}\right)}{9}+\frac{4\left(H^q_{(1,1)}+J^q_{(1,1)}\right)}{9}+\frac{8F^q_{(1,1)}}{9}\right]^{\frac{1}{q}}, \quad (2.27)$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 7.

It is easy to see that, when $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, $q \geq 1$, is convex on the co-ordinates on Δ, m, $n \in \mathbb{N}$, $m,n \geq 1$, then

$$2\left(G^q_{(n,m)}+I^q_{(n,m)}\right) \leq B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)},$$
$$2\left(H^q_{(n,m)}+J^q_{(n,m)}\right) \leq B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)}$$

and

$$4F^q_{(n,m)} \leq B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)}.$$

Substituting these inequalities in corollary 3, we get the following corollary which is [24, Theorem 2.3, page12]:

Corollary 5. *Let $f : \Delta \subset [0,\infty)\times[0,\infty) \to [0,\infty)$, $a < b$, $c < d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exist on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m} \in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, $q \geq 1$, is convex on the co-ordinates on Δ, m, $n \in \mathbb{N}$, $m,n \geq 1$. Then*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt\right.$$
$$-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}$$
$$+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr$$
$$\left.+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\right|$$
$$\leq \frac{(b-a)^n(d-c)^m}{2^{m+n+\frac{2}{q}}(n+1)!\,(m+1)!}\sqrt[q]{B^q_{(n,m)}+C^q_{(n,m)}+D^q_{(n,m)}+E^q_{(n,m)}}, \quad (2.28)$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 7.

A different approach leads us to the following result:

Theorem 8. *Let $f:\Delta\subset[0,\infty)\times[0,\infty)\to[0,\infty)$, $a<b$, $c<d$, be a continuous mapping such that $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}$ exist on Δ° and $\frac{\partial^{m+n}f}{\partial t^n\partial r^m}\in L(\Delta)$. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$, $q\geq 1$, is s-convex on the co-ordinates on Δ, $s_1,s_2\in(0,1]$ with $s=\frac{s_1+s_2}{2}$, m, $n\in\mathbb{N}$, $m,n\geq 1$. Then*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt\right.\\
&\quad-\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l}\\
&\quad+\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr\\
&\quad\left.+\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\right|\\
&\leq\frac{1}{4n!m!}\left(\frac{1}{(n+1)(m+1)}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{2}\right)^n\left(\frac{d-c}{2}\right)^m\\
&\times\left\{\left[B^q_{(n,m)}B(n+1,s_1+1)B(m+1,s_2+1)+\frac{G^q_{(n,m)}B(n+1,s_1+1)}{m+s_2+1}\right.\right.\\
&\quad\left.+\frac{H^q_{(n,m)}B(m+1,s_2+1)}{n+s_1+1}+\frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}\right]^{\frac{1}{q}}\\
&+\left[\frac{H^q_{(n,m)}B(m+1,s_2+1)}{n+s_1+1}+D^q_{(n,m)}B(n+1,s_1+1)B(m+1,s_2+1)\right.\\
&\quad\left.+\frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}+\frac{I^q_{(n,m)}B(n+1,s_1+1)}{m+s_2+1}\right]^{\frac{1}{q}}\\
&+\left[\frac{G^q_{(n,m)}B(n+1,s_1+1)}{m+s_2+1}+C^q_{(n,m)}B(n+1,s_1+1)B(m+1,s_2+1)\right.\\
&\quad\left.+\frac{J^q_{(n,m)}B(m+1,s_2+1)}{n+s_1+1}+\frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}\right]^{\frac{1}{q}}\\
&+\left[\frac{F^q_{(n,m)}}{(n+s_1+1)(m+s_2+1)}+\frac{I^q_{(n,m)}B(n+1,s_1+1)}{m+s_2+1}+\frac{J^q_{(n,m)}B(m+1,s_2+1)}{n+s_1+1}\right.\\
&\quad\left.\left.+E^q_{(n,m)}B(n+1,s_1+1)B(m+1,s_2+1)\right]^{\frac{1}{q}}\right\},
\end{aligned}\tag{2.29}$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 7.

Proof. By letting $x \mapsto \frac{a+b}{2}$ and $y \mapsto \frac{c+d}{2}$ in Theorem 4, using the properties of the absolute value, we obtain

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt \right. \\
&\quad - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}} \frac{(b-a)^k (d-c)^l}{(k+1)!\,(l+1)!} \frac{\partial^{k+l} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l} \\
&\quad + \frac{(-1)^{m+1}}{(d-c)\, m!} \sum_{k=0}^{n-1} \frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, r\right)}{\partial x^k \partial r^m} dr \\
&\quad \left. + \frac{(-1)^{n+1}}{(b-a)\, n!} \sum_{l=0}^{m-1} \frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \right| \\
&\le \frac{1}{(b-a)(d-c)\, m!n!} \left[\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} (t-a)^n (r-c)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| dr dt \right. \\
&\qquad + \int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} (b-t)^n (r-c)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| dr dt \\
&\qquad + \int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d (t-a)^n (d-r)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| dr dt \\
&\qquad \left. + \int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d (b-t)^n (d-r)^m \left| \frac{\partial^{n+m} f(t,r)}{\partial t^n \partial r^m} \right| dr dt \right]. \qquad (2.30)
\end{aligned}$$

Using the power-mean inequality for each integral on the right-side of (2.30) and by the similar arguments as in proving Theorem 7, we get (2.29). □

Corollary 6. *If the conditions of Theorem 8 are satisfied and if $m = n = 1$ and $s_1 = s_2 = 1$, then we have the inequality*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt + \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right. \\
&\quad \left. - \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, r \right) dr - \frac{1}{2(b-a)} \int_a^b f\left(t, \frac{c+d}{2} \right) dt \right| \\
&\le \left(\frac{1}{4} \right)^{2-\frac{1}{q}} \left(\frac{b-a}{2} \right) \left(\frac{d-c}{2} \right) \left\{ \left[\frac{1}{36} B^q_{(1,1)} + \frac{1}{18} G^q_{(1,1)} + \frac{1}{18} H^q_{(1,1)} + \frac{1}{9} F^q_{(1,1)} \right]^{\frac{1}{q}} \right. \\
&\qquad + \left[\frac{1}{18} H^q_{(1,1)} + \frac{1}{36} D^q_{(1,1)} + \frac{1}{9} F^q_{(1,1)} + \frac{1}{18} I^q_{(1,1)} \right]^{\frac{1}{q}} \\
&\qquad + \left[\frac{1}{18} G^q_{(1,1)} + \frac{1}{36} C^q_{(1,1)} + \frac{1}{18} J^q_{(1,1)} + \frac{1}{9} F^q_{(1,1)} \right]^{\frac{1}{q}} \\
&\qquad \left. + \left[\frac{1}{9} F^q_{(1,1)} + \frac{1}{18} I^q_{(1,1)} + \frac{1}{18} J^q_{(1,1)} + \frac{1}{36} E^q_{(1,1)} \right]^{\frac{1}{q}} \right\}. \qquad (2.31)
\end{aligned}$$

If we use the Hölder's inequality instead of the power-mean inequality we get the following result:

Theorem 9. *Let $f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty)$, $a < b$, $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^n \partial r^m}$ exist on Δ° and $\frac{\partial^{m+n} f}{\partial t^n \partial r^m} \in L(\Delta)$. If $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|^p$, $p > 1$, is s-convex on the co-ordinates on Δ, $s_1, s_2 \in (0, 1]$ with $s = \frac{s_1+s_2}{2}$, m, $n \in \mathbb{N}$, $m, n \geq 1$. Then*

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt \right.$$
$$- \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{\left[1 + (-1)^k\right]\left[1 + (-1)^l\right]}{2^{k+l+2}} \frac{(b-a)^k (d-c)^l}{(k+1)!\,(l+1)!} \frac{\partial^{k+l} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)}{\partial x^k \partial y^l}$$
$$+ \frac{(-1)^{m+1}}{(d-c)\, m!} \sum_{k=0}^{n-1} \frac{\left[1 + (-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} \int_c^d Q(r) \frac{\partial^{k+m} f\left(\frac{a+b}{2}, r\right)}{\partial x^k \partial r^m} dr$$
$$\left. + \frac{(-1)^{n+1}}{(b-a)\, n!} \sum_{l=0}^{m-1} \frac{\left[1 + (-1)^l\right](d-c)^l}{2^{l+1}(l+1)!} \int_a^b P(t) \frac{\partial^{n+l} f\left(t, \frac{c+d}{2}\right)}{\partial t^n \partial y^l} dt \right|$$
$$\leq \frac{(b-a)^n (d-c)^m}{2^{n+m} n! m! \left[(np+1)(mp+1)\right]^{\frac{1}{p}}} \left[\frac{1}{2}\left(\frac{1}{(s_1+1)^2} + \frac{1}{(s_2+1)^2}\right)\right]^{\frac{1}{q}}$$
$$\times \left[B^q_{(n,m)} + C^q_{(n,m)} + D^q_{(n,m)} + E^q_{(n,m)}\right]^{\frac{1}{q}}, \qquad (2.32)$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 7 and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The inequality (2.32) follows using the Hölder's inequality and the inequality (1.9). □

Corollary 7. *Under the assumptions of Theorem 9, if $m = n = 1$ and $s_1 = s_2 = 1$, then we have the inequality*

$$\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,r)\, dr dt + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.$$
$$\left. - \frac{1}{2(d-c)} \int_c^d f\left(\frac{a+b}{2}, r\right) dr - \frac{1}{2(b-a)} \int_a^b f\left(t, \frac{c+d}{2}\right) dt \right|$$
$$\leq \frac{(b-a)(d-c)}{2^{2+\frac{2}{q}}(p+1)^{\frac{2}{p}}} \sqrt[q]{\left|\frac{\partial^2 f(a,c)}{\partial t \partial r}\right|^q + \left|\frac{\partial^2 f(b,c)}{\partial t \partial r}\right|^q + \left|\frac{\partial^2 f(a,d)}{\partial t \partial r}\right|^q + \left|\frac{\partial^2 f(b,d)}{\partial t \partial r}\right|^q}, \qquad (2.33)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Our last result is for the s-concave functions can be stated as follows:

Theorem 10. *Let $f : \Delta \subset [0, \infty) \times [0, \infty) \to [0, \infty)$, $a < b$, $c < d$, be a continuous mapping such that $\frac{\partial^{m+n} f}{\partial t^n \partial r^m}$ exist on Δ° and $\frac{\partial^{m+n} f}{\partial t^n \partial r^m} \in L(\Delta)$. If $\left|\frac{\partial^{n+m} f}{\partial t^n \partial s^m}\right|^p$, $p > 1$,* *is s-concave on the co-ordinates on Δ, $s_1, s_2 \in (0,1]$ with $s = \frac{s_1+s_2}{2}$, m, $n \in \mathbb{N}$, $m, n \geq 1$. Then*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt \\
&\quad -\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^k\right]\left[1+(-1)^l\right]}{2^{k+l+2}}\frac{(b-a)^k(d-c)^l}{(k+1)!\,(l+1)!}\frac{\partial^{k+l}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial x^k\partial y^l} \\
&\quad +\frac{(-1)^{m+1}}{(d-c)\,m!}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}\int_c^d Q(r)\frac{\partial^{k+m}f\left(\frac{a+b}{2},r\right)}{\partial x^k\partial r^m}dr \\
&\quad +\frac{(-1)^{n+1}}{(b-a)\,n!}\sum_{l=0}^{m-1}\frac{\left[1+(-1)^l\right](d-c)^l}{2^{l+1}(l+1)!}\int_a^b P(t)\frac{\partial^{n+l}f\left(t,\frac{c+d}{2}\right)}{\partial t^n\partial y^l}dt\Bigg| \\
&\leq \frac{(b-a)^n(d-c)^m}{2^{n+m}n!m!\left[(np+1)(mp+1)\right]^{\frac{1}{p}}}\left[\frac{4^{s_1+1}+4^{s_2+1}}{2}\right]^{\frac{1}{q}}\left|\frac{\partial^{n+m}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t^n\partial r^m}\right|, \qquad (2.34)
\end{aligned}$$

where $P(t)$ and $Q(r)$ are as defined in Theorem 7 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The inequality (2.34) follows using the Hölder's inequality and the inequality (1.9) with inequalities in reversed direction. □

Corollary 8. *If the conditions of Theorem 10 are satisfied and if $m = n = 1$ and $s_1 = s_2 = 1$, then we have the inequality*

$$\begin{aligned}
&\Bigg|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,r)\,drdt + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
&\quad -\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},r\right)dr - \frac{1}{2(b-a)}\int_a^b f\left(t,\frac{c+d}{2}\right)dt\Bigg| \\
&\qquad \leq \frac{(b-a)(d-c)}{2^{2-\frac{4}{q}}(p+1)^{\frac{2}{p}}}\left|\frac{\partial^2 f\left(\frac{a+b}{2},\frac{c+d}{2}\right)}{\partial t\partial r}\right|, \qquad (2.35)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

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