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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE (α, m)-CONVEX FUNCTIONS WITH APPLICATIONS TO SPECIAL MEANS

M. A. LATIF AND SABIR HUSSAIN

ABSTRACT. Several new inequalities are obtained for n-times differenitable (α, m)-convex functions that are connected with the Hermite-Hadamard's inequality. Applications of the established results to special means of positive real numbers are given as well.

1. INTRODUCTION

The following definition for convex functions is well known in the mathematical literature: A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard inequality [11] has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see, [1, 2, 7, 8, 9, 15, 16, 17, 24, 25, 31] and the references therein.

Let $[0, b]$, where b is greater than 0, be an interval of the real line $\mathbb{R}$, and let $K(b)$ denote the class of all functions $f : [0, b] \to \mathbb{R}$ which are continuous and nonnegative on $[0, b]$ and such that $f(0) = 0$. A function f is said to be convex on $[0, b]$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

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holds for all $x, y \in [0, b]$ and $t \in [0, 1]$. Let $K_C(b)$ denote the class of all functions $f \in K(b)$ convex on $[0, b]$, and let $K_F(b)$ be the class of all functions $f \in K(b)$ convex in mean on $[0, b]$, that is, the class of all functions $f \in K(b)$ for which $F \in K_C(b)$, where the mean function F of the function $f \in K(b)$ is defined by

$$F(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)dt, & x \in (0, b] \\ 0, & x = 0 \end{cases}$$

Let $K_S(b)$ denote the class of all functions $f \in K(b)$ which are starshaped with respect to the origin on $[0, b]$, that is, the class of all functions f with the property that

$$f(tx) \leq tf(x)$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$. In [6], Bruckner and Ostrow, among others, proved that

$$K_C(b) \subset K_F(b) \subset K_S(b).$$

In [29], G. Toader, defined m-convexity: another intermediate between the usual convexity and starshaped convexity.

Definition 1. [29] *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ for which $f(0) \leq 0$. Obviously, for $m = 1$, m-convexity is the standard convexity of functions on $[0, b]$, and for $m = 0$ the concept of starshaped functions. The following lemmas hold (see [29]).

Lemma 1. [29] *If f is in the class $\mathrm{K}_m(b)$, then it is starshaped.*

Lemma 2. [29] *If f is in the class $\mathrm{K}_m(b)$ and $0 < n < m \leq 1$, then f is in the class $\mathrm{K}_n(b)$.*

From Lemma 2 and Lemma 3 it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is, $K_1(b)$ is a proper subclass of the class of convex functions on $[0, b]$. It is interesting to point out that for any $m \in (0, 1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense (see [30]). The notion of m-convexity was further generalized by [19] in the following definition.

Definition 2. [19] *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^\alpha(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$. It can be easily seen that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex functions respectively. Note that in the class $K_1^1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \le 0$, that is $K_1^1(b)$ is a proper subclass of the class of all convex functions on $[0, b]$.

For further results on Hermite-Hadamard type inequalities related to m-convex and (α, m)-convex functions we refer the readers [3, 10, 18, 21, 22, 27].

In recent papers, Dah-Yang Hwang [12], Wei-Dong Jiang et. al [14] and Shu-Hong et al. [17] established several new inequalities of Hermite-Hadamard type for n-times differentiable convex functions, s-convex functions and m-convex functions respectively (see the references in these papers as well).

The main result from [12] is pointed out as follows:

Theorem 1. [27]*Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, a, $b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L(a, b)$ for $n \in \mathbb{N}$, $n \ge 1$ and $\left|f^{(n)}\right|^q$, $q \ge 1$, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| = \frac{(n-1)^{1-\frac{1}{q}}(b-a)^n}{2(n+1)!}\left[\frac{(n^2-2)\left|f'(a)\right|^q + n\left|f'(b)\right|^q}{n+2}\right]^{\frac{1}{q}}. \tag{1.2}$$

The following lemma was used to establish the above result:

Lemma 3. [12]*Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, a, $b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I° and $f^{(n)} \in L(a, b)$ for $n \in \mathbb{N}$, $n \ge 1$, then we have the identity:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a) = \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt. \tag{1.3}$$

Motivated by the above results, the main purpose of the present paper is to establish new Hermite-Hadamard type inequalities that are connected with the right side of (1.1) for n-times differentiable (α, m)-convex functions that will generalize the above.

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function on I° such that $f^{(n)} \in L[a, mb]$, where a, $mb \in I^\circ$ with $a < mb$, $n \in \mathbb{N}$, $n \ge 1$ and $m \in (0, 1]$. Then we have the identity:*

$$\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a) = \frac{(mb-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)\,dt \tag{2.1}$$

where the sum above takes 0 for $n = 1$ and $n = 2$.

Proof. For $n=1$, by integration by parts and by the change of variables, we have the following equality:

$$\begin{aligned}&\frac{mb-a}{2}\int_0^1 (1-2t)\, f^{'}(ta+m(1-t)b)\,dt\\&=\frac{mb-a}{2}\left[\left.\frac{(1-2t)\,f(ta+m(1-t)b)}{a-mb}\right|_0^1+\frac{2}{a-mb}\int_0^1 f(ta+m(1-t)b)\,dt\right]\\&\qquad=\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx,\end{aligned}$$

which coincides with the left-hand side of (2.1) for $n=1$.
Suppose (2.1) is true for $n-1$, i.e.

$$\begin{aligned}\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx-\sum_{k=2}^{n-2}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\\=\frac{(mb-a)^{n-1}}{2(n-1)!}\int_0^1 t^{n-2}(n-1-2t)\,f^{(n-1)}(ta+m(1-t)b)\,dt.\end{aligned}\tag{2.2}$$

Now by integration by parts and (2.2), we have

$$\begin{aligned}&\frac{(mb-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(ta+m(1-t)b)\,dt\\&=-\frac{(n-2)(mb-a)^{n-1}f^{(n-1)}(a)}{2n!}\\&+\frac{(mb-a)^{n-1}}{2(n-1)!}\int_0^1 t^{n-2}(n-1-2t)\,f^{(n-1)}(ta+m(1-t)b)\,dt\\&=\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx\\&-\sum_{k=2}^{n-2}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)-\frac{(n-2)(mb-a)^{n-1}f^{(n-1)}(a)}{2n!}\\&=\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a).\end{aligned}$$

This completes the proof of the lemma. □

Remark 1. *For $m=1$, Lemma 4 reduces to Lemma 3.*

Now we set off some new integral inequalities of Hermite-Hadamard type for n-times differentiable (α,m)-convex functions.

Theorem 2. *Let $f: I\subset[0,b^*]\to\mathbb{R}$, $b^*>0$, be an n-times differentiable function on I° such that $f^{(n)}\in L[a,mb]$, where a, $mb\in I^\circ$ with $a<mb$, $n\in\mathbb{N}$, $n\geq 2$ and $m\in(0,1]$. If $\left|f^{(n)}\right|^q$ is (α,m)-convex on $[a,mb]$ for $q\in[1,\infty)$, $\alpha\in[0,1]$,*

$m\in(0,1]$, then we have the inequality:

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\right| \tag{2.3}$$
$$\leq\frac{(mb-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left[\frac{n(n+\alpha-1)-2\alpha}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|^q\right.$$
$$\left.+m\left[\frac{n-1}{n+1}-\frac{n(n+\alpha-1)-2\alpha}{(n+\alpha)(n+\alpha+1)}\right]\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}.$$

Proof. Suppose $n\geq2$. From Lemma 4 and by the power-mean integral inequality, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\right| \tag{2.4}$$
$$\leq\frac{(mb-a)^n}{2n!}\int_0^1t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|dt$$
$$\leq\frac{(mb-a)^n}{2n!}\left(\int_0^1t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}$$

By the (α,m)-convexity of $\left|f^{(n)}\right|^q$ on $[a,mb]$, we have

$$\int_0^1t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|^q dt \tag{2.5}$$
$$\leq\int_0^1t^{n-1}(n-2t)\left(t^\alpha\left|f^{(n)}(a)\right|^q+m(1-t^\alpha)\left|f^{(n)}(b)\right|^q\right)dt$$
$$=\frac{n(n+\alpha-1)-2\alpha}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|^q+m\left[\frac{n-1}{n+1}-\frac{n(n+\alpha-1)-2\alpha}{(n+\alpha)(n+\alpha+1)}\right]\left|f^{(n)}(b)\right|^q.$$

We also have

$$\int_0^1t^{n-1}(n-2t)\,dt=\frac{n-1}{n+1} \tag{2.6}$$

Substitution of (2.5) and (2.6) in (2.4), we get the desired inequality. □

Corollary 1. *Under the assumptions of Theorem 2 for $q=1$, we have the following inequality:*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\right| \tag{2.7}$$
$$\leq\frac{(mb-a)^n}{2n!}\left[\frac{n(n+\alpha-1)-2\alpha}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|\right.$$
$$\left.+m\left[\frac{n-1}{n+1}-\frac{n(n+\alpha-1)-2\alpha}{(n+\alpha)(n+\alpha+1)}\right]\left|f^{(n)}(b)\right|\right].$$

Corollary 2. *Under the assumptions of Theorem 2 for $q = 1$, $n = 2$, we have the following inequality:*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx\right| \leq \frac{(mb-a)^2}{4}\left[\frac{2\left|f''(a)\right|}{(\alpha+2)(\alpha+3)}+m\left(\frac{1}{3}-\frac{2}{(2+\alpha)(\alpha+3)}\right)\left|f''(b)\right|\right]. \tag{2.8}$$

Corollary 3. [12, Theorem 3.1]*Under the assumptions of Theorem 2 for $m = 1$, $\alpha = 1$, we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b} f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n(n-1)^{1-\frac{1}{q}}}{2(n+1)!}\left[\frac{(n^2-2)\left|f^{(n)}(a)\right|^q+n\left|f^{(n)}(b)\right|^q}{n+2}\right]^{\frac{1}{q}}. \tag{2.9}$$

Corollary 4. *Under the assumptions of Theorem 2 for $m = 1$, $\alpha = 1$ and $n = 2$, we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b} f(x)\,dx\right| \leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.10}$$

The inequality (2.10) reduces to the following inequality if we choose $q = 1$:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b} f(x)\,dx\right| \leq \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right]. \tag{2.11}$$

Theorem 3. *Let $f : I \subset [0, b^*] \to \mathbb{R}$, $b^* > 0$, be an n-times differentiable function on I°such that $f^{(n)} \in L[a, mb]$, where a, $mb \in I^\circ$ with $a < mb$, $n \in \mathbb{N}$, $n \geq 2$ and $m \in (0, 1]$. If $\left|f^{(n)}\right|^q$ is (α, m)-convex on $[a, mb]$ for $q \in (1, \infty)$, $\alpha \in [0, 1]$, $m \in (0, 1]$, then we have the inequality:*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(mb-a)^n\left(n^{p+1}-(n-2)^{p+1}\right)^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}n!(\alpha+nq-q+1)^{\frac{1}{q}}} \times\left[\frac{(nq-q+1)\left|f^{(n)}(a)\right|^q+m\alpha\left|f^{(n)}(b)\right|^q}{nq-q+1}\right]^{\frac{1}{q}}, \tag{2.12}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Suppose $n \geq 2$. By Lemma 4 and the Hölder's integral inequality, we have

$$\begin{aligned}
(2.13)\quad & \left| \frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx - \sum_{k=2}^{n-1} \frac{(k-1)(mb-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\
& \leq \frac{(mb-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right| dt \\
& \leq \frac{(mb-a)^n}{2n!}\left(\int_0^1 (n-2t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

By the (α, m)-convexity of $\left|f^{(n)}\right|^q$ on $[a, mb]$, we have for every $t \in [0,1]$ that

$$\left|f^{(n)}(ta+m(1-t)b)\right|^q \leq t^\alpha\left|f^{(n)}(a)\right|^q + m(1-t^\alpha)\left|f^{(n)}(b)\right|^q.$$

Now

$$\begin{aligned}
(2.14)\quad & \int_0^1 t^{q(n-1)}\left|f^{(n)}(ta+m(1-t)b)\right|^q dt \\
& \leq \int_0^1 \left[t^{q(n-1)+\alpha}\left|f^{(n)}(a)\right|^q + m\left(t^{q(n-1)} - t^{q(n-1)+\alpha}\right)\left|f^{(n)}(b)\right|^q\right] dt \\
& = \frac{\left|f^{(n)}(a)\right|^q}{\alpha+nq-q+1} + \frac{m\alpha\left|f^{(n)}(b)\right|^q}{(nq-q+1)(\alpha+nq-q+1)}
\end{aligned}$$

Using (2.14) and

$$\int_0^1 (n-2t)^p\,dt = \frac{n^{p+1}-(n-2)^{p+1}}{2(p+1)}$$

in (2.13), we get the required inequality. □

Corollary 5. *Suppose the assumptions of Theorem 3 are satisfied and if we choose* $m = \alpha = 1$*, then we have the following inequality:*

$$\begin{aligned}
(2.15)\quad & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^{b} f(x)\,dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\
& \leq \frac{(b-a)^n\left(n^{p+1}-(n-2)^{p+1}\right)^{\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}\,n!\,(nq-q+2)^{\frac{1}{q}}} \\
& \qquad \times \left[\frac{(nq-q+1)\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{nq-q+1}\right]^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 6. *Suppose the assumptions of Theorem 3 are satisfied and if we choose $m=\alpha=1$ and $n=2$, then we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}(q+2)^{\frac{1}{q}}}\left[\frac{(q+1)\left|f''(a)\right|^q+\left|f''(b)\right|^q}{q+1}\right]^{\frac{1}{q}}, \tag{2.16}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4. *Let $f:I\subset[0,b^*]\to\mathbb{R}$, $b^*>0$, be an n-times differentiable function on I° such that $f^{(n)}\in L[a,mb]$, where $a,\ mb\in I^\circ$ with $a<mb$, $n\in\mathbb{N}$, $n\geq 2$ and $m\in(0,1]$. If $\left|f^{(n)}\right|^q$ is (α,m)-convex on $[a,mb]$ for $q\in[1,\infty)$, $\alpha\in[0,1]$, $m\in(0,1]$, then we have the inequality:*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(mb-a)^n(n-1)^{1-\frac{1}{q}}}{2n!}\left[P\left|f^{(n)}(a)\right|^q+mQ\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}, \tag{2.17}$$

where

$$P=\frac{n}{\alpha+nq-q+1}-\frac{2}{\alpha+nq-q+2}$$

and

$$Q=\frac{n\alpha}{(nq-q+1)(\alpha+nq-q+1)}-\frac{2\alpha}{(nq-q+2)(\alpha+nq-q+2)}.$$

Proof. Suppose $n\geq 2$. By Lemma 4 and the Hölder's integral inequality, we have

$$\begin{aligned}\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!}f^{(k)}(a)\right| &\leq \frac{(mb-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|dt \\ &\leq \frac{(mb-a)^n}{2n!}\left(\int_0^1(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1(n-2t)\,t^{q(n-1)}\left|f^{(n)}(ta+m(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{2.18}$$

By the (α,m)-convexity of $\left|f^{(n)}\right|^q$ on $[a,mb]$, we have for every $t\in[0,1]$ that

$$\left|f^{(n)}(ta+m(1-t)b)\right|^q\leq t^\alpha\left|f^{(n)}(a)\right|^q+m(1-t^\alpha)\left|f^{(n)}(b)\right|^q.$$

Now

$$\begin{aligned}\int_0^1(n-2t)\,t^{q(n-1)}\left|f^{(n)}(ta+m(1-t)b)\right|^q dt &\leq \left(\frac{n}{\alpha+nq-q+1}-\frac{2}{\alpha+nq-q+2}\right)\left|f^{(n)}(a)\right|^q \\ &+\left(\frac{n\alpha}{(nq-q+1)(\alpha+nq-q+1)}-\frac{2\alpha}{(nq-q+2)(\alpha+nq-q+2)}\right)\left|f^{(n)}(b)\right|^q\end{aligned} \tag{2.19}$$

Using (2.19) and

$$\int_0^1 (n-2t)\,dt = n-1$$

in (2.18), we get the required inequality. □

Corollary 7. *If the conditions of the Theorem 4 are fulfilled and if we take $q=1$, then we have the inequality:*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{2(k+1)!} f^{(k)}(a)\right| \le \frac{(mb-a)^n}{2n!}\left[P\left|f^{(n)}(a)\right| + mQ\left|f^{(n)}(b)\right|\right], \tag{2.20}$$

where

$$P = \frac{n}{\alpha+n} - \frac{2}{\alpha+n+1}$$

and

$$Q = \frac{\alpha}{\alpha+n} - \frac{2\alpha}{(n+1)(\alpha+n+1)}$$

Corollary 8. *If the conditions of the Theorem 4 are fulfilled and if we take $m=\alpha=1$, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^{b} f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right| \le \frac{(b-a)^n}{2n!}\left[P\left|f^{(n)}(a)\right|^q + Q\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}, \tag{2.21}$$

where

$$P = \frac{n}{nq-q+2} - \frac{2}{nq-q+3}$$

and

$$Q = \frac{n\alpha}{(nq-q+1)(nq-q+2)} - \frac{2\alpha}{(nq-q+2)(nq-q+3)}.$$

Corollary 9. *If the conditions of the Theorem 4 are fulfilled and if we take $m=\alpha=1$, $n=2$, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^{b} f(x)\,dx\right| \le \frac{(b-a)^2}{2^{2-\frac{1}{q}}}\left[\frac{(q+1)\left|f''(a)\right|^q + 2\left|f''(b)\right|^q}{(q+1)(q+2)(q+3)}\right]^{\frac{1}{q}}. \tag{2.22}$$

The inequality (2.22) reduces to the following inequality if we take $q=1$:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^{b} f(x)\,dx\right| \le \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right| + \left|f''(b)\right|}{4}\right]. \tag{2.23}$$

Remark 2. *It can be observed that the inequality (2.23) gives better bound than the one given by (2.11).*

3. Applications to Special Means

We consider some means for arbitrary positive real numbers α, β (see for instance [4]).

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}; \alpha, \beta \in \mathbb{R} \text{ with } \alpha, \beta > 0$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}; \alpha, \beta \in \mathbb{R} \text{ with } \alpha, \beta > 0$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}; \alpha, \beta \in \mathbb{R} \text{ with } \alpha, \beta > 0$$

(4) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}; \alpha, \beta \in \mathbb{R} \text{ with } \alpha, \beta > 0$$

(5) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln \alpha - \ln \beta}, \ \alpha, \beta \in \mathbb{R} \text{ with } \alpha \neq \beta, \alpha, \beta > 0$$

(6) The generalized log-mean:

$$L_r := L_r(\alpha, \beta) = \left[\frac{\beta^{r+1} - \alpha^{r+1}}{(r+1)(\beta - \alpha)}\right]^{\frac{1}{r}}; \alpha, \beta \in \mathbb{R}, \alpha, \beta > 0, \ \alpha \neq \beta, \ r \in \mathbb{R}\backslash\{-1, 0\}.$$

We have the following propositions concerning the means:

Proposition 1. *Let* $r \in (-\infty, 0) \cup [1, \infty) \backslash \{-1\}$ *and* $[a, b] \subset [0, b^*]$, $b^* > 0$. *Then for* $p, q > 1$, *we have the inequality:*

$$\text{(3.1)} \quad |A(a^r, b^r) - L_r^r(a, b)| \leq \frac{(b-a)^2 r(r-1)}{2(p+1)^{\frac{1}{p}}(q+2)^{\frac{1}{q}}} \left[\frac{(q+1)a^{(r-2)q} + b^{(r-2)q}}{q+1}\right]^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The assertion follows from the inequality (2.16) of the corollary 6 for $f(x) = x^r$ and r as specified above. □

Proposition 2. *Let* $r \in (-\infty, 0) \cup [1, \infty) \backslash \{-1\}$ *and* $[a, b] \subset [0, b^*]$, $b^* > 0$. *Then for* $q \geq 1$, *we have the inequality:*

$$\text{(3.2)} \quad |A(a^r, b^r) - L_r^r(a, b)| \leq \frac{(b-a)^2 r(r-1)}{2^{2-\frac{1}{q}}} \left[\frac{(q+1)a^{(r-2)q} + 2b^{(r-2)q}}{(q+1)(q+2)(q+3)}\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from the inequality (2.22) of the corollary 9 for $f(x) = x^r$ and r as specified above. □

Proposition 3. *Let* $r \in (-\infty, 0) \cup [1, \infty) \backslash \{-1\}$ *and* $[a, b] \subset [0, b^*]$, $b^* > 0$. *Then we have the inequality:*

$$\text{(3.3)} \quad |A(a^r, b^r) - L_r^r(a, b)| \leq \frac{(b-a)^2 r(r-1)}{24} A\left(a^{r-2}, b^{r-2}\right).$$

Proof. The assertion follows from the inequality (2.23) of the corollary 9 for $f(x) = x^r$ and r as specified above. □

Proposition 4. *Let $a, b \in (0, b^*]$, $b^* > 0$. Then for $p, q > 1$, we have the inequality:*

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{(p+1)^{\frac{1}{p}}(q+2)^{\frac{1}{q}}}\left[\frac{(q+1)a^{-3q}+b^{-3q}}{q+1}\right]^{\frac{1}{q}}, \tag{3.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The assertion follows from the inequality (2.16) of the corollary 6 for $f(x) = \frac{1}{x}$. □

Proposition 5. *Let $a, b \in (0, b^*]$, $b^* > 0$. Then for $q \geq 1$, we have the inequality:*

$$\begin{aligned} &\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \\ &\leq \frac{(b-a)^2 r(r-1)}{2^{1-\frac{1}{q}}}\left[\frac{a^{-3q}}{(q+2)(q+3)}+\frac{2b^{-3q}}{(q+1)(q+2)(q+3)}\right]^{\frac{1}{q}}. \end{aligned} \tag{3.5}$$

Proof. The assertion follows from the inequality (2.22) of the corollary 9 for $f(x) = x^r$ and r as specified above. □

Proposition 6. *Let $a, b \in (0, b^*]$, $b^* > 0$. Then we have the inequality:*

$$\left|\ln I(a,b) - \ln G(a,b)\right| \leq \frac{(b-a)^2}{24} A\left(a^{-2}, b^{-2}\right). \tag{3.6}$$

Proof. The assertion follows from the inequality (2.23) of the corollary 9 for $f(x) = -\ln x$. □

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College of Science, Department of Mathematics,, University of Hail, Hail 2440, Saudi Arabia

E-mail address: m_amer_latif@hotmail.com

Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan

E-mail address: sabirhus@gmail.com