

**ON HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES
FOR n -TIMES DIFFERENTIABLE PREINVEX FUNCTIONS
WITH APPLICATIONS**

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ABSTRACT. In this paper some new Hermite-Hadamard type inequalities for n -times differentiable preinvex functions are established. Our established results generalize some of those results proved in recent papers for differentiable preinvex functions and n -times differentiable convex functions. Applications to some special means of our results are given as well.

1. INTRODUCTION

The following definition for convex functions is well known in the mathematical literature: A function $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follows:

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard inequality (see [11]) has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f . A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see [6]-[8], [10, 12], [14]-[16], [23]-[29] and the references therein.

By using the following result:

Lemma 1. [6] *Suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^\circ \rightarrow \mathbb{R}$ is differentiable. If $f' \in L(a, b)$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{b-a}{2} \int_0^1 (1-2t) f'(ta + (1-t)b) dt.$$

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Dragomir and Agarwal [6], established the following results connected with the right part of (1.1) and applied them for some elementary inequalities for real numbers and in numerical integration:

Theorem 1. [6] *Suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^\circ \rightarrow \mathbb{R}$ is differentiable. If $f' \in L(a, b)$ and $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} \left(|f'(a)| + |f'(b)| \right). \quad (1.2)$$

Theorem 2. [6] *Suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^\circ \rightarrow \mathbb{R}$ is differentiable. If $f' \in L(a, b)$ and $|f'|^{\frac{p}{p-1}}$, $p > 1$, is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2(1+p)^{\frac{1}{p}}} \left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right]^{\frac{p-1}{p}}. \quad (1.3)$$

Pearce and Pečarić [21], established the following result that gave an improvement and simplification of the constant in Theorem 2 and consolidate this result with Theorem 1 as Theorem 3 below. An analogous result, Theorem 4, is developed which relates in the same way to the first inequality in (1.1). Also, they develop analogous results base on concavity and apply them to special means and to estimates of the error term in the trapezoidal formula.

Theorem 3. [21] *Suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^\circ \rightarrow \mathbb{R}$ is differentiable. If $f' \in L(a, b)$ and $|f'|^q$, $q \geq 1$, is convex on $[a, b]$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}. \quad (1.4)$$

Theorem 4. [21] *Suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^\circ \rightarrow \mathbb{R}$ is differentiable. If $f' \in L(a, b)$ and $|f'|^q$, $q \geq 1$, is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}. \quad (1.5)$$

In a recent paper, Dah-Yang Hwang [12], established new inequalities of Hermite-Hadamard type for n -times differentiable convex and concave functions and obtained better estimates of those results established in Theorem 3 and Theorem 4.

The main result from [12] is pointed out as follows:

Theorem 5. [12] Suppose $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I° , $f^{(n)} \in L(a, b)$ for $n \in \mathbb{N}$, $n \geq 1$ and $|f^{(n)}|^q$, $q \geq 1$, then we have the inequality:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\ &= \frac{(n-1)^{1-\frac{1}{q}}(b-a)^n}{2(n+1)!} \left[\frac{(n^2-2)|f'(a)|^q + n|f'(b)|^q}{n+2} \right]^{\frac{1}{q}}. \end{aligned} \quad (1.6)$$

The following lemma, which generalize Lemma 1, was used to establish the above result:

Lemma 2. [12] Suppose $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$, $a, b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I° and $f^{(n)} \in L(a, b)$ for $n \in \mathbb{N}$, $n \geq 1$, then we have the identity:

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \\ &= \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(ta + (1-t)b) dt. \end{aligned}$$

For more recent results for n -times differentiable functions we refer the readers to the latest research work of Wei-Dong Jiang et. al [15] and Shu-Hong et al. [21] concerning inequalities for n -times differentiable s -convex and m -convex functions respectively (see the references in these papers as well).

In recent years, lot of efforts have been made by many mathematicians to extend and to generalize the classical convexity. These studies include among others the work of Hanson in [9], Ben-Israel and Mond [5] and Pini [22]. Hanson in [9], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [5], gave the concept of preinvex functions. Pini [22], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning invexity and preinvexity

Let K be a closed set in \mathbb{R}^n and let $f : K \rightarrow \mathbb{R}$ and $\eta : K \times K \rightarrow \mathbb{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot, \cdot)$, if

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η -connected set.

Definition 1. [30] The function f on the invex set K is said to be preinvex with respect to η , if

$$f(u + t\eta(v, u)) \leq (1-t)f(u) + tf(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every preinvex function is convex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [30].

Barani, Ghazanfari and Dragomir in [4], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved. These results generalize the results given above in Theorem 1 and Theorem 2.

Theorem 6. [4] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$. Suppose that $f : K \rightarrow \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:

$$\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \leq \frac{\eta(b, a)}{2(1+p)^{\frac{1}{p}}} \left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2} \right]^{\frac{p-1}{p}}. \quad (1.7)$$

Theorem 7. [4] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$. Suppose that $f : K \rightarrow \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:

$$\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \leq \frac{\eta(b, a)}{8} \left(\left|f'(a)\right| + \left|f'(b)\right| \right). \quad (1.8)$$

For more results on Hermite-Hadamard type inequalities for preinvex and log-preinvex functions, we refer the readers to the latest papers of M. Z. Sarikaya et al, [27] and Noor [18]-[20].

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities in Section 2 that are connected with the right-side and left-side of Hermite-Hadamard inequality for preinvex functions but for n -times differentiable preinvex functions which generalize those results established for differentiable preinvex functions and n -times differentiable convex functions.

2. MAIN RESULTS

In order to prove our main results, we need the following lemmas:

Lemma 3. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following equality holds:

$$\begin{aligned} & - \frac{f(a) + f(a + \eta(b, a))}{2} + \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ & + \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \\ & = \frac{(-1)^{n-1} (\eta(b, a))^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(a + t\eta(b, a)) dt, \quad (2.1) \end{aligned}$$

where the sum above takes 0 when $n = 1$ and $n = 2$.

Proof. The case $n = 1$ is the Lemma 2.1 from [4]. Suppose (2.1) holds for $n - 1$, i.e.

$$\begin{aligned} & -\frac{f(a) + f(a + \eta(b, a))}{2} + \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ & \quad + \sum_{k=2}^{n-2} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \\ & = \frac{(-1)^{n-2} (\eta(b, a))^{n-1}}{2(n-1)!} \int_0^1 t^{n-2} ((n-1) - 2t) f^{(n-1)}(a + t\eta(b, a)) dt. \end{aligned} \quad (2.2)$$

Now integrating by parts and using (2.2), we have

$$\begin{aligned} & \frac{(-1)^{n-1} (\eta(b, a))^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(a + t\eta(b, a)) dt \\ & = \frac{(-1)^{n-1} (n-2) (\eta(b, a))^{n-1}}{2n!} f^{(n-1)}(a + \eta(b, a)) \\ & \quad + \frac{(-1)^{n-2} (\eta(b, a))^{n-1}}{2(n-1)!} \int_0^1 t^{n-2} ((n-1) - 2t) f^{(n-1)}(a + t\eta(b, a)) dt \\ & = \frac{(-1)^{n-1} (n-2) (\eta(b, a))^{n-1}}{2n!} f^{(n-1)}(a + \eta(b, a)) \\ & \quad - \frac{f(a) + f(a + \eta(b, a))}{2} + \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ & \quad + \sum_{k=2}^{n-2} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \\ & = -\frac{f(a) + f(a + \eta(b, a))}{2} + \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ & \quad + \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)). \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 4. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ the following equality holds:

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b, a)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ & = \frac{(-1)^{n+1} (\eta(b, a))^n}{n!} \int_0^1 K_n(t) f^{(n)}(a + t\eta(b, a)) dt, \end{aligned} \quad (2.3)$$

where

$$K_n(t) := \begin{cases} t^n, & t \in [0, \frac{1}{2}] \\ (t-1)^n, & t \in (\frac{1}{2}, 1] \end{cases}.$$

Proof. For $n = 1$, we have by integration by parts that

$$\begin{aligned} & \eta(b, a) \int_0^1 K_1(t) f'(a + t\eta(b, a)) dt \\ &= \eta(b, a) \left[\int_0^{\frac{1}{2}} t f'(a + t\eta(b, a)) dt + \int_{\frac{1}{2}}^1 (t-1) f'(a + t\eta(b, a)) dt \right] \\ &= f\left(a + \frac{1}{2}\eta(b, a)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx. \end{aligned}$$

which is true. Suppose now that (2.3) is true for $n - 1$, i.e.

$$\begin{aligned} & \sum_{k=0}^{n-2} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b, a)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ &= \frac{(-1)^n (\eta(b, a))^{n-1}}{(n-1)!} \int_0^1 K_{n-1}(t) f^{(n-1)}(a + t\eta(b, a)) dt. \quad (2.4) \end{aligned}$$

Now by integration by parts and using (2.4), we have

$$\begin{aligned} & \frac{(-1)^{n+1} (\eta(b, a))^n}{n!} \int_0^1 K_n(t) f^{(n)}(a + t\eta(b, a)) dt \\ &= \frac{(-1)^{n+1} (\eta(b, a))^n}{n!} \left[\int_0^{\frac{1}{2}} t^n f^{(n)}(a + t\eta(b, a)) dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 (t-1)^n f^{(n)}(a + t\eta(b, a)) dt \right] \\ &= \frac{(-1)^{n+1} (\eta(b, a))^n}{n!} \left[\frac{f^{(n-1)}(a + \frac{1}{2}\eta(b, a))}{2^n \eta(b, a)} - \frac{n}{\eta(b, a)} \int_0^{\frac{1}{2}} t^{n-1} f^{(n-1)}(a + t\eta(b, a)) dt \right. \\ & \quad \left. - \frac{(-1)^n f^{(n-1)}(a + \frac{1}{2}\eta(b, a))}{2^n \eta(b, a)} - \frac{n}{\eta(b, a)} \int_{\frac{1}{2}}^1 (t-1)^{n-1} f^{(n-1)}(a + t\eta(b, a)) dt \right] \\ &= \frac{(-1)^{n-1} (\eta(b, a))^{n-1} f^{(n-1)}(a + \frac{1}{2}\eta(b, a))}{2^n n!} + \frac{(\eta(b, a))^{n-1} f^{(n-1)}(a + \frac{1}{2}\eta(b, a))}{2^n n!} \\ & \quad + \frac{(-1)^n (\eta(b, a))^{n-1}}{(n-1)!} \int_0^1 K_{n-1}(t) f^{(n-1)}(a + t\eta(b, a)) dt \\ &= \frac{[(-1)^{n-1} + 1] (\eta(b, a))^{n-1}}{2^n n!} f^{(n-1)}\left(a + \frac{1}{2}\eta(b, a)\right) \\ & \quad + \sum_{k=0}^{n-2} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b, a)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \\ &= \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b, a)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \end{aligned}$$

This completes the proof of the lemma. \square

We are now ready to give our first result.

Theorem 8. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $|f^{(n)}|$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have the following inequality:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right. \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right| \\ & \leq \frac{(\eta(b, a))^n}{2(n+1)!} \left(\frac{n |f^{(n)}(a)| + (n^2 - 2) |f^{(n)}(b)|}{n+2} \right). \end{aligned} \quad (2.5)$$

Proof. Suppose $n \geq 2$. Let $a, b \in K$. Since K is an invex set with respect to η , for every $t \in [0, 1]$ we have $a + t\eta(b, a) \in K$. By preinvexity of $|f^{(n)}|$ and Lemma 3, we get that

$$\begin{aligned} & \left| -\frac{f(a) + f(a + \eta(b, a))}{2} + \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right. \\ & \quad \left. + \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right| \\ & \leq \frac{(\eta(b, a))^n}{2n!} \int_0^1 t^{n-1} (n-2t) |f^{(n)}(a + t\eta(b, a))| dt \\ & \leq \frac{(\eta(b, a))^n}{2n!} \int_0^1 t^{n-1} (n-2t) (t |f^{(n)}(a)| + (1-t) |f^{(n)}(b)|) dt \\ & = \frac{(\eta(b, a))^n}{2n!} \left(|f^{(n)}(b)| \int_0^1 t^n (n-2t) dt + |f^{(n)}(a)| \int_0^1 t^{n-1} (n-2t) (1-t) dt \right). \end{aligned} \quad (2.6)$$

Since

$$\begin{aligned} & \int_0^1 t^n (n-2t) dt \\ & = \frac{n^2 - 2}{(n+1)(n+2)} \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t^{n-1} (n-2t) (1-t) dt \\ & = \frac{n}{(n+1)(n+2)}, \end{aligned}$$

we have from (2.6) the desired inequality (2.5).

This completes the proof of the theorem \square

Theorem 9. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $|f^{(n)}|^q$,

$q \geq 1$, is preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have the following inequality:

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right. \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right| \\ & \leq \frac{(\eta(b, a))^n (n-1)^{1-\frac{1}{q}}}{2(n+1)!} \left(\frac{(n^2-2) |f^{(n)}(a)|^q + n |f^{(n)}(b)|^q}{n+2} \right)^{\frac{1}{q}}. \end{aligned} \quad (2.7)$$

Proof. Suppose that $n \geq 2$. For $q = 1$, we get the inequality (2.5). Assume now that $q > 1$, then by the preinvexity of $|f^{(n)}|^q$ on K , Lemma 3 and the Hölder's inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right. \\ & \quad \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right| \\ & \leq \frac{(\eta(b, a))^n}{2n!} \left(\int_0^1 t^{n-1} (n-2t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-1} (n-2t) |f^{(n)}(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(\eta(b, a))^n}{2n!} \left(\int_0^1 t^{n-1} (n-2t) dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(|f^{(n)}(b)|^q \int_0^1 t^n (n-2t) dt + |f^{(n)}(a)|^q \int_0^1 t^{n-1} (n-2t) (1-t) dt \right)^{\frac{1}{q}}. \end{aligned} \quad (2.8)$$

Since

$$\int_0^1 t^n (n-2t) dt = \frac{n^2-2}{(n+1)(n+2)},$$

$$\int_0^1 t^n (n-2t) dt = \frac{n-1}{n+1}$$

and

$$\int_0^1 t^{n-1} (n-2t) (1-t) dt = \frac{n}{(n+1)(n+2)},$$

we have from (2.8) that

$$\begin{aligned}
 & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right. \\
 & \quad \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right| \\
 & \leq \frac{(\eta(b, a))^n}{2n!} \left(\int_0^1 t^{n-1} (n-2t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-1} (n-2t) |f^{(n)}(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\
 & \leq \frac{(\eta(b, a))^n (n-1)^{1-\frac{1}{q}}}{2(n+1)!} \left(\frac{n|f^{(n)}(a)|^q + (n^2-2)|f^{(n)}(b)|^q}{n+2} \right)^{\frac{1}{q}}. \quad (2.9)
 \end{aligned}$$

Hence the proof of the theorem is complete. \square

Corollary 1. *Suppose the assumptions of Theorem 9 are satisfied then for $n = 2$, we have*

$$\begin{aligned}
 & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\
 & \leq \frac{(\eta(b, a))^2}{12} \left(\frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}. \quad (2.10)
 \end{aligned}$$

Corollary 2. *If we take $q = 1$ in (2.10), we obtain*

$$\begin{aligned}
 & \left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx \right| \\
 & \leq \frac{(\eta(b, a))^2}{12} \left(\frac{|f''(a)| + |f''(b)|}{2} \right). \quad (2.11)
 \end{aligned}$$

We note that the bound in (2.11) may be better than the bound in (1.8).

Corollary 3. *If $|f^{(n)}|^q$ is preinvex on K with respect to the function $\eta(y, x) = y - x$, $q \geq 1$, $n \in \mathbb{N}$, $n \geq 2$. Then $|f^{(n)}|^q$ is convex on K , $q \geq 1$, $n \in \mathbb{N}$, $n \geq 2$ and hence the inequality (2.7) becomes*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (b-a)^k}{2(k+1)!} f^{(k)}(b) \right| \\
 & \leq \frac{(b-a)^n (n-1)^{1-\frac{1}{q}}}{2(n+1)!} \left(\frac{n|f^{(n)}(a)|^q + (n^2-2)|f^{(n)}(b)|^q}{n+2} \right)^{\frac{1}{q}}. \quad (2.12)
 \end{aligned}$$

Now we give some results related to left-side of Hermite-Hadamard's inequality for n -times differentiable preinvex functions.

Theorem 10. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $|f^{(n)}|$ is*

preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have the following inequality:

$$\left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \leq \frac{(\eta(b, a))^n}{2^{n+1} (n+1)!} \left[|f^{(n)}(a)| + |f^{(n)}(b)| \right]. \quad (2.13)$$

Proof. Suppose $n \geq 1$. By using Lemma 4 and the preinvexity of $|f^{(n)}|$ on K for $n \in \mathbb{N}$, $n \geq 1$, we have

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^n}{n!} \left[\int_0^{\frac{1}{2}} t^n |f^{(n)}(a + t\eta(b, a))| dt + \int_{\frac{1}{2}}^1 (1-t)^n |f^{(n)}(a + t\eta(b, a))| dt \right] \\ & \leq \frac{(\eta(b, a))^n}{n!} \left[\int_0^{\frac{1}{2}} t^n \left((1-t) |f^{(n)}(a)| + t |f^{(n)}(b)| \right) dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 (1-t)^n \left((1-t) |f^{(n)}(a)| + t |f^{(n)}(b)| \right) dt \right] \\ & = \frac{(\eta(b, a))^n}{n!} \left[|f^{(n)}(a)| \left(\int_0^{\frac{1}{2}} t^n (1-t) dt + \int_{\frac{1}{2}}^1 (1-t)^{n+1} dt \right) \right. \\ & \quad \left. + |f^{(n)}(b)| \left(\int_0^{\frac{1}{2}} t^{n+1} dt + \int_{\frac{1}{2}}^1 t (1-t)^n dt \right) \right]. \quad (2.14) \end{aligned}$$

Since

$$\int_0^{\frac{1}{2}} t^n (1-t) dt + \int_{\frac{1}{2}}^1 (1-t)^{n+1} dt = \frac{1}{2^{n+1} (n+1)}$$

and

$$\int_0^{\frac{1}{2}} t^{n+1} dt + \int_{\frac{1}{2}}^1 t (1-t)^n dt = \frac{1}{2^{n+1} (n+1)},$$

we get from (2.14) the desired inequality (2.13). This completes the proof of the theorem. \square

The following results contains the powers of the absolute values of the n the derivative of the preinvex function.

Theorem 11. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $|f^{(n)}|^{\frac{p}{p-1}}$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $p > 1$, then for every $a, b \in K$ with*

$\eta(b, a) \neq 0$ we have the following inequality:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^n}{2^{n+\frac{1}{p}} (np+1)^{\frac{1}{p}} n!} \left[\left(\frac{3 |f^{(n)}(a)|^{\frac{p}{p-1}} + |f^{(n)}(b)|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right. \\ & \quad \left. + \left(\frac{|f^{(n)}(a)|^{\frac{p}{p-1}} + 3 |f^{(n)}(b)|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right]. \quad (2.15) \end{aligned}$$

Proof. From Lemma 4 and the Hölder's integral inequality, we have

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^n}{n!} \left[\left(\int_0^{\frac{1}{2}} t^{np} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f^{(n)}(a + t\eta(b, a))|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 (1-t)^{np} dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |f^{(n)}(a + t\eta(b, a))|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \right]. \quad (2.16) \end{aligned}$$

Since $|f^{(n)}|^{\frac{p}{p-1}}$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $p > 1$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$, we have

$$\begin{aligned} & \int_0^{\frac{1}{2}} |f^{(n)}(a + t\eta(b, a))|^{\frac{p}{p-1}} dt \\ & \leq |f^{(n)}(a)|^{\frac{p}{p-1}} \int_0^{\frac{1}{2}} (1-t) dt + |f^{(n)}(b)|^{\frac{p}{p-1}} \int_0^{\frac{1}{2}} t dt \\ & = \frac{3}{8} |f^{(n)}(a)|^{\frac{p}{p-1}} + \frac{1}{8} |f^{(n)}(b)|^{\frac{p}{p-1}} \quad (2.17) \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^1 |f^{(n)}(a + t\eta(b, a))|^{\frac{p}{p-1}} dt \\ & \leq |f^{(n)}(a)|^{\frac{p}{p-1}} \int_{\frac{1}{2}}^1 (1-t) dt + |f^{(n)}(b)|^{\frac{p}{p-1}} \int_{\frac{1}{2}}^1 t dt \\ & = \frac{1}{8} |f^{(n)}(a)|^{\frac{p}{p-1}} + \frac{3}{8} |f^{(n)}(b)|^{\frac{p}{p-1}}. \quad (2.18) \end{aligned}$$

Also

$$\int_0^{\frac{1}{2}} t^{np} dt = \int_{\frac{1}{2}}^1 (1-t)^{np} dt = \frac{1}{2^{np+1} (np+1)}. \quad (2.19)$$

Using (2.17), (2.18) and (2.19) in (2.16), we get the required inequality (2.15). This completes the proof of the theorem. \square

Theorem 12. Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < \eta(b, a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $|f^{(n)}|^q$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $q \geq 1$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have the following inequality:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k f^{(k)}(a + \frac{1}{2}\eta(b, a))}{2^{k+1} (k+1)!} \right. \\ & \quad \left. - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^n}{2^{n+1} (n+1)!} \left[\left(\frac{(n+3) |f^{(n)}(a)|^q + (n+1) |f^{(n)}(b)|^q}{2(n+2)} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(n+1) |f^{(n)}(a)|^q + (n+3) |f^{(n)}(b)|^q}{2(n+2)} \right)^{\frac{1}{q}} \right]. \quad (2.20) \end{aligned}$$

Proof. The case $q = 1$ is the Theorem 10. Suppose $q > 1$, then from Lemma 4 and the power-mean integral inequality, we have

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k f^{(k)}(a + \frac{1}{2}\eta(b, a))}{2^{k+1} (k+1)!} \right. \\ & \quad \left. - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^n}{n!} \left[\left(\int_0^{\frac{1}{2}} t^n dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} t^n |f^{(n)}(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt \right)^{\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 |f^{(n)}(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right]. \quad (2.21) \end{aligned}$$

Since $|f^{(n)}|^q$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $q > 1$, then for every $a, b \in K$ with $\eta(b, a) \neq 0$, we have

$$\begin{aligned} & \int_0^{\frac{1}{2}} t^n |f^{(n)}(a + t\eta(b, a))|^q dt \\ & \leq |f^{(n)}(a)|^q \int_0^{\frac{1}{2}} t^n (1-t) dt + |f^{(n)}(b)|^q \int_0^{\frac{1}{2}} t^{n+1} dt \\ & = \frac{(n+3) |f^{(n)}(a)|^q + (n+1) |f^{(n)}(b)|^q}{2^{n+2} (n+1) (n+2)} \quad (2.22) \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\frac{1}{2}} (1-t)^n \left| f^{(n)}(a+t\eta(b,a)) \right|^q dt \\ & \leq \left| f^{(n)}(a) \right|^q \int_{\frac{1}{2}}^1 (1-t)^{n+1} dt + \left| f^{(n)}(b) \right|^q \int_{\frac{1}{2}}^1 t(1-t)^n dt \\ & = \frac{(n+1) \left| f^{(n)}(a) \right|^q + (n+3) \left| f^{(n)}(b) \right|^q}{2^{n+2} (n+1) (n+2)}. \end{aligned} \quad (2.23)$$

Also

$$\int_0^{\frac{1}{2}} t^n dt = \int_{\frac{1}{2}}^1 (1-t)^n dt = \frac{1}{2^{n+1} (n+1)}. \quad (2.24)$$

A usage of (2.22), (2.23) and (2.24) in (2.21) gives us the required inequality (2.20). This completes the proof of the theorem. \square

Remark 1. If we take $n = 1$ in Theorem 10, Theorem 11 and Theorem 12, we obtain those results proved in [27] for differentiable preinvex functions.

Corollary 4. Under the assumptions of Theorem 11, if f is preinvex on K with respect to the function $\eta(y, x) = y - x$, $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$. Then $\left| f^{(n)} \right|^{\frac{p-1}{p}}$ is convex on K , $p > 1$, $p \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 1$ and hence we have the following inequality:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (b-a)^k}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^n}{2^{n+\frac{1}{p}} (np+1)^{\frac{1}{p}} n!} \left[\left(\frac{3 \left| f^{(n)}(a) \right|^{\frac{p}{p-1}} + \left| f^{(n)}(b) \right|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right. \\ & \quad \left. + \left(\frac{\left| f^{(n)}(a) \right|^{\frac{p}{p-1}} + 3 \left| f^{(n)}(b) \right|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right]. \end{aligned} \quad (2.25)$$

Corollary 5. Under the assumptions of Theorem 12, if f is preinvex on K with respect to the function $\eta(y, x) = y - x$, $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$. Then $\left| f^{(n)} \right|^q$ is convex on K , $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$ and hence we have the following inequality:

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^n}{2^{n+1} (n+1)!} \left[\left(\frac{(n+3) \left| f^{(n)}(a) \right|^q + (n+1) \left| f^{(n)}(b) \right|^q}{2(n+2)} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{(n+1) \left| f^{(n)}(a) \right|^q + (n+3) \left| f^{(n)}(b) \right|^q}{2(n+2)} \right)^{\frac{1}{q}} \right]. \end{aligned} \quad (2.26)$$

Corollary 6. *If we take $q = 1$ in (2.26), we get the following inequality:*

$$\left| \sum_{k=0}^{n-1} \frac{[(-1)^k + 1] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^n}{2^n n!} \left[\frac{|f^{(n)}(a)| + |f^{(n)}(b)|}{2(n+1)} \right]. \quad (2.27)$$

Corollary 7. *If the conditions of Theorem 10, Theorem 11 and Theorem 12 are satisfied then for $n = 2$, we have the following inequalities respectively:*

$$\left| f \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \leq \frac{(\eta(b, a))^2}{48} [|f''(a)| + |f''(b)|]. \quad (2.28)$$

$$\begin{aligned} & \left| f \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^2}{8 \cdot 2^{\frac{1}{p}} (2p+1)^{\frac{1}{p}}} \left[\left(\frac{3 |f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right. \\ & \quad \left. + \left(\frac{|f''(a)|^{\frac{p}{p-1}} + 3 |f''(b)|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right], p > 1. \quad (2.29) \end{aligned}$$

$$\begin{aligned} & \left| f \left(a + \frac{1}{2} \eta(b, a) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\ & \leq \frac{(\eta(b, a))^2}{48} \left[\left(\frac{5 |f^{(n)}(a)|^q + 3 |f^{(n)}(b)|^q}{8} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{3 |f''(a)|^q + 5 |f''(b)|^q}{8} \right)^{\frac{1}{q}} \right], q \geq 1. \quad (2.30) \end{aligned}$$

Remark 2. *It may be noted that the inequalities (2.28), (2.29) and (2.30) may give better bounds than those proved in [27].*

3. APPLICATIONS TO SPECIAL MEANS

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 2. [2] *A function $M : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:*

- (1) *Homogeneity:* $M(ax, ay) = aM(x, y)$, for all $a > 0$,
- (2) *Symmetry:* $M(x, y) = M(y, x)$,

- (3) *Reflexivity* : $M(x, x) = x$,
 (4) *Monotonicity*: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
 (5) *Internality*: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β (see for instance [2]).

- (1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

- (2) The The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

- (3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

- (4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1$$

- (5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right), & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}$$

- (6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|$$

- (7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right], \quad \alpha \neq \beta, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \rightarrow \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtained variant inequalities for these means as follows:

Setting $\eta(b, a) = M(b, a)$ in (2.11), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} \left| \frac{f(a) + f(a + M(b, a))}{2} - \frac{1}{M(b, a)} \int_a^{a+M(b, a)} f(x) dx \right| \\ \leq \frac{(M(b, a))^2}{24} \left(|f''(a)| + |f''(b)| \right). \quad (3.1) \end{aligned}$$

$$\left| f\left(a + \frac{1}{2}M(b, a)\right) - \frac{1}{M(b, a)} \int_a^{a+M(b, a)} f(x) dx \right| \leq \frac{(M(b, a))^2}{48} \left[|f''(a)| + |f''(b)| \right]. \quad (3.2)$$

$$\left| f\left(a + \frac{1}{2}M(b, a)\right) - \frac{1}{M(b, a)} \int_a^{a+M(b, a)} f(x) dx \right| \leq \frac{(M(b, a))^2}{8 \cdot 2^{\frac{1}{p}} (2p+1)^{\frac{1}{p}}} \left[\left(\frac{3|f''(a)|^{\frac{p}{p-1}} + |f''(b)|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} + \left(\frac{|f''(a)|^{\frac{p}{p-1}} + 3|f''(b)|^{\frac{p}{p-1}}}{8} \right)^{\frac{p-1}{p}} \right], p > 1. \quad (3.3)$$

$$\left| f\left(a + \frac{1}{2}M(b, a)\right) - \frac{1}{M(b, a)} \int_a^{a+M(b, a)} f(x) dx \right| \leq \frac{(M(b, a))^2}{48} \left[\left(\frac{5|f^{(n)}(a)|^q + 3|f^{(n)}(b)|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{3|f''(a)|^q + 5|f''(b)|^q}{8} \right)^{\frac{1}{q}} \right], q \geq 1. \quad (3.4)$$

Letting $M = A, G, H, P_r, I, L, L_p$ in (3.1), (3.2), (3.3) and (3.4), we get the inequalities involving means, and the details are left to the interested reader.

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