Received 20/02/13

SOME APPLICATIONS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS (I)

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Abstract. Some applications of Fejér's inequality for convex functions are explored. Upper and lower bounds for the weighted integral

$$\int_a^b (b-x)(x-a) f(x)\,dx$$

under various assumptions for f with applications to the trapezoidal quadrature rule are given. Some inequalities for special means are also provided.

1. Introduction

The *Hermite-Hadamard* integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2} \tag{HH}$$

is well known in the literature and has many applications for special means.

For related results, see for instance the research papers [1], [8], [9], [10], [12], [11], [13], [14], [15], the monograph online [7] and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(a+t)=w(b-t),\ \ 0\le t\le\frac{1}{2}(a+b),$$

i.e., $y=w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\le\int_a^b h(x)\,w(x)\,dx\le\frac{h(a)+h(b)}{2}\int_a^b w(x)\,dx. \tag{1.1}$$

If h is concave on (a,b), then the inequalities reverse in (1.1).

Clearly, for $w(x)\equiv 1$ on $[a,b]$ we get HH.

We observe that, if we take $w(x)=(b-x)(x-a),x\in[a,b]$, then w satisfies the conditions in Theorem 1,

$$\int_a^b (b-x)(x-a)\,dx=\frac{1}{6}(b-a)^3$$

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Convex functions, Hermite-Hadamard inequality, Fejér's Inequality, Special means.

and by 1.1 we have the following inequality

$$\frac{1}{6}h\left(\frac{a+b}{2}\right)(b-a)^{3}\leq\int_{a}^{b}(b-x)(x-a)h(x)\,dx \leq\frac{h(a)+h(b)}{12}(b-a)^{3}, \tag{1.2}$$

for any convex function $h:[a,b]\to\mathbb{R}$. If the function h is concave the inequalities in (1.2) reverse.

In this paper we establish amongst other some better bounds for the weighted integral

$$\int_{a}^{b}(b-x)(x-a)h(x)\,dx$$

in the case of convex functions $h:[a,b]\to\mathbb{R}$. We also investigate the connection with the trapezoid rule and apply some of the obtained results for special means.

2. The Results

The following result holds.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and such that the second derivative f'' is convex on (a,b). Then*

$$\frac{1}{12}f''\left(\frac{a+b}{2}\right)(b-a)^{2} \leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx \leq \frac{f''(a)+f''(b)}{24}(b-a)^{2}. \tag{2.1}$$

Proof. We know, see for instance [7, Lemma 4, p. 38], that

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx=\frac{1}{2(b-a)}\int_{a}^{b}(x-a)(b-x)f''(x)\,dx. \tag{2.2}$$

Since f'' is convex on (a,b), then by (1.2) we have

$$\frac{1}{6}f''\left(\frac{a+b}{2}\right)(b-a)^{3}\leq\int_{a}^{b}(b-x)(x-a)f''(x)\,dx \leq\frac{f''(a)+f''(b)}{12}(b-a)^{3}. \tag{2.3}$$

Utilising (2.2) and (2.3) we deduce the desired result (2.1). □

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b).*

If there exists a real number m such that $f''(x)\geq m$ for any $x\in(a,b)$, then

$$\begin{aligned}&\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^{3}+\frac{1}{240}m(b-a)^{5}\\&\leq\int_{a}^{b}(b-x)(x-a)f(x)\,dx\\&\leq\frac{f(a)+f(b)}{12}(b-a)^{3}-\frac{1}{60}m(b-a)^{5},\end{aligned} \tag{2.4}$$

If there exists a real number M such that $f''(x) \le M$ for any $x \in (a,b)$, then

$$\begin{aligned}(2.5)\qquad & \frac{f(a)+f(b)}{12}(b-a)^3 - \frac{1}{60}M(b-a)^5 \\ & \le \int_a^b (b-x)(x-a)f(x)\,dx \\ & \le \frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{240}M(b-a)^5 .\end{aligned}$$

Proof. Define the function $h_m : [a,b] \to \mathbb{R}$ by

$$h_m(x) := f(x) + \frac{1}{2}m(x-a)(b-x).$$

This function is twice differentiable and the second derivative is

$$h_m''(x) = f''(x) - m \ge 0,\ x \in (a,b)$$

showing that h_m is convex on $[a,b]$.

If we apply the inequality (1.2) for h_m, then we have

$$\begin{aligned}(2.6)\qquad & \frac{1}{6}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{8}m(b-a)^2\right](b-a)^3 \\ & \le \int_a^b (b-x)(x-a)f(x)\,dx + \frac{1}{2}m\int_a^b (b-x)^2(x-a)^2\,dx \\ & \le \frac{f(a)+f(b)}{12}(b-a)^3 .\end{aligned}$$

Observe that

$$\begin{aligned}& \frac{1}{6}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{8}m(b-a)^2\right](b-a)^3 \\ & = \frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{48}m(b-a)^5 .\end{aligned}$$

We also have

$$\begin{aligned}\int_a^b (b-x)^2(x-a)^2\,dx &= \frac{1}{3}(x-a)^3(b-x)^2\Big|_a^b + \frac{2}{3}\int_a^b (b-x)(x-a)^3\,dx \\ &= \frac{2}{3}\left[\frac{1}{4}(b-x)(x-a)^4\Big|_a^b + \frac{1}{4}\int_a^b (x-a)^4\,dx\right] \\ &= \frac{1}{30}(b-a)^5 .\end{aligned}$$

Then (2.6) becomes

$$\begin{aligned}& \frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{48}m(b-a)^5 \\ & \le \int_a^b (b-x)(x-a)f(x)\,dx + \frac{1}{60}m(b-a)^5 \\ & \le \frac{f(a)+f(b)}{12}(b-a)^3\end{aligned}$$

which is equivalent with (2.4).

Now define the function $h_M : [a,b] \to \mathbb{R}$ by

$$h_M(x) := -f(x) - \frac{1}{2}M(x-a)(b-x).$$

This function is twice differentiable and

$$h''_M(x) := M - f''(x) \geq 0, \ x \in (a,b)$$

showing that h_M is convex on $[a,b]$.

If we apply the inequality (1.2) for h_M, then we have

$$\begin{aligned}
&\frac{1}{6}\left[-f\left(\frac{a+b}{2}\right) - \frac{1}{8}M(b-a)^2\right](b-a)^3 \\
&\leq \int_a^b (b-x)(x-a)\left[-f(x) - \frac{1}{2}M(x-a)(b-x)\right]dx \\
&\leq \frac{-f(a)-f(b)}{12}(b-a)^3,
\end{aligned}$$

which, by multiplication with -1, produces

$$\begin{aligned}
&\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{48}M(b-a)^5 \\
&\geq \int_a^b (b-x)(x-a)f(x)\,dx + \frac{1}{2}M\int_a^b (x-a)^2(b-x)^2\,dx \\
&\geq \frac{f(a)+f(b)}{12}(b-a)^3
\end{aligned}$$

that is equivalent with

$$\begin{aligned}
&\frac{f(a)+f(b)}{12}(b-a)^3 - \frac{1}{60}M(b-a)^5 \\
&\leq \int_a^b (b-x)(x-a)f(x)\,dx \\
&\leq \frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 + \frac{1}{240}M(b-a)^5
\end{aligned}$$

and the inequality (2.5) is proved. □

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b). If there exists a $K > 0$ such that $|f''(x)| \leq K$ for any $x \in (a,b)$, then*

$$\begin{aligned}
&\left|\int_a^b (b-x)(x-a)f(x)\,dx - \frac{1}{12}(b-a)^3\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\right| \\
&\leq \frac{1}{96}K(b-a)^5.
\end{aligned} \tag{2.7}$$

Proof. If we write the inequality (2.4) for $m = -K$ and the inequality (2.5) for $M = K$ we have

$$\begin{aligned}
&\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3-\frac{1}{240}K(b-a)^5 \\
&\leq\int_a^b(b-x)(x-a)f(x)\,dx \\
&\leq\frac{f(a)+f(b)}{12}(b-a)^3+\frac{1}{60}K(b-a)^5,
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
&\frac{f(a)+f(b)}{12}(b-a)^3-\frac{1}{60}K(b-a)^5 \\
&\leq\int_a^b(b-x)(x-a)f(x)\,dx \\
&\leq\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{1}{240}K(b-a)^5.
\end{aligned}\tag{2.9}$$

If we add the inequality (2.8) with (2.8) and divide the sum by 2 we get

$$\begin{aligned}
&\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{f(a)+f(b)}{24}(b-a)^3-\frac{1}{96}K(b-a)^5 \\
&\leq\int_a^b(b-x)(x-a)f(x)\,dx \\
&\leq\frac{1}{12}f\left(\frac{a+b}{2}\right)(b-a)^3+\frac{f(a)+f(b)}{24}(b-a)^3+\frac{1}{96}K(b-a)^5,
\end{aligned}$$

which is equivalent with the desired result (2.7). □

Remark 1. *We observe that the case $m>0$ in the inequality (2.4) produces a better result than (1.2).*

For twice differentiable functions we can provide the following *perturbed trapezoid quadrature rule*

$$\begin{aligned}
\int_a^b f(x)\,dx &\simeq\frac{f(a)+f(b)}{2}(b-a) \\
&\quad-\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right].
\end{aligned}\tag{2.10}$$

Denote $R_{P,T}(f;a,b)$ the error in approximating the integral as in (2.10), namely

$$\begin{aligned}
R_{P,T}(f;a,b)&:=\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}(b-a) \\
&\quad+\frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right].
\end{aligned}$$

The following result that provides an *a priory* error bound for functions whose forth derivatives are bounded, holds.

Proposition 1. *Let $f:[a,b]\to\mathbb{R}$ be a four time differentiable function on (a,b). If there exists a $K>0$ such that $\left|f^{(4)}(x)\right|\leq K$ for any $x\in(a,b)$, then*

$$|R_{P,T}(f;a,b)|\leq\frac{1}{192}K(b-a)^5.\tag{2.11}$$

Proof. Writing the inequality (2.7) for the second derivative f'' we have

$$\left|\int_a^b (b-x)(x-a) f''(x)\,dx - \frac{1}{12}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right]\right| \le \frac{1}{96}K(b-a)^5.$$

Dividing this inequality by 2 and utilizing the representation (2.2) we have

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx - \frac{1}{24}(b-a)^3\left[f''\left(\frac{a+b}{2}\right)+\frac{f''(a)+f''(b)}{2}\right]\right| \le \frac{1}{192}K(b-a)^5,$$

and the inequality (2.11) is proved. □

The following result that improves the inequality (1.2) also holds.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then*

$$\begin{aligned}(2.12)\qquad \frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 &\le 2\int_a^b \left(x-\frac{a+b}{2}\right)^2 f\left(\frac{x+\frac{a+b}{2}}{2}\right)dx \\ &\le \int_a^b (b-x)(x-a) f(x)\,dx \\ &\le \int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx + \frac{(b-a)^3}{12} f\left(\frac{a+b}{2}\right) \\ &\le \frac{(b-a)^3}{12}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\le \frac{f(a)+f(b)}{12}(b-a)^3.\end{aligned}$$

Proof. Denote, as usual, $F(x) := \int_a^x f(t)\,dt$, $x\in[a,b]$. By the Hermite-Hadamard inequality we have for any $x\in[a,b]$, $x\ne\frac{a+b}{2}$ that

$$f\left(\frac{x+\frac{a+b}{2}}{2}\right) \le \frac{F(x)-F\left(\frac{a+b}{2}\right)}{x-\frac{a+b}{2}} \le \frac{1}{2}\left[f(x)+f\left(\frac{a+b}{2}\right)\right],$$

which, by multiplication with $\left(x-\frac{a+b}{2}\right)^2\ge 0$ implies

$$\begin{aligned}(2.13)\qquad & f\left(\frac{x+\frac{a+b}{2}}{2}\right)\left(x-\frac{a+b}{2}\right)^2 \\ &\le \left[F(x)-F\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right) \\ &\le \frac{1}{2}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)^2,\end{aligned}$$

that holds for any $x \in [a,b]$.

Integrating the inequality (2.13) on the interval $[a,b]$ we get

$$\begin{aligned}
&\int_a^b \left(x-\frac{a+b}{2}\right)^2 f\left(\frac{x+\frac{a+b}{2}}{2}\right) dx \qquad (2.14)\\
&\le \int_a^b \left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right) dx\\
&\le \frac{1}{2}\int_a^b \left[f(x) + f\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)^2 dx\\
&= \frac{1}{2}\left[\int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx + f\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{12}\right].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\int_a^b \left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right) dx\\
&= \int_a^b F(x)\left(x-\frac{a+b}{2}\right) dx = \frac{1}{2}\int_a^b F(x)\, d\left(x-\frac{a+b}{2}\right)^2\\
&= \frac{1}{2}\left[F(x)\left(x-\frac{a+b}{2}\right)^2\Big|_a^b - \int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx\right]\\
&= \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 \int_a^b f(x) - \int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx\right]\\
&= \frac{1}{2}\int_a^b \left[\left(\frac{b-a}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)^2\right] f(x)\,dx\\
&= \frac{1}{2}\int_a^b (b-x)(x-a) f(x)\,dx
\end{aligned}$$

and by (2.14) we have

$$\begin{aligned}
&\int_a^b \left(x-\frac{a+b}{2}\right)^2 f\left(\frac{x+\frac{a+b}{2}}{2}\right) dx\\
&\le \frac{1}{2}\int_a^b (b-x)(x-a) f(x)\,dx\\
&= \frac{1}{2}\left[\int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx + f\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{12}\right],
\end{aligned}$$

which proves the second and the third inequality in (2.12).

The function $g(x) := f\left(\frac{x+\frac{a+b}{2}}{2}\right)$ is convex on $[a,b]$ and $w(x) := \left(x - \frac{a+b}{2}\right)^2$ is nonnegative and symmetric on $[a,b]$. Applying Fejér's first inequality we have

$$f\left(\frac{\frac{a+b}{2}+\frac{a+b}{2}}{2}\right)\int_a^b \left(x-\frac{a+b}{2}\right)^2 dx \le \int_a^b f\left(\frac{x+\frac{a+b}{2}}{2}\right)\left(x-\frac{a+b}{2}\right)^2 dx$$

i.e.

$$\frac{(b-a)^3}{12} f\left(\frac{a+b}{2}\right) \le \int_a^b \left(x - \frac{a+b}{2}\right)^2 f\left(\frac{x+\frac{a+b}{2}}{2}\right) dx,$$

which proves the first inequality in (2.12).

From the Fejér's second inequality for the convex function f function and the weight $w(x) := \left(x - \frac{a+b}{2}\right)^2$ we also have

$$\begin{aligned}\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx &\le \frac{f(a)+f(b)}{2} \int_a^b \left(x - \frac{a+b}{2}\right)^2 dx \\ &= \frac{f(a)+f(b)}{24}(b-a)^3,\end{aligned}$$

which proves the fourth inequality in (2.12).

The last inequality is obvious. □

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b) and such that the second derivative f'' is convex on (a,b). Then*

(2.15)

$$\begin{aligned}\frac{1}{12} f''\left(\frac{a+b}{2}\right)(b-a)^2 &\le \int_a^b \left(x - \frac{a+b}{2}\right)^2 f''\left(\frac{x+\frac{a+b}{2}}{2}\right) dx \\ &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{1}{2}\int_a^b \left(x - \frac{a+b}{2}\right)^2 f''(x)\,dx + \frac{(b-a)^3}{24} f''\left(\frac{a+b}{2}\right) \\ &\le \frac{(b-a)^3}{24}\left[f''\left(\frac{a+b}{2}\right) + \frac{f''(a)+f''(b)}{2}\right] \\ &\le \frac{f''(a)+f''(b)}{24}(b-a)^3.\end{aligned}$$

We observe that the inequality (2.15) is a better result than (2.1).

3. Applications for Special Means

Let us recall the following means for two positive numbers.

(1) *The Arithmetic mean*

$$A = A(a,b) := \frac{a+b}{2},\ a,b > 0;$$

(2) *The Geometric mean*

$$G = G(a,b) := \sqrt{ab},\ a,b > 0;$$

(3) *The Harmonic mean*

$$H = H(a,b) := \frac{2ab}{a+b},\ a,b > 0;$$

(4) *The Logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \ne b; \end{cases},\ a,b > 0,$$

(5) *The Identric mean*

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b \end{cases}, \; a,b > 0;$$

(6) *The p-Logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} a & \text{if} \quad a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \end{cases}, \quad a,b > 0.$$

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A. \tag{3.1}$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = x^p$ for $p \geq 3$. We have the fourth derivative of the function given by

$$f^{(4)}(x) = p(p-1)(p-2)(p-3)x^{p-4}$$

which shows that the second derivative f'' is convex on $[a,b]$. Applying the inequality (2.1) we have

$$\begin{aligned} \frac{1}{12}p(p-1)\left(\frac{a+b}{2}\right)^{p-2}(b-a)^2 &\leq \frac{a^p+b^p}{2} - \frac{1}{b-a}\int_a^b x^p dx \\ &\leq p(p-1)\frac{a^{p-2}+b^{p-2}}{24}(b-a)^2, \end{aligned}$$

which in terms of the special means define above can be written as

$$\begin{aligned} \frac{1}{12}p(p-1)A^{p-2}(a,b)(b-a)^2 &\leq A(a^p,b^p) - L_p^p(a,b) \\ &\leq \frac{1}{12}p(p-1)A\left(a^{p-2},b^{p-2}\right)(b-a)^2, \end{aligned} \tag{3.2}$$

that holds for any $a,b > 0$ and $p \geq 3$.

Consider the function $f : [a,b] \subset (0,\infty) \to (0,\infty)$, $f(x) = \frac{1}{x}$. Then $f''(x) = \frac{2}{x^3}$ and $f^{(4)}(x) = \frac{24}{x^5}$ showing that the second derivative is convex on $[a,b]$. Applying the inequality (2.1) we have

$$\begin{aligned} \frac{1}{6}\frac{(b-a)^2}{A^3(a,b)} &\leq \frac{\frac{1}{a}+\frac{1}{b}}{2} - \frac{\ln b - \ln a}{b-a} \\ &\leq \frac{\frac{2}{a^3}+\frac{2}{b^3}}{24}(b-a)^2. \end{aligned}$$

which is equivalent with

$$\frac{1}{6}\frac{(b-a)^2}{A^3(a,b)} \leq \frac{L(a,b) - H(a,b)}{L(a,b)H(a,b)} \leq \frac{1}{6}\frac{(b-a)^2}{H(a^3,b^3)} \tag{3.3}$$

that holds for any $a,b > 0$.

Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=-\ln x$. Then $f''(x)=\frac{1}{x^2}$ and $f^{(4)}(x)=\frac{6}{x^4}$ showing that the second derivative is convex on $[a,b]$. Applying the inequality (2.1) we have

$$\begin{aligned}\frac{1}{12}\frac{(b-a)^2}{A^2(a,b)} &\le \frac{-\ln a-\ln b}{2}+\frac{1}{b-a}\int_a^b \ln x dx \\ &\le \frac{\frac{1}{a^2}+\frac{1}{b^2}}{24}(b-a)^2 .\end{aligned}$$

Observe that

$$\begin{aligned}\frac{1}{b-a}\int_a^b \ln x dx &= \frac{1}{b-a}\left[x\ln x|_a^b-(b-a)\right]= \\ &= \left[\ln\left(\frac{b^b}{a^a}\right)^{1/(b-a)}-1\right]=\ln I(a,b),\end{aligned}$$

and

$$\frac{-\ln a-\ln b}{2}=\ln\frac{1}{G(a,b)}.$$

Then we get

$$\frac{1}{12}\frac{(b-a)^2}{A^2(a,b)}\le\ln\left(\frac{I(a,b)}{G(a,b)}\right)\le\frac{1}{12}\frac{(b-a)^2}{H(a^2,b^2)} \tag{3.4}$$

that holds for any $a,b>0$.

The interested reader may apply the inequality (2.11) or (2.15) to obtain other similar results. However, the details are omitted here.

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