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SOME NEW FRACTIONAL INTEGRAL HERMITE−HADAMARD TYPE INEQUALITIES

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Abstract. We establish some new fractional integral inequalities of Hermite−Hadamard type for functions whose absolute values of second derivatives are convex and concave . The obtained results generalize the existing Hermite−Hadamard type integral inequalities.

1. Introduction and Defintions

[5] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

Both inequalities hold in the reversed direction for f to be concave. It is well known that the Hermite−Hadamard inequality plays an important role in nonlinear analysis. In the recent years, this classical inequality has been improved and generalized in a number of ways and a large number of research papers have been written on this inequality, (see, [1]−[5] and [7]−[10]) and the references therein.

In recent paper, [10] Sarikaya et. al. proved a variant of Hermite−Hadamard's inequalities in fractional integral forms as follows:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2}$$

Remark 1. *For $\alpha = 1$, inequality 2 reduces to inequality 1.*

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let $f \in L[a, b]$, the Reimann-Liouville integrals $J^\alpha_{a^+}$ and $J^\alpha_{b^-}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a^+} = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > \alpha$$

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and

$$J_{b^-}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < \alpha$$

respectively. Here, $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$ *is the Gamma function and* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found in [6].

In this paper, we establish some new estimates of right Hermite−Hadamard inequality in the form of fractional integrals for functions whose absolute values of second derivatives are convex and concave.

2. Main Results

In order to prove our main results, we modified [7, Lemma 2]:

Lemma 1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I°*,the interior of* I*. Assume that* $a, b \in I^\circ$ *with* $a < b$ *and* $f'' \in L[a, b]$*, then the following identity for fractional integral with* $\alpha > 0$ *holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \\ &\qquad = \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 t(1-t^\alpha) \left[f''(ta+(1-t)b) + f''((1-t)a+tb)\right] dt, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} u^{\alpha-1} du$.

Proof. Integrating by parts, we can state

$$\begin{aligned} I_1 &= \int_0^1 \left(t - t^{\alpha+1}\right) f''(ta+(1-t)b)dt \\ &= \frac{1}{b-a} \int_0^1 \left[1-(\alpha+1)t^\alpha\right] f'(ta+(1-t)b)dt \\ &= \frac{\alpha f(a)+f(b)}{(b-a)^2} - \frac{\alpha(\alpha+1)}{(b-a)^2} \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt, \end{aligned}$$

now making substitution $u = ta + (1-t)b$ and using the reduction formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ $(\alpha > 0)$ for Euler gamma function,we have

$$I_1 = \frac{\alpha f(a)+f(b)}{(b-a)^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} J_{a^+}^{\alpha} f(b),$$

analogously:

$$I_2 = \frac{f(a)+\alpha f(b)}{(b-a)^2} - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha+2}} J_{b^-}^{\alpha} f(a),$$

we obtain the desired result. □

Using this lemma, we can obtain the following fractional integral inequalities.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $|f''|$ is a convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \leq \frac{(b-a)^2}{\alpha+1}\beta(2,\alpha+1)\left[\frac{|f''(a)|+|f''(b)|}{2}\right], \tag{3}$$

where β is Euler Beta function.

Proof. From lemma 1, using the convexity of $|f''|$ with properties of modulus, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| &\leq \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)| + |f''((1-t)a+tb)|\right)dt \\ &\leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|f''(a)|+|f''(b)|}{2}\right],\end{aligned}$$

where we have used the fact that

$$\int_0^1 t^2(1-t^\alpha)dt = \frac{1}{3} - \frac{1}{\alpha+3} \quad \text{and} \quad \int_0^1 t(1-t)(1-t^\alpha)dt = \frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}.$$

To prove the second inequality we used the fact that

$$|{t_1}^\alpha + {t_2}^\alpha| \leq |t_1+t_2|^\alpha, \quad \text{for } \alpha \in [0,1] \text{ and } \forall\, t_1, t_2 \in [0,1],$$

and the Beta function,

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \qquad p, q > 0,$$

we obtaine

$$\int_0^1 t^2(1-t^\alpha)dt + \int_0^1 t(1-t)(1-t^\alpha)dt \leq \int_0^1 t(1-t)^\alpha dt = \beta(2,\alpha+1)$$

□

Remark 2. *For $\alpha = 1$, Theorem 2 reduces to* [9, Theorem 2].

The corresponding versions for powers of the absolute value of the second derivative is incorporated in the following theorems.

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°. Assume that $p \in \mathbf{R}, p > 1$ such that $|f''|^{\frac{p}{p-1}}$ is convex function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \leq \frac{(b-a)^2}{\alpha+1}\beta^{1/p}(p+1,\alpha p+1)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{1/q}, \tag{4}$$

where β is Euler,s Beta function.

Proof. From lemma 1,using the convexity of $|f''|^q$ and the Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right)dt \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^\alpha)^p dt\right)^{1/p} \\
&\times\left[\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{1/q}+\left(\int_0^1 |f''((1-t)a+tb)|^q dt\right)^{1/q}\right] \\
&\leq \frac{(b-a)^2}{\alpha+1}\beta^{1/p}(p+1,\alpha p+1)\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{1/q},
\end{aligned}$$

where $p^{-1}+q^{-1}=1$.

∎

Remark 3. *For $\alpha=1$, Theorem 3 reduces to* [7, Theorem 10] *for $s=1$.*

Theorem 4. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°. Assume that $q\geq 1$ such that $|f''|^q$ is convex function on I. Suppose that $a,b\in I^\circ$ with $a<b$ and $f''\in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \leq \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|f''(a)|^q\right.\right. \\
&\left.\left.+\frac{\alpha+5}{3\alpha+9}|f''(b)|^q\right)^{1/q}+\left(\frac{\alpha+5}{3\alpha+9}|f''(a)|^q+\frac{2\alpha+4}{3\alpha+9}|f''(b)|^q\right)^{1/q}\right]. \qquad (5)
\end{aligned}$$

Proof. Suppose that $a,b\in I^\circ$. From lemma 1and using the well-known power mean integral inequality with convexity of $|f''|^q$ for $q>1$ we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)|+|f''((1-t)a+tb)|\right)dt \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t(1-t^\alpha)dt\right)^{1-1/q}\left[\left(\int_0^1 t(1-t^\alpha)|f''(ta+(1-t)b)|^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 t(1-t^\alpha)|f''((1-t)a+tb)|^q dt\right)^{1/q}\right] \\
&\leq \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}\left[\left(\frac{2\alpha+4}{3\alpha+9}|f''(a)|^q+\frac{\alpha+5}{3\alpha+9}|f''(b)|^q\right)^{1/q}\right. \\
&\left.+\left(\frac{\alpha+5}{3\alpha+9}|f''(a)|^q+\frac{2\alpha+4}{3\alpha+9}|f''(b)|^q\right)^{1/q}\right],
\end{aligned}$$

which completes the proof. ∎

Remark 4. *For $\alpha=1$, Theorem 4 reduces* [7, Theorem 8] *for $s=1$.*

Other similar results for concave functions may be extended in the following theorems.

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°. Assume that $p \in \mathbf{R}, p > 1$ with $q = \frac{p}{p-1}$ such that $|f''|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \leq \frac{(b-a)^2}{\alpha+1}\beta^{1/p}(1+p, 1+\alpha p)\left|f''\left(\frac{a+b}{2}\right)\right|, \tag{6}$$

where β is Euler,s Beta function.

Proof. By assumption, lemma 1,and the Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 |t(1-t^\alpha)|\left(|f''(ta+(1-t)b)| + |f''((1-t)a+tb)|\right)dt \\
&\leq \frac{(b-a)^2}{2(\alpha+1)}\left(\int_0^1 t^p(1-t^\alpha)^p dt\right)^{1/p} \\
&\times\left[\left(\int_0^1 |f''(ta+(1-t)b)|^q dt\right)^{1/q} + \left(\int_0^1 |f''((1-t)a+tb)|^q dt\right)^{1/q}\right],
\end{aligned}$$

Since $|f''|^q$ is concave on $[a,b]$; we can use the integral Jensen's inequality to obtain

$$\begin{aligned}
\int_0^1 |f''(ta+(1-t)b)|^q dt &= \int_0^1 t^0 |f''(ta+(1-t)b)|^q dt \\
&\leq \left(\int_0^1 t^0 dt\right)\left|f''\left(\frac{\int_0^1 t^0(ta+(1-t)b)dt}{\int_0^1 t^0 dt}\right)\right|^q = \left|f''\left(\frac{a+b}{2}\right)\right|^q
\end{aligned}$$

Analogously:

$$\int_0^1 |f''((1-t)a+tb)|^q dt \leq \left|f''\left(\frac{a+b}{2}\right)\right|,$$

which completes the proof. □

Remark 5. *For $\alpha = 1$, Theorem 5 reduces to* [7, Theorem 9].

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°. Assume that $p \geq 1$ with $q = \frac{p}{p-1}$, such that $|f''|^q$ is concave function on I. Suppose that $a, b \in I^\circ$ with $a < b$ and $f'' \in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \leq \frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)} \\
&\left[\left|f''\left(\frac{2\alpha+4}{3\alpha+9}a + \frac{\alpha+5}{3\alpha+9}b\right)\right| + \left|f''\left(\frac{\alpha+5}{3\alpha+9}a + \frac{2\alpha+4}{3\alpha+9}b\right)\right|\right].
\end{aligned} \tag{7}$$

Proof. Using the concavity of $|f''|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}|f''(tx+(1-t)y)|^q &> t|f''(x)|^q + (1-t)|f''(y)|^q \\ &\geq (t|f''(x)| + (1-t)|f''(y)|)^q.\end{aligned}$$

Hence

$$|f''(tx+(1-t)y)| \geq t|f''(x)| + (1-t)|f''(y)|,$$

so, $|f''|$ is also concave. By the Jensen integral inequality, we have

$$\begin{aligned}&\left|\int_0^1 t(1-t^\alpha)f''(ta+(1-t)b)dt\right| \\ &\leq \left(\int_0^1 t(1-t^\alpha)dt\right)\left|f''\left(\frac{\int_0^1 t(1-t^\alpha)(ta+(1-t)b)dt}{\int_0^1 t(1-t^\alpha)dt}\right)\right|^q \\ &= \frac{\alpha}{2(\alpha+2)}\left|f''\left(\frac{2(\alpha+2)a+(\alpha+5)b}{3(\alpha+3)}\right)\right|^q.\end{aligned}$$

Analogously:

$$\left|\int_0^1 t(1-t^\alpha)f''((1-t)a+tb)dt\right| \leq \frac{\alpha}{2(\alpha+2)}\left|f''\left(\frac{(\alpha+5)a+2(\alpha+2)b}{3(\alpha+3)}\right)\right|^q,$$

which completes the proof. ∎

Remark 6. *For* $\alpha = 1$ *with* $\|f''\|_\infty := \sup_{x\in[a,b]}|f''(x)| \leq \infty$, *inequality* (7) *becomes inequality* (2.48) *as obtained in* [5].

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