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SOME INEQUALITIES FOR DIFFERENTIABLE PREQUASIINVEX FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this paper, we present several inequalities of Hermite-Hadamard type for differentiable prequasiinvex functions.Our results generalize those results proved in [2] and hence generalize those given in [7], [11] and [23]. Applications of the obtained results are given as well.

1. INTRODUCTION

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as (see [25]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [7, 8, 9], [11]-[14], [23, 24], [27]-[32].

In [7], Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in (1.1):

Theorem 1. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f' \in L(a, b)$. If $\left|f'\right|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 2. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$,and $f' \in L(a, b)$. If $\left|f'\right|^{\frac{p}{p-1}}$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right], \tag{1.3}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

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In [23], Pearce and J. Pečarić gave an improvment and simplication of the constant in Theorem 2 and consolidated this results with Theorem 1. The following is the main result from [23]:

Theorem 3. [23] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f^{'} \in L(a, b)$. If $\left|f^{'}\right|^{q}$ is convex function on $[a, b]$, for some $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{4}\left[\frac{\left|f^{'}(a)\right|^{q}+\left|f^{'}(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

If $\left|f^{'}\right|^{q}$ is concave on $[a, b]$, for some $q \geq 1$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{4}\left|f^{'}\left(\frac{a+b}{2}\right)\right|. \tag{1.5}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f\left(tx+(1-t)y\right)\leq\max\left\{f\left(x\right),f\left(y\right)\right\}$$

for all $x; y \in [a; b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex
functions which are not convex, (see [11]).

Recently, Ion [11] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follows:

Theorem 4. [11] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f^{'}\right|$ is quasi-convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{4}\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}(b)\right|\right\} \tag{1.6}$$

Theorem 5. [11] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f^{'}\right|^{p}$ is quasi-convex function on $[a, b]$, for some $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}\int_{a}^{b}g(x)dx-\frac{1}{b-a}\int_{a}^{b}f(x)g\left(x\right)dx\right| \tag{1.7}$$
$$\leq\frac{b-a}{2\left(p+1\right)^{\frac{1}{p}}}\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [2], Alomari, Darus and Kirmaci established Hermite-Hadamard-type inequalities for quasi-convex functions which give refiments of those given above in Theorem 4 and Theorem 5.

Theorem 6. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If the mapping $\left|f^{'}\right|$ is quasi-convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\max\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f^{'}(b)\right|,\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right\}\right]. \tag{1.8}$$

Theorem 7. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f^{'}\right|^{p}$ is convex function on $[a, b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\max\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\left.+\left(\max\left\{\left|f^{'}(b)\right|^{\frac{p}{p-1}},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{1.9}$$

Theorem 8. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f^{'}\right|^{q}$ is convex function on $[a, b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{8}\left[\left(\max\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f^{'}(b)\right|^{q},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{1.10}$$

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [10], Ben-Israel and Mond [5], Pini [22], M.A.Noor [19, 20], Yang and Li [34] and Weir [33]. Mond [5], Weir [32] and Noor [18, 19], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [10], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4], gave the concept of preinvex function which is special case of invexity. Pini [22], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning preinvexity and quasi-preinvexity.

Let K be a closed set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot, \cdot)$, if

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1. [33] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (1 - t) f(u) + t f(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [32].

Definition 2. [21] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq \max\{f(u), f(v)\}, \forall u, v \in K, t \in [0, 1].$$

Also Every quasi-convex function is a prequasiinvex with respect to the map $\eta(v, u)$ but the converse does not holds, see for example [35].

In the recent paper, Noor [17] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 9. [17]*Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b,a)} f(x)\, dx \leq \frac{f(a) + f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [4], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 10. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b, a)}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.11}$$

Theorem 11. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function .Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the* *following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f^{'}(a)\right|^{\frac{p}{p-1}}+\left|f^{'}(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.12}$$

In [3], Barani, Ghazanfari and Dragomir gave similar results for quasi-preinvex functions as follows:

Theorem 12. [3] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function. If $\left|f^{'}\right|$ is qusi-preinvex on K then, for every $a,b\in K$ with $\eta(b,a)\neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{\eta(b,a)}{8}\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}(b)\right|\right\}. \tag{1.13}$$

Theorem 13. [3] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function .Assume $p\in\mathbb{R}$ with $p>1$. If $\left|f^{'}\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a,b\in K$ with $\eta(b,a)\neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \tag{1.14}$$

For several new results on inequalities for preinvex functions we refer the interested reader to [4, 21, 26] and the references therein.

In the present paper we give new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are preinvex and quasi-preinvex. Our results generalize those results presented in a very recent paper of Alomari, Darus and Kirmaci [2].

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 1. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and a, $b\in K$ with $a<a+\eta(b,a)$ Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f^{'}\in L([a,a+\eta(b,a)])$. Then for every $a,b\in K$ with $\eta(b,a)\neq 0$ the*

following equality holds:

$$\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx=\frac{\eta(b,a)}{4}$$
$$\times\left[\int_0^1(-t)f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)dt+\int_0^1 tf'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)dt\right],$$

Proof. It suffices to note that

$$\begin{aligned}I_1&=\int_0^1(-t)f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\\&=\left.\frac{2(-t)f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)}{-\eta(b,a)}\right|_0^1-\frac{2}{\eta(b,a)}\int_0^1 f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\\&=\frac{2f(a)}{\eta(b,a)}-\frac{2}{\eta(b,a)}\int_0^1 f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right).\end{aligned}$$

Setting $x=a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $dx=-\frac{\eta(b,a)}{2}dt$, which gives

$$I_1=\frac{2f(a)}{\eta(b,a)}-\frac{4}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}f(x)\,dx.$$

Similarly, we also have

$$I_2=\frac{2f(a+\eta(b,a))}{\eta(b,a)}-\frac{4}{(\eta(b,a))^2}\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)}f(x)\,dx.$$

Thus

$$\frac{\eta(b,a)}{4}[I_1+I_2]=\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)}f(x)\,dx.$$

which is the required result. □

Remark 1. *If we take $\eta(b,a)=b-a$, then Lemma 1 reduces to Lemma 2.1 from* [2]*.*

Now using Lemma 1, we shall propose some new upper bound for the right-hand side of Hadamard's inequality for quasi-preinvex mappings, which is better than the inequality had done in [3]. our results generalize those reults proved in [2] as well.

Theorem 14. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\,b\in K$ with $a<a+\eta(b,a)$ Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $\left|f'\right|$ is preinvex on K, then for every $a,b\in K$ with $\eta(b,a)\neq 0$ we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\&\le\frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\right.\\&\left.\qquad+\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right].\end{aligned}\tag{2.1}$$

Proof. From Lemma 1 and by using the quasi-preinvexity of $\left|f^{'}\right|$ is preinvex on K, for any $t\in[0,1]$ we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]$$
$$\leq\frac{\eta(b,a)}{4}\left[\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\int_0^1 tdt\right.$$
$$\left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f^{'}(a+\eta(b,a))\right|\right\}\int_0^1 tdt\right]$$
$$=\frac{\eta(b,a)}{8}\left[\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\right.$$
$$\left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f^{'}(a+\eta(b,a))\right|\right\}\right].$$

This completes the proof of the theorem. □

Corollary 1. *Let f be as in Theorem 14, if in addition*

(1) $\left|f^{'}\right|$ *is increasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\leq\frac{\eta(b,a)}{8}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right]\tag{2.2}$$

(2) $\left|f^{'}\right|$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\leq\frac{\eta(b,a)}{8}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]\tag{2.3}$$

Proof. The proof follows directly from Theorem 14. □

Remark 2. *We note that the inequalities (2.2) and (2.3) are two new refinements of the trapezoid inequality for quasi-preinvex functions, and thus for preinvex functions.*

Remark 3. *If we take $\eta(b,a)=b-a$ in Theorem 14, then the inequality reduces to the inequality (1.8). If we take $\eta(b,a)=b-a$ in corollary 1, then (2.2) and (2.3) reduce to the related corollary of Theorem 6 from* [2].

Another similar result may be extended in the following theorem.

Theorem 15. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping*

on K such that $f^{'}\in L\left([a,a+\eta(b,a)]\right)$. If $\left|f^{'}\right|^{p}$ is quasi-preinvex on K, from some $p>1$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$ we have the following inequality:

$$
\begin{aligned}
(2.4)\quad & \left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\
&\qquad \left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},\left|f^{'}(a+\eta(b,a))\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right].
\end{aligned}
$$

Proof. From Lemma 1 and using the well konwn Hölder's inequality, we have

$$
\begin{aligned}
(2.5)\quad & \left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right] \\
&\leq \frac{\eta(b,a)}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad \left.+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

By the quasi-preinvexity of $\left|f^{'}\right|^{p}$ on K, from some $p>1$, we have for every $a,b\in K$ with $\eta(b,a)\neq 0$ and $t\in[0,1]$ that

$$\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q\leq \sup\left\{\left|f^{'}(a)\right|^q,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}$$

and

$$\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q\leq \sup\left\{\left|f^{'}(a+\eta(b,a))\right|^q,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the above inequalities in (2.5), we get the required result. This completes the proof of the theorem as well. □

Corollary 2. *Let f be as in Theorem 15, if in addition*

(1) $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is increasing, then we have*

$$
\begin{aligned}
(2.6)\quad & \left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right]
\end{aligned}
$$

(2) $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right] \tag{2.7}$$

Proof. It is a direct consequence of Theorem 15. □

Remark 4. *If we take $\eta(b,a)=b-a$ in Theorem 15, then the inequality reduces to the inequality (1.9). If we take $\eta(b,a)=b-a$ in corollary 2, then (2.6) and (2.7) reduce to the related corollary of Theorem 7 from* [2].

An improvement of the constants in Theorem 15 and a consolidation of this result with Theorem 14 are given in the following theorem.

Theorem 16. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $a<a+\eta(b,a)$ Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f^{'}\in L([a,a+\eta(b,a)])$. If $\left|f^{'}\right|^{q}$, $q\geq 1$, is quasi-preinvex on K, then for every $a,b\in K$ with $\eta(b,a)\neq 0$ we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}} + \sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right]. \tag{2.8}$$

Proof. From Lemma **??**, using the power-mean integral inequality and using the quasi-preinvexity of $\left|f^{'}\right|^{q}$ on K for $q\geq 1$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \qquad (2.9)\\
&\leq \frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]\\
&\leq \frac{\eta(b,a)}{4}\left[\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]\\
&\leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.\\
&\quad\left.++\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

which completes the proof □

Corollary 3. *Let f be as in Theorem 16, if in addition*

(1) $\left|f^{'}\right|^{\frac{1}{q}}$ *is increasing, then we have the iequality (2.2).*
(2) $\left|f^{'}\right|^{\frac{1}{q}}$ *is decreasing, then we have the iequality (2.3).*

Remark 5. *If we take $\eta(b,a)=b-a$ in Theorem 16, then the inequality reduces to the inequality (1.10). If we take $\eta(b,a)=b-a$ in corollary 3, then we get the results of the related corollary of Theorem 8 from* [2].

Remark 6. *For $q=1$, (2.8) reduces to Theorem 14. For $q=\frac{p}{p-1}$ $(p>1)$ we have an improvement of the constants in Theorem 15, since $4^p>p+1$ if $p>1$ and accordingly*

$$\frac{1}{8}<\frac{1}{(p+1)^{\frac{1}{p}}}.$$

3. Applications to Special Means

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3. [6]*A function $M:\mathbb{R}_+^2\to\mathbb{R}_+$, is called a Mean function if it has the following properties:*

(1) *Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,*
(2) *Symmetry : $M(x,y)=M(y,x)$,*
(3) *Reflexivity : $M(x,x)=x$,*
(4) *Monotonicity: If $x\leq x^{'}$ and $y\leq y^{'}$, then $M(x,y)\leq M(x^{'},y^{'})$,*
(5) *Internality: $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.*

We consider some means for arbitrary positive real numbers α, β (see for instance [6]).

(1) The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}$$
(2) The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}$$
(3) The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
(4) The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\ r\geq 1$$
(5) The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha\neq\beta\\ \alpha, & \alpha=\beta\end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right], \ \alpha \neq \beta, \ p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b, a) = M(b, a)$ in (2.1), (2.4) and (2.8), one can obtain the following interesting inequalities involving means:

$$\begin{aligned} &\left|\frac{f(a) + f(a + M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)\,dx\right| \\ &\quad \leq \frac{M(b,a)}{8} \left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|\right\}\right. \\ &\qquad \left. + \sup\left\{\left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|, \left|f'(a + M(b,a))\right|\right\}\right]. \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\left|\frac{f(a) + f(a + M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)\,dx\right| \\ &\quad \leq \frac{M(b,a)}{4(p+1)^{\frac{1}{p}}} \left[\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\qquad \left. + \sup\left\{\left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}, \left|f'(a + M(b,a))\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right], \end{aligned} \tag{3.2}$$

for $p > 1$, and

$$\begin{aligned} &\left|\frac{f(a) + f(a + M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)\,dx\right| \\ &\quad \leq \frac{M(b,a)}{8} \left[\left(\sup\left\{\left|f'(a)\right|^q, \left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\qquad \left. + \sup\left\{\left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|^q, \left|f'(a + M(b,a))\right|^q\right\}^{\frac{1}{q}}\right], \end{aligned} \tag{3.3}$$

for $q \geq 1$. Letting $M = A$, G, H, P_r, I, L, L_p in (3.1), (3.2) and (3.3), we can get the required inequalities, and the details are left to the interested reader.

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