Received 26/02/13

OPERATOR INEQUALITIES INVOLVING SUPERQUADRATIC FUNCTIONS

M. KIAN AND S. S. DRAGOMIR

ABSTRACT. By the use of some integral inequalities containing superquadratic functions we obtain an operator inequality which generalize some previous results. We also present an inequality for positive linear mappings of operators on Hilbert spaces. Some applications and examples are given as well.

1. INTRODUCTION

Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$ and I denote the identity operator. If $dim\mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with complex entries.

We denote by $\sigma(J)$ the set of all self-adjoint operators on $\mathscr{H}$ whose spectra are contained in an interval $J \subseteq \mathbb{R}$. Let $f : J \to \mathbb{R}$ be a continuous real function. For $A \in \sigma(J)$, we mean by $f(A)$ the continuous functional calculus at A. Let $A \in \sigma([m, M])$ and $\{E_t\}$ be its spectral family. Then $f(A)$ can be represented via the well known spectral representation as

$$f(A) = \int_{m-0}^{M} f(t)dE_t, \tag{1}$$

in which the integral is in terms of the Riemann-Stieltjes integral. If $x, y \in \mathscr{H}$, then

$$\langle f(A)x, y\rangle = \int_{m-0}^{M} f(t)d\langle E_t x, y\rangle.$$

Mond and Pečarić [10] showed that if $f : J \to \mathbb{R}$ is convex, then

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{2}$$

for all $A \in \sigma(J)$ and every unit vector $x \in \mathscr{H}$. Regarding the possible converse of (2), Dragomir [3] proved the following result.

Lemma A.([3, Theorem 5]) Let $f : J \to \mathbb{R}$ be a convex and differentiable function on

2010 *Mathematics Subject Classification.* 47A63, 26D15.

Key words and phrases. Superquadratic function, integral inequality, convex function, positive linear mapping, matrix inequality.

the interior J° of J, whose derivative f' is continuous on J°. Then

$$0 \leq \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle$$

for every $A \in \sigma(J)$ and every unit vector $x \in \mathscr{H}$.

The authors of [8] proved a similar inequality to (2) for positive linear mappings.

Lemma B.([8, Lemma 2.4]) If $f : J \to \mathbb{R}$ is convex with $f(0) \leq 0$ and A is a Hermitian matrix, then for every vector $x \in \mathscr{H}$ with $\|x\| \leq 1$ and every positive linear map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ with $0 \leq \Phi(I) \leq I$, the inequality

$$f(\langle \Phi(A)x, x\rangle) \leq \langle \Phi(f(A))x, x\rangle \tag{3}$$

holds true.

We would like to refer the reader to [3, 6, 9, 11] and references therein for a collection of such inequalities.

A function $f : [0, \infty) \to \mathbb{R}$ is said to be superquadratic provided that for all $s \geq 0$ there exists a constant $C_s \in \mathbb{R}$ such that

$$f(t) \geq f(s) + C_s(t - s) + f(|t - s|) \tag{4}$$

for all $t \geq 0$. This notion was introduced in [2]. It was also shown that:

Lemma C.([2, Lemma 2.2]) If f is a superquadratic function with C_s as in (4), then

(i) $f(0) \leq 0$;

(ii) If $f(0) = f'(0) = 0$ and f is differentiable at s, then $C_s = f'(s)$;

(iii) If $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

The reader is referred to [1, 2, 4] for more information about superquadratic fuctions. Kian [7] presented a Jensen operator inequality for superquadratic functions.

Lemma D.([7, Theorem 2.1]) If $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle - \langle f(|A - \langle Ax, x\rangle|)x, x\rangle \tag{5}$$

for any positive operator A and any unit vector $x \in \mathscr{H}$. This inequality improves (2) for some convex functions.

In section 2, we present an integral inequality for superquadratic functions and apply it to generalize inequality (5). In section 3, we establish another generalization using positive linear mappings. More precisely, an improvement of (3) is proved. In section 4, we give an improvement of Lemma A for superquadratic functions. Section 5 is devoted to some applications and related results.

2. Main Results

To prove our result we need the following lemma.

Lemma 2.1. *([2, Theorem 2.3]) Let $f:[0,\infty)\to\mathbb{R}$ be a superquadratic function and μ be a probability measure on a μ-measurable set Ω. Then*

$$f\left(\int_\Omega g\ d\mu\right)\leq\int_\Omega (f\circ g)\ d\mu-\int_\Omega f\left(\left|g(s)-\int_\Omega g\ d\mu\right|\right)\ d\mu(s)$$

for all non-negative μ-integrable function g on Ω.

Now we are ready to prove our first generalization of (5).

Theorem 2.2. *Let $g,\omega:[a,b]\to\mathbb{R}_+$ be continuous functions. If $f:[0,\infty)\to\mathbb{R}$ is a continuous superquadratic function, then*

$$\begin{aligned}&f\left(\frac{\langle g(A)\omega(A)x,x\rangle}{\langle\omega(A)x,x\rangle}\right)\\&\leq\frac{1}{\langle\omega(A)x,x\rangle}\langle(fog)(A)\omega(A)x,x\rangle\\&\quad-\frac{1}{\langle\omega(A)x,x\rangle}\left\langle f\left(\left|g(A)-\frac{1}{\langle\omega(A)x,x\rangle}\langle g(A)\omega(A)x,x\rangle I_{\mathscr{H}}\right|\right)x,x\right\rangle\end{aligned}\tag{6}$$

for every $A\in\sigma([a,b])$ and every $x\in\mathscr{H}$.

Proof. Assume that $\nu:[a,b]\to\mathbb{R}$ is monotone non-decreasing and $\omega:[a,b]\to\mathbb{R}$ is continuous and $\omega(s)\geq 0$ for all $s\in[a,b]$. Define

$$\mu(t):=\frac{1}{\int_a^b\omega(s)d\nu(s)}\int_a^t\omega(s)d\nu(s),\qquad t\in[a,b]\tag{7}$$

where the integral is in terms of the Riemann-Stieltjes integral. Then μ is a probability measure. Applying Lemma 2.1 with μ as in (7) we obtain

$$\begin{aligned}&f\left(\frac{1}{\int_a^b\omega(s)d\nu(s)}\int_a^b g(t)\omega(t)d\nu(t)\right)\\&\leq\frac{1}{\int_a^b\omega(s)d\nu(s)}\int_a^b(fog)(t)\omega(t)d\nu(t)\\&\quad-\frac{1}{\int_a^b\omega(s)d\nu(s)}\int_a^b f\left(\left|g(t)-\frac{1}{\int_a^b\omega(s)d\nu(s)}\int_a^b g(s)\omega(s)d\nu(s)\right|\right)\omega(t)d\nu(t).\end{aligned}\tag{8}$$

Now assume that A is a self-adjoint operator with $m=\min\mathrm{sp(A)}$ and $M=\max\mathrm{sp(A)}$ and $m,M\in[a,b]$. Let $\{E_t\}$ be the spectral family of A. Fix $x\in\mathscr{H}$ and let $\epsilon>0$. Now if the function ν is defined on $[m-\epsilon, M]$ by $\nu(t)=\langle E_t x, x\rangle$, then (8) implies that

$$\begin{aligned} & f\left(\frac{1}{\int_{m-\epsilon}^{M}\omega(s)d\langle E_s x,x\rangle}\int_{m-\epsilon}^{M} g(t)\omega(t)d\langle E_t x,x\rangle\right) \\ & \leq \frac{1}{\int_{m-\epsilon}^{M}\omega(s)d\langle E_s x,x\rangle}\int_{m-\epsilon}^{M}(fog)(t)\omega(t)d\langle E_t x,x\rangle \qquad (9) \\ & \quad -\frac{1}{\int_{m-\epsilon}^{M}\omega(s)d\langle E_s x,x\rangle}\int_{m-\epsilon}^{M} f\left(\left|g(t)-\frac{1}{\int_{m-\epsilon}^{M}\omega(s)d\langle E_s x,x\rangle}\int_{m-\epsilon}^{M} g(s)\omega(s)d\langle E_s x,x\rangle\right|\right)\omega(t)d\langle E_t x,x\rangle. \end{aligned}$$

Letting $\epsilon\to 0^+$ and utilizing the spectral representation for the continuous function f we obtain the desired inequality (6). □

Remark 2.3. (i) Note that Lemma D is a special case of Theorem 2.2. In fact if $g(t)=t$ and $w(t)=1$, then inequality (6) turns out to be (5).
(ii) Inequality (8) is a generalization of [2, Theorem 2.3].

3. Inequalities containing positive linear mappings

We are going to obtain a similar result to (3) for superquadratic functions. Let $f:[0,\infty)\to\mathbb{R}$ be a continuous superquadratic function. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{f(a)+f(b)}{2}-f\left(\left|\frac{a-b}{2}\right|\right) \qquad (10)$$

for all $a,b\geq 0$. Now assume that the positive linear map $\Phi:\mathcal{M}_2\to\mathcal{M}_2$ is defined by $\Phi(A)=1/2\ \mathrm{tr(A)}\ \mathrm{I}_2$, where I_2 is the 2×2 identity matrix. Put

$$A=\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} \qquad \text{and} \qquad x=\begin{pmatrix}\sqrt{\lambda} \\ \sqrt{1-\lambda}\end{pmatrix}.$$

Then $f(\langle\Phi(A)x,x\rangle)=f\left(\frac{a+b}{2}\right)$, $\Phi(f(A))x,x\rangle=\frac{f(a)+f(b)}{2}$ and

$$f\left(\left|\frac{a-b}{2}\right|\right)=\langle\Phi\left(f\left(|A-\langle\Phi(A)x,x\rangle|\right)\right)x,x\rangle\,.$$

Therefore, for a superquadratic function f, inequality (3) turns to be

$$f(\langle\Phi(A)x,x\rangle)\leq\langle\Phi(f(A))x,x\rangle-\langle\Phi\left(f\left(|A-\langle\Phi(A)x,x\rangle|\right)\right)x,x\rangle$$

for every positive matrix A and every unit vector x.

We need some lemmas to prove our result.

Lemma 3.1. *Every unital positive linear map on a commutative C^*-algebra is completely positive.*

Lemma 3.2. *(*[5, Theorem 3.1.2]*) Let Φ be a unital completely positive linear map from a C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}_n(\mathbb{C})$ into $\mathcal{M}_m(\mathbb{C})$. Then there exists a Hilbert space $\mathscr{K}$, an isometry $V : \mathbb{C}^m \to \mathscr{K}$ and a unital $*$-homomorphism π from $\mathcal{A}$ into the C^*-algebra $\mathbb{B}(\mathscr{K})$ such that $\Phi(A) = V^*\pi(A)V$.*

The next theorem extend (3) for superquadratic functions.

Theorem 3.3. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous superquadratic function and let $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a unital positive linear map. Then*

$$f(\langle\Phi(A)x, x\rangle) \leq \langle\Phi(f(A))x, x\rangle - \langle\Phi\left(f\left(|A - \langle\Phi(A)x, x\rangle|\right)\right)x, x\rangle \tag{11}$$

for every positive matrix $A \in \mathcal{M}_n(\mathbb{C})$ and every unit vector $x \in \mathbb{C}^m$.

Proof. Let $A \in \mathcal{M}_n(\mathbb{C})$ be positive. Assume that $\mathcal{A}$ is the C^*-subalgebra of $\mathcal{M}_n(\mathbb{C})$ generated by A and I. Without loss of generality we may assume that Φ is defined on $\mathcal{A}$. It follows from Lemma 3.1 that Φ is completely positive. So there exists, by Lemma 3.2, a Hilbert space $\mathscr{K}$, an isometry $V : \mathbb{C}^m \to \mathscr{K}$ and a unital $*$-homomorphism π from $\mathcal{A}$ into the C^*-algebra $\mathbb{B}(\mathscr{K})$ such that $\Phi(A) = V^*\pi(A)V$. Clearly $f(\pi(A)) = \pi(f(A))$. Moreover, for any $\alpha \in \mathbb{C}$, it is easy to see that

$$f(|\pi(A - \alpha I)|) = \pi(f(|A - \alpha I|)). \tag{12}$$

If $x \in \mathbb{C}^m$ is a unit vector, then $\|Vx\| = 1$. Hence

$$\begin{aligned}
f(\langle\Phi(A)x, x\rangle) &= f\left(\langle V^*\pi(A)Vx, x\rangle\right) \\
&= f\left(\langle\pi(A)Vx, Vx\rangle\right) \\
&\leq \langle f(\pi(A))Vx, Vx\rangle - \langle f(|\pi(A) - \langle\pi(A)Vx, Vx\rangle|)Vx, Vx\rangle \quad (\text{ By Lemma D}) \\
&= \langle f(\pi(A))Vx, Vx\rangle - \langle\pi(f(|A - \langle\pi(A)Vx, Vx\rangle|))Vx, Vx\rangle \quad (\text{ By (12)}) \\
&= \langle V^*\pi(f(A))Vx, x\rangle - \langle V^*\pi(f(|A - \langle V^*\pi(A)Vx, x\rangle|))Vx, x\rangle \\
&= \langle\Phi(f(A))x, x\rangle - \langle\Phi(f(|A - \langle\Phi(A)x, x\rangle|))x, x\rangle.
\end{aligned}$$

This completes the proof. □

Remark 3.4. If the superquadratic function f is non-negative, then Theorem 3.3 gives an improvement of (3) for the convex function f. In fact

$$f(\langle\Phi(A)x, x\rangle) \leq \langle\Phi(f(A))x, x\rangle - \langle\Phi(f(|A - \langle\Phi(A)x, x\rangle|))x, x\rangle \leq \langle\Phi(f(A))x, x\rangle. \tag{13}$$

Furthermore, if $f \leq 0$ is superquadratic and $-f$ is convex, then Theorem 3.3 provides a converse of (3) for the concave function f. More precisely if $f \leq 0$ is superquadratic and $-f$ is convex, then

$$\langle\Phi(f(A))x, x\rangle \leq f(\langle\Phi(A)x, x\rangle) \leq \langle\Phi(f(A))x, x\rangle - \langle\Phi(f(|A - \langle\Phi(A)x, x\rangle|))x, x\rangle. \tag{14}$$

Example 3.5. Assume that $\Phi : \mathcal{M}_3 \to \mathcal{M}_2$ is defined by $\Phi((a_{ij})) = (a_{ij})_{1\leq i,j\leq 2}$. Let

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If $f(t) = t^3$, then

$$f(\langle \Phi(A)x, x\rangle) = 8, \quad \langle \Phi(f(A))x, x\rangle = 13, \quad \langle \Phi(f(|A - \langle \Phi(A)x, x\rangle|))x, x\rangle = 1.4142,$$

which shows that inequality (13) is really an improvement of (3). If $g(t) = -t\sqrt{t}$, then g is superquadratic as well as concave. With the same values for A and x as above we have

$$g(\langle \Phi(A)x, x\rangle) = -2\sqrt{2}, \qquad \langle \Phi(g(A))x, x\rangle = -3.1305$$

and

$$\langle \Phi(g(|A - \langle \Phi(A)x, x\rangle|))x, x\rangle = -0.8409,$$

whence

$$g(\langle \Phi(A)x, x\rangle) + \langle \Phi(g(|A - \langle \Phi(A)x, x\rangle|))x, x\rangle \lneqq \langle \Phi(g(A))x, x\rangle \lneqq g(\langle \Phi(A)x, x\rangle).$$

Corollary 3.6. *Let $\Phi_1, \cdots, \Phi_k : \mathcal{M}_n \to \mathcal{M}_m$ be positive linear mappings with $\sum_{i=1}^k \Phi_i(I) = I$. If $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then*

$$f\left(\left\langle \sum_{i=1}^k \Phi_i(A_i)x, x \right\rangle\right) \leq \left\langle \sum_{i=1}^k \Phi_i(f(A_i))x, x \right\rangle - \left\langle \sum_{i=1}^k \Phi_i\left(f\left(\left|A_i - \left\langle \sum_{j=1}^k \Phi_j(A_j)x, x\right\rangle\right|\right)\right) x, x\right\rangle \tag{15}$$

for all positive matrices $A_1, \cdots, A_k \in \mathcal{M}_n$ and every unit vector $x \in \mathbb{C}^m$.

Proof. If $A_1, \cdots, A_k \in \mathcal{M}_n$ are positive matrices, then $A = A_1 \oplus \cdots \oplus A_k$ is a positive matrix in $\mathcal{M}_k(\mathcal{M}_n)$. Let the unital positive linear mapping $\Phi : \mathcal{M}_k(\mathcal{M}_n) \to \mathcal{M}_m$ be defined by $\Phi(A) = \sum_{i=1}^k \Phi_i(A_i)$. Utilizing Theorem 3.3 we obtain desired inequality. □

Remark 3.7. We can prove a more general result than Theorem 3.3. If $x \in \mathscr{H}$ is a vector with $\|x\| \leq 1$ and if $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then it was shown [7] that

$$f(\langle Ax, x\rangle) \leq \frac{1}{2 - \|x\|^2}\left(\langle f(A)x, x\rangle - \left\langle f\left(\left|A - \frac{1}{\|x\|^2}\langle Ax, x\rangle\right|\right)x, x\right\rangle - \|x\|^2 f\left(\frac{1 - \|x\|^2}{\|x\|^2}\langle Ax, x\rangle\right)\right)$$

for every positive operator A. Using the last inequality and applying a similar argument as in the proof of Theorem 3.3 we can prove the next theorem. We omit its proof.

Theorem 3.8. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function and let $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a positive linear map with $0 < \Phi(I) \leq I$. Then*

$$\begin{aligned} f(\langle\Phi(A)x,x\rangle) \leq & \frac{1}{2-\langle\Phi(I)x,x\rangle}\{\langle\Phi(f(A))x,x\rangle \\ & - \left\langle \Phi\left(f\left(\left|A - \frac{1}{\langle\Phi(I)x,x\rangle}\langle\Phi(A)x,x\rangle\right|\right)\right)x,x\right\rangle \\ & - \langle\Phi(I)x,x\rangle f\left(\frac{1-\langle\Phi(I)x,x\rangle}{\langle\Phi(I)x,x\rangle}\langle\Phi(A)x,x\rangle\right)\} \end{aligned}$$

for every positive matrix $A \in \mathcal{M}_n(\mathbb{C})$ and every vector $x \in \mathbb{C}^m$ with $\|x\| \leq 1$.

4. A Reverse Inequality

Lemma A can be improved for non-negative superquadratic functions. First we give a reverse inequality for (11).

Theorem 4.1. *Let $f : [0,\infty) \to [0,\infty)$ be a differentiable superquadratic function whose derivative f' is continuous. If $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is a positive linear map, then*

$$\begin{aligned} 0 &\leq \langle\Phi(f(A))x,x\rangle - f(\langle\Phi(A)x,x\rangle) \\ &\leq \langle\Phi(f'(A)A)x,x\rangle - \langle\Phi(A)x,x\rangle\langle\Phi(f'(A))x,x\rangle - \langle\Phi(f(|A-\langle\Phi(A)x,x\rangle|))x,x\rangle \end{aligned}$$

for every positive matrix $A \in \mathcal{M}_n(\mathbb{C})$ and every unit vector $x \in \mathbb{C}^m$.

Proof. Let $s \geq 0$. Since f is superquadratic, there exists $C_s \in \mathbb{R}$ such that

$$f(t) \geq f(s) + C_s(t-s) + f(|t-s|) \tag{16}$$

for every $t \geq 0$. As $f \geq 0$, it follows from Lemma C that f is convex and $C_s = f'(s)$. So the first inequality follows from (3).

Assume that $x \in \mathbb{C}^m$ with $\|x\| = 1$ and $A \geq 0$. Utilizing functional calculus for (16) with $s = A$ and $t = \langle\Phi(A)x,x\rangle$ we obtain

$$f(\langle\Phi(A)x,x\rangle) \geq f(A) + f'(A)\langle\Phi(A)x,x\rangle - f'(A)A + f(|A-\langle\Phi(A)x,x\rangle|).$$

Applying the positive linear map Φ to both sides of the last inequality we get

$$f(\langle\Phi(A)x,x\rangle) \geq \Phi(f(A)) + \Phi(f'(A))\langle\Phi(A)x,x\rangle - \Phi(f'(A)A) + \Phi(f(|A-\langle\Phi(A)x,x\rangle|))$$

from which we have the desired result. □

With $\Phi(A) = A$, Theorem 4.1 gives an improvement of Lemma A.

Corollary 4.2. *Let f be as in the Theorem 4.1. Then*

$$\begin{aligned}0 &\le \langle f(A)x, x\rangle - f(\langle Ax, x\rangle)\\ &\le \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle - \langle f(|A - \langle Ax, x\rangle|)x, x\rangle\end{aligned}$$

for every $A \ge 0$ and every unit vector $x \in \mathscr{H}$.

Example 4.3. If $r \ge 2$, then $f(t) = t^r$ is a non-negative superquadratic function on $[0, \infty)$. Let $A \ge 0$ and $x \in \mathscr{H}$ be a unit vector. Applying Corollary 4.2 we get

$$\begin{aligned}0 &\le \langle A^r x, x\rangle - \langle Ax, x\rangle^r\\ &\le r\langle A^r x, x\rangle - r\langle Ax, x\rangle\langle A^{r-1}x, x\rangle - \langle |A - \langle Ax, x\rangle|^r x, x\rangle\end{aligned}$$

which is a reverse Hölder–McCarthy's inequality(see e.g. [3]).

5. Some Applications and Related Results

Using Theorem 2.2 when $f(t) = t^r$, $(r \ge 2$ or $1 < r \le 2)$, we obtain the following improvement of Hölder–McCarthy's inequality (see e.g. [6, Theorem 32]) , which is also a generalization of [7, Corollary 3.1].

Corollary 5.1. *Let $g, \omega : [a, b] \to \mathbb{R}_+$ be continuous functions. If $r \ge 2$, then*

$$\begin{aligned}\frac{\langle g(A)\omega(A)x, x\rangle^r}{\langle \omega(A)x, x\rangle^r} &\le \frac{1}{\langle \omega(A)x, x\rangle}\langle g(A)^r\omega(A)x, x\rangle\\ &\quad - \frac{1}{\langle \omega(A)x, x\rangle}\left\langle \left|g(A) - \frac{1}{\langle \omega(A)x, x\rangle}\langle g(A)\omega(A)x, x\rangle I_{\mathscr{H}}\right|^r x, x\right\rangle\end{aligned}$$

for every $A \in \sigma([a, b])$ and every $x \in \mathscr{H}$. If $1 < r \le 2$, then

$$\begin{aligned}\frac{1}{\langle \omega(A)x, x\rangle}\langle g(A)^r\omega(A)x, x\rangle &\le \frac{\langle g(A)\omega(A)x, x\rangle^r}{\langle \omega(A)x, x\rangle^r}\\ &\quad + \frac{1}{\langle \omega(A)x, x\rangle}\left\langle \left|g(A) - \frac{1}{\langle \omega(A)x, x\rangle}\langle g(A)\omega(A)x, x\rangle I_{\mathscr{H}}\right|^r x, x\right\rangle\end{aligned}$$

for every $A \in \sigma([a, b])$ and every $x \in \mathscr{H}$.

Utilizing Theorem 2.2 for some special functions, gives applicable inequalities.

Corollary 5.2. *If $f : [0, \infty) \to \mathbb{R}$ is a continuous superquadratic function, then*

$$\langle Ax, x\rangle f\left(\frac{\langle A\ln Ax, x\rangle}{\langle Ax, x\rangle}\right) \le \langle Af(\ln A)x, x\rangle - \left\langle f\left(\left|\ln A - \frac{1}{\langle Ax, x\rangle}\langle A\ln Ax, x\rangle I_{\mathscr{H}}\right|\right) x, x\right\rangle$$

for every $A \ge 0$ and every $x \in \mathscr{H}$.

Corollary 5.3. *Let $p>0$ and $f:[0,\infty)\to\mathbb{R}$ be a nonnegative superquadratic function. If $A\geq 0$, then*

$$\begin{aligned} &f\left(\langle Ax,x\rangle^{p}\right)\\ &\leq f\left(\frac{\langle A^{p+1}x,x\rangle}{\langle Ax,x\rangle}\right)\\ &\leq \frac{1}{\langle Ax,x\rangle}\left\{\langle Af\left(A^{p}\right)x,x\rangle-\left\langle f\left(\left|A^{p}-\frac{1}{\langle Ax,x\rangle}\langle A^{p+1}x,x\rangle\right|\right)x,x\right\rangle\right\} \end{aligned} \tag{17}$$

for every unit vector $x\in\mathscr{H}$.

Proof. Applying Theorem 2.2 to $g(t)=t^{p}$ and $w(t)=t$ we obtain the second inequality of (17). To get the first one note that convexity of the function t^{p+1} for $p>0$, implies that $(\langle Ax,x\rangle)^{p+1}\leq\langle A^{p+1}x,x\rangle$. Using the fact that every nonnegative superquadratic function is non-decreasing, we get the first inequality. □

Example 5.4. Assume that $p>0$, $r\geq 2$ and $f(t)=t^{r}$. On making use of (17) we obtain

$$\begin{aligned} \langle Ax,x\rangle^{p+r} &\leq \frac{\langle A^{p+1}x,x\rangle^{r}}{\langle Ax,x\rangle^{r}}\\ &\leq \frac{1}{\langle Ax,x\rangle}\left\{\langle A^{p+r+1}x,x\rangle-\left\langle\left|A^{p}-\frac{1}{\langle Ax,x\rangle}\langle A^{p+1}x,x\rangle\right|^{r}x,x\right\rangle\right\}. \end{aligned}$$

Multiplying the last inequality by $\langle Ax,x\rangle^{r}$ and using the convexity of t^{r-1} we obtain

$$\begin{aligned} \langle Ax,x\rangle^{p+2r} &\leq \langle A^{p+1}x,x\rangle^{r}\\ &\leq \langle A^{r-1}x,x\rangle\left\{\langle A^{p+r+1}x,x\rangle-\left\langle\left|A^{p}-\frac{1}{\langle Ax,x\rangle}\langle A^{p+1}x,x\rangle\right|^{r}x,x\right\rangle\right\}. \end{aligned}$$

References

1. S. Abramovich, S. Ivelić, J. Pečarić, *Improvement of Jensen-Steffensen's inequality for superquadratic functions*, Banach J. Math. Anal. **4** (2010), no. 1, 146–158.
2. S. Abramovich, G. Jameson, G. Sinnamon, *Refining Jensen's inequality*, Bull. Math. Soc. Sci. Math. Roumanie **47** (2004), 3–14.
3. R. P. Agarwal, S. S. Dragomir, *A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces*, Comput. Math. Appl. **59** (2010), 3785–3812.
4. J. Barić, A. Matković, J. Pečarić, *A variant of the Jensen–Mercer operator inequality for superquadratic functions*, Math. Comput. Modelling, **51** (2010), 1230–1239.
5. R. Bhatia, *Positive definite matrices*, Princeton Uuniversity Press, Princeton, NJ, 2007.
6. S. S. Dragomir, *Inequalities for Functions of Selfadjoint Operators on Hilbert Spaces*, arXiv:1203.1667v1[math.FA].

7. M. Kian, *Operator Jensen inequality for superquadratic functions*, Linear Algebra Appl. (2013), http://dx.doi.org/10.1016/j.laa.2012.12.011.
8. J. S. Matharua, M. S. Moslehian and J. S. Aujla, *Eigenvalue extensions of Bohrs inequality*, Linear Algebra Appl. **435** (2011), 270–276.
9. J. Mićić, J. Pečarić, J. Perić, *Extension of the refined Jensen's operator inequality with condition on spectra*, Ann. Funct. Anal. **3** (2012), no. 1, 67–85.
10. B. Mond and J. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math., **19** (1993), 405–420.
11. M. S. Moslehian and M. Kian, *Non-commutative f-divergence functional*, Math. Nachr., 1–16 (2013), DOI 10.1002/mana.201200194.

Mohsen Kian: Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran.

E-mail address: kian@member.ams.org and kian@ub.ac.ir

S. S. Dragomir: Mathematics, School of Engineering and Science, Victoria University, P.O. Box 14428, Melbourne City, MC 8001, Australia

E-mail address: sever.dragomir@vu.edu.au