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SOME LIPSCHITZ TYPE INEQUALITIES FOR COMPLEX FUNCTIONS

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Abstract. In this paper we establish some Lipschitz type inequalities for complex functions when some convexity properties for powers of the absolute value of the derivative are assumed. The case of functions defined by power series is analysed. Some applications in bounding the Jensen difference for complex functions are also provided.

1. Introduction

The *Lipschitz inequality* for a real variable function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ on the interval $[a,b]$

$$|f(x)-f(y)|\le L|x-y| \tag{L}$$

that holds for a given $L>0$ and any $x,y\in[a,b]$ is an important tool in obtaining various famous inequalities in the literature, such as the *Hermite-Hadamard type inequalities*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}(b-a)L,\ \ [8] \tag{1.1}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}(b-a)L,\ \ [14] \tag{1.2}$$

or the *Čebyšev type inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right|\le\frac{1}{12}LK(b-a)^2, \tag{1.3}$$

where g satisfies a Lipschitz condition with the constant K.

In (1.1) and (1.2) the constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity. The same applies for the constant $\frac{1}{12}$ in (1.3).

The interested reader may find many other inequalities involving Lipschitzian functions in the papers [1]-[11], [13]-[20] and in the references therein.

In this paper we establish some Lipschitz type inequalities for complex functions when some convexity properties for powers of the absolute value of the derivative are assumed. The case of functions defined by power series is analysed. Some

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applications in bounding the *Jensen difference* for complex functions $f : C \subset \mathbb{C} \to \mathbb{C}$,

$$\sum_{k=1}^{n} p_k f(z_k) - f\left(\sum_{j=1}^{n} p_j z_j\right)$$

where $\{p_k\}_{k=1,\dots,n}$ are probabilities and $\{z_k\}_{k=1,\dots,n} \subset C$, C a convex set, are also provided.

2. Some Results

Lemma 1. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a differentiable function on the convex domain D. Then for any $z, w \in D$ we have the inequality*

$$\begin{aligned} |f(w) - f(z)| &\le |w - z| \int_0^1 |f'((1-t)z + tw)|\, dt \\ &\le |w - z| \times \begin{cases} ess\sup_{t\in[0,1]} |f'((1-t)z + tw)| \\ \left(\int_0^1 |f'((1-t)z + tw)|^{\alpha} dt\right)^{1/\alpha}, & \alpha > 1 \end{cases} \\ &\le |w - z|\, ess\sup_{u\in D} |f'(u)|. \end{aligned} \tag{2.1}$$

Proof. Due to the convexity of D, for any $z, w \in D$ we can define the function $\varphi_{z,w} : [0,1] \to \mathbb{R}$ by $\varphi_{z,w}(t) := f((1-t)z + tw)$. The function $\varphi_{z,w}$ is differentiable on $(0,1)$ and

$$\frac{d\varphi_{z,w}(t)}{dt} = (w - z) f'((1-t)z + tw) \text{ for } t \in (0,1).$$

We have

$$\begin{aligned} |f(w) - f(z)| = |\varphi_{z,w}(1) - \varphi_{z,w}(0)| &= \left|\int_0^1 \frac{d\varphi_{z,w}(t)}{dt} dt\right| \\ &= \left|(w - z)\int_0^1 f'((1-t)z + tw)\, dt\right| \\ &\le |w - z| \int_0^1 |f'((1-t)z + tw)|\, dt \end{aligned}$$

for any $z, w \in D$.

Utilising the Hölder integral inequality for the Lebesgue integral

$$\left|\int_a^b g(s)\, ds\right| \le \begin{cases} (b-a)\, ess\sup_{s\in[a,b]} |g(s)| \\ (b-a)^{1/\beta} \left(\int_a^b |g(s)|^{\alpha} ds\right)^{1/\alpha} & \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \end{cases}$$

we deduce the second part of (2.1).

The last part is obvious. □

We recal that, for a *convex function* $g:[a,b]\to\mathbb{R}$ we have the following *Hermite-Hadamard type inequality*

$$g\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{2}\right)\right]\leq\frac{1}{b-a}\int_a^b g(s)\,ds\leq\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}\right]\leq\frac{g(a)+g(b)}{2}. \tag{2.2}$$

If the function is *concave*, the inequalities in (2.2) reverse.

For other related results see the monograph [10].

Theorem 1. *If the function* $|f'|^{\alpha}$ *with* $\alpha\geq 1$ *is convex on* D, *then we have the inequalities*

$$\begin{aligned}|f(w)-f(z)| &\leq |w-z|\left(\int_0^1|f'((1-t)z+tw)|^{\alpha}dt\right)^{1/\alpha}\\ &\leq |w-z|\frac{1}{2^{1/\alpha}}\left[\left|f'\left(\frac{z+w}{2}\right)\right|^{\alpha}+\frac{|f'(z)|^{\alpha}+|f'(w)|^{\alpha}}{2}\right]^{1/\alpha}\\ &\leq |w-z|\left[\frac{|f'(z)|^{\alpha}+|f'(w)|^{\alpha}}{2}\right]^{1/\alpha}\end{aligned} \tag{2.3}$$

for any $z,w\in D$.

If the function $|f'|^{\alpha}$ *with* $\alpha\geq 1$ *is concave on* D, *then we have the inequalities*

$$\begin{aligned}|f(w)-f(z)| &\leq |w-z|\left(\int_0^1|f'((1-t)z+tw)|^{\alpha}dt\right)^{1/\alpha}\\ &\leq |w-z|\frac{1}{2^{1/\alpha}}\left[\left|f\left(\frac{3z+w}{4}\right)\right|^{\alpha}+\left|f\left(\frac{z+3w}{2}\right)\right|^{\alpha}\right]\\ &\leq |w-z|\left|f\left(\frac{w+z}{2}\right)\right|^{\alpha}\end{aligned} \tag{2.4}$$

for any $z,w\in D$.

Proof. For $z,w\in D$, consider the function $\psi_{z,w,\alpha}:[0,1]\to\mathbb{R}$ given by $\psi_{z,w,\alpha}(t):=|f'((1-t)z+tw)|^{\alpha}$. For $t_1,t_2\in[0,1]$ and $\lambda\in[0,1]$, by the convexity of $|f'|^{\alpha}$ on D we have

$$\begin{aligned}\psi_{z,w,\alpha}((1-\lambda)t_1+\lambda t_2) &= |f'((1-(1-\lambda)t_1-\lambda t_2)z+[(1-\lambda)t_1+\lambda t_2]w)|^{\alpha}\\ &= |f'((1-\lambda)((1-t_1)z+t_1w)+\lambda((1-t_2)z+t_2w))|^{\alpha}\\ &\leq (1-\lambda)|f'((1-t_1)z+t_1w)|^{\alpha}+\lambda|f'((1-t_2)z+t_2w)|^{\alpha}\\ &= (1-\lambda)\psi_{z,w,\alpha}(t_1)+\lambda\psi_{z,w,\alpha}(t_2),\end{aligned}$$

which proves the convexity of $\psi_{z,w,\alpha}$.

Applying the Hermite-Hadamard type inequality (2.2) for the function $\psi_{z,w,\alpha}$ on the interval $[0,1]$ we deduce the desired result (2.3). □

Remark 1. *For $z \in \mathbb{C}$ we have*

$$\begin{aligned}|\exp(z)| &= |\exp(\operatorname{Re} z + i \operatorname{Im} z)| = |\exp(\operatorname{Re} z)\exp(i \operatorname{Im} z)| \\ &= |\exp(\operatorname{Re} z)|\,|\exp(i \operatorname{Im} z)| = \exp(\operatorname{Re} z)\,|\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z)| \\ &= \exp(\operatorname{Re} z).\end{aligned}$$

Then for any $t \in [0,1]$ and for any $z, w \in \mathbb{C}$ we have

$$\begin{aligned}|\exp((1-t)z + tw)|^{\alpha} &= \exp[\alpha(\operatorname{Re}((1-t)z + tw))] \\ &= \exp[(1-t)\alpha \operatorname{Re} z + t\alpha \operatorname{Re} w] \\ &\leq (1-t)\exp(\alpha \operatorname{Re} z) + t\exp(\alpha \operatorname{Re} w) \\ &= (1-t)|\exp(z)|^{\alpha} + t|\exp(w)|^{\alpha}\end{aligned}$$

which shows that the function $g(z) = |\exp(z)|^{\alpha}$ is convex for any $\alpha \in \mathbb{R} \setminus \{0\}$.

Utilising the inequality (2.3) for the exponential function $f(z) = \exp z$ we have

$$\begin{aligned}&|\exp(w) - \exp(z)| \\ &\leq |w-z|\left(\int_0^1 |\exp((1-t)z + tw)|^{\alpha} dt\right)^{1/\alpha} \\ &\leq |w-z|\frac{1}{2^{1/\alpha}}\left[\left|\exp\left(\frac{z+w}{2}\right)\right|^{\alpha} + \frac{|\exp(z)|^{\alpha} + |\exp(w)|^{\alpha}}{2}\right]^{1/\alpha} \\ &\leq |w-z|\left[\frac{|\exp(z)|^{\alpha} + |\exp(w)|^{\alpha}}{2}\right]^{1/\alpha}\end{aligned} \tag{2.5}$$

for any $z, w \in \mathbb{C}$ and $\alpha \geq 1$.

Observe that

$$\begin{aligned}\int_0^1 |\exp((1-t)z + tw)|^{\alpha} dt &= \int_0^1 \exp[(1-t)\alpha \operatorname{Re} z + t\alpha \operatorname{Re} w]\,dt \\ &= \begin{cases} \exp(\alpha \operatorname{Re} z) & \text{for } \operatorname{Re} w = \operatorname{Re} z \\ \frac{\exp(\alpha \operatorname{Re} z) - \exp(\alpha \operatorname{Re} w)}{\alpha(\operatorname{Re} z - \operatorname{Re} w)} & \text{for } \operatorname{Re} w \neq \operatorname{Re} z \end{cases}\end{aligned}$$

for any $\alpha \geq 1$.

If $z, w \in \mathbb{C}$ are such that $\operatorname{Re} z, \operatorname{Re} w \leq K$, then

$$\exp[(1-t)\alpha \operatorname{Re} z + t\alpha \operatorname{Re} w] \leq \exp(\alpha K)$$

and we have from (2.5) the following coarser but simpler inequality

$$|\exp(z) - \exp(w)| \leq |w-z|\exp(K). \tag{2.6}$$

Then from (2.5), for any $z, w \in \mathbb{C}$ *with* $\operatorname{Re} w \neq \operatorname{Re} z$ *and* $\alpha \geq 1$, *we have*

$$\begin{aligned}(2.7)\qquad & \left|\frac{\exp(w)-\exp(z)}{w-z}\right| \\ & \leq \left[\frac{\exp(\alpha \operatorname{Re} z)-\exp(\alpha \operatorname{Re} w)}{\alpha(\operatorname{Re} z-\operatorname{Re} w)}\right]^{1/\alpha} \\ & \leq \frac{1}{2^{1/\alpha}}\left[\left|\exp\left(\alpha \frac{\operatorname{Re} z+\operatorname{Re} w}{2}\right)\right|^{\alpha}+\frac{\exp(\alpha \operatorname{Re} z)+\exp(\alpha \operatorname{Re} w)}{2}\right]^{1/\alpha} \\ & \leq \left[\frac{\exp(\alpha \operatorname{Re} z)+\exp(\alpha \operatorname{Re} w)}{2}\right]^{1/\alpha}.\end{aligned}$$

If we take $z = e^{is}, w = e^{it}$ *with* $s, t \in \mathbb{R}$ *in the first inequality in (2.5), then we get*

$$(2.8)\qquad \left|\exp\left(e^{it}\right)-\exp\left(e^{is}\right)\right| \leq \left|e^{it}-e^{it}\right| \int_0^1 \exp\left[(1-t)\,\alpha \cos s+t\alpha \cos t\right] dt.$$

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2 & =\left|e^{is}\right|^2-2 \operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2 \\ & =2-2 \cos(s-t)=4 \sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t, s \in \mathbb{R}$, *then*

$$(2.9)\qquad \left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t, s \in \mathbb{R}$.

We then have for $\alpha \geq 1$

$$\begin{aligned}(2.10)\qquad & \left|\exp\left(e^{is}\right)-\exp\left(e^{it}\right)\right| \\ & \leq 2\left|\sin\left(\frac{s-t}{2}\right)\right| \int_0^1 \exp\left[(1-t)\,\alpha \cos s+t\alpha \cos t\right] dt \\ & \leq 2^{\frac{\alpha-1}{\alpha}}\left|\sin\left(\frac{s-t}{2}\right)\right| \\ & \times\left[\exp\left[\alpha\left(\frac{\cos s+\cos t}{2}\right)\right]+\frac{\exp(\alpha \cos s)+\exp(\alpha \cos t)}{2}\right]^{1/\alpha} \\ & \leq 2\left|\sin\left(\frac{s-t}{2}\right)\right|\left[\frac{\exp(\alpha \cos s)+\exp(\alpha \cos t)}{2}\right]^{1/\alpha}\end{aligned}$$

for any $t, s \in \mathbb{R}$.

If $t,s\in[0,2\pi]$ *such that* $\cos s\neq\cos t$, *then*

$$\begin{aligned}(2.11)\qquad &\left|\exp\left(e^{is}\right)-\exp\left(e^{it}\right)\right|\\ &\leq 2\left|\sin\left(\frac{s-t}{2}\right)\right|\left[\frac{\exp\left(\cos s\right)-\exp\left(\cos t\right)}{\cos s-\cos t}\right]\\ &\leq\left|\sin\left(\frac{s-t}{2}\right)\right|\left[\exp\left(\frac{\cos s+\cos t}{2}\right)+\frac{\exp\left(\cos s\right)+\exp\left(\cos t\right)}{2}\right]\\ &\leq 2\left|\sin\left(\frac{s-t}{2}\right)\right|\left[\frac{\exp\left(\cos s\right)+\exp\left(\cos t\right)}{2}\right].\end{aligned}$$

Finally, we notice that, since $\cos s,\cos t\leq 1$ *we have from (2.11) the coarser but simpler inequality*

$$(2.12)\qquad \left|\exp\left(e^{is}\right)-\exp\left(e^{it}\right)\right|\leq 2e\left|\sin\left(\frac{s-t}{2}\right)\right|$$

for any $t,s\in[0,2\pi]$.

Remark 2. *For* $n\geq 1$ *consider the power function* $f(z)=z^n$. *Then for* $\alpha\geq 1$

$$\left|f'(z)\right|^{\alpha}=n^{\alpha}\left|z\right|^{(n-1)\alpha}.$$

Define $g:\mathbb{C}\to[0,\infty)$ *by* $g(z):=n^{\alpha}|z|^{(n-1)\alpha}$. *For any* $t\in[0,1]$ *and any* $z,w\in\mathbb{C}$ *we have*

$$\begin{aligned}g\left((1-t)z+tw\right)&=n^{\alpha}\left|(1-t)z+tw\right|^{(n-1)\alpha}\\ &\leq n^{\alpha}\left[(1-t)\left|z\right|^{(n-1)\alpha}+t\left|w\right|^{(n-1)\alpha}\right]\\ &=(1-t)n^{\alpha}\left|z\right|^{(n-1)\alpha}+tn^{\alpha}\left|w\right|^{(n-1)\alpha}\\ &=(1-t)g(z)+tg(w)\end{aligned}$$

showing that $|f'|^{\alpha}$ *is convex.*

Applying the inequality (2.3) for the power function $f(z)=z^n$ *we have*

$$\begin{aligned}(2.13)\qquad \left|w^n-z^n\right|&\leq n\left|w-z\right|\left(\int_0^1\left|(1-t)z+tw\right|^{(n-1)\alpha}dt\right)^{1/\alpha}\\ &\leq n\left|w-z\right|\frac{1}{2^{1/\alpha}}\left[\left|\frac{z+w}{2}\right|^{(n-1)\alpha}+\frac{|z|^{(n-1)\alpha}+|w|^{(n-1)\alpha}}{2}\right]^{1/\alpha}\\ &\leq n\left|w-z\right|\left[\frac{|z|^{(n-1)\alpha}+|w|^{(n-1)\alpha}}{2}\right]^{1/\alpha}\end{aligned}$$

for any $z,w\in D$ *and* $\alpha\geq 1$.

If we take $z=e^{it}$ *and* $w=e^{is}$ *with* $s,t\in\mathbb{R}$, *then we get*

$$\begin{aligned}(2.14)\qquad \left|e^{int}-e^{ins}\right|&\leq n\left|e^{it}-e^{is}\right|\left(\int_0^1\left|(1-\lambda)e^{it}+\lambda e^{is}\right|^{(n-1)\alpha}d\lambda\right)^{1/\alpha}\\ &\leq n\left|e^{it}-e^{is}\right|\frac{1}{2^{1/\alpha}}\left[\frac{1}{2^{(n-1)\alpha}}\left|e^{it}+e^{is}\right|^{(n-1)\alpha}+1\right]^{1/\alpha}\\ &\leq n\left|e^{it}-e^{is}\right|.\end{aligned}$$

Since

$$\left|e^{int}-e^{ins}\right|=2\left|\sin\left[n\left(\frac{t-s}{2}\right)\right]\right|$$

and

$$\begin{aligned}\left|e^{it}+e^{is}\right| &= \left[\left|e^{is}\right|^{2}+2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^{2}\right]^{1/2}\\ &= \left[2+2\cos\left(s-t\right)\right]^{1/2}=2\left|\cos\left(\frac{s-t}{2}\right)\right|,\end{aligned}$$

then by (2.15) we have

$$\begin{aligned}\left|\sin\left[n\left(\frac{t-s}{2}\right)\right]\right| &\le n\left|\sin\left(\frac{s-t}{2}\right)\right|\left(\int_{0}^{1}\left|\left(1-\lambda\right)e^{it}+\lambda e^{is}\right|^{(n-1)\alpha}d\lambda\right)^{1/\alpha}\\ &\le n\left|\sin\left(\frac{s-t}{2}\right)\right|\frac{1}{2^{1/\alpha}}\left[\left|\cos\left(\frac{s-t}{2}\right)\right|+1\right]^{1/\alpha}\\ &\le n\left|\sin\left(\frac{s-t}{2}\right)\right|\end{aligned} \tag{2.15}$$

for any $s,t\in\mathbb{R}$.

Now, by the help of power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z)=\sum_{n=0}^{\infty}\left|a_n\right|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n\ge 0$ then $f_a=f$.

We notice that if

$$\begin{aligned}f\left(z\right) &= \sum_{n=1}^{\infty}\frac{\left(-1\right)^{n}}{n}z^{n}=\ln\frac{1}{1+z},\ z\in D\left(0,1\right);\\ g\left(z\right) &= \sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{(2n)!}z^{2n}=\cos z,\ z\in\mathbb{C};\\ h\left(z\right) &= \sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{(2n+1)!}z^{2n+1}=\sin z,\ z\in\mathbb{C};\\ l\left(z\right) &= \sum_{n=0}^{\infty}\left(-1\right)^{n}z^{n}=\frac{1}{1+z},\ z\in D\left(0,1\right);\end{aligned} \tag{2.16}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z) = \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \ln \frac{1}{1-z}, \ z \in D(0,1); \tag{2.17}$$

$$g_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C};$$

$$h_a(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C};$$

$$l_a(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \ z \in D(0,1).$$

Theorem 2. *Consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that is convergent on the open disck $D(0,R)$. Then we have the inequalities*

$$\begin{aligned} |f(w) - f(z)| &\le |w-z| \int_0^1 |f'((1-t)z + tw)| \, dt \\ &\le |w-z| \int_0^1 f_a'(|(1-t)z + tw|) \, dt \end{aligned} \tag{2.18}$$

for any $z, w \in D$.

Moreover, we have

$$\begin{aligned} \int_0^1 f_a'(|(1-t)z + tw|) \, dt &\le \frac{1}{2} \left[\left| f_a' \left(\frac{z+w}{2} \right) \right| + \frac{|f_a'(z)| + |f_a'(w)|}{2} \right] \\ &\le \frac{|f_a'(z)| + |f_a'(w)|}{2} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \int_0^1 f_a'(|(1-t)z + tw|) \, dt &\le \begin{cases} \frac{f_a(|z|) - f_a(|w|)}{|z| - |w|} & \text{if } |z| \ne |w| \\ f_a'(|z|) & \text{if } |z| = |w| \end{cases} \\ &\le \frac{1}{2} \left[f_a' \left(\frac{|z| + |w|}{2} \right) + \frac{f_a'(|z|) + f_a'(|w|)}{2} \right] \\ &\le \frac{f_a'(|z|) + f_a'(|w|)}{2}, \end{aligned} \tag{2.20}$$

for any $z, w \in D$.

Proof. We have $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ and $f_a'(z) = \sum_{n=1}^{\infty} n |a_n| z^{n-1}$. For $m \ge 1$, by using the generalized triangle inequality we have

$$\left| \sum_{n=1}^{m} n a_n z^{n-1} \right| \le \sum_{n=1}^{m} n |a_n| z^{n-1}. \tag{2.21}$$

Since the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n |a_n| z^{n-1}$ are convergent, then by letting $m \to \infty$ in (2.21) we get

$$|f'(z)| \le f_a'(|z|) \text{ for any } z \in D(0,R).$$

This prove the second inequality in (2.18).

We observe that, since f_a' has nonnegative coefficients, then this functions is convex as a real variable functions on the interval $(-R,R)$ and increasing on $[0,R)$.

For $z,w\in D$, consider the function $h_{z,w}:[0,1]\to[0,\infty)$, $h_{z,w}(t):=f_a'(|(1-t)z+tw|)$. For $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$ we have

$$\begin{aligned}h_{z,w}(\alpha t_1+\beta t_2)&=f_a'(|(1-\alpha t_1+\beta t_2)z+\alpha t_1+\beta t_2 w|)\\&=f_a'[|\alpha((1-t_1)z+t_1w)+\beta((1-t_2)z+t_2w)|]\\&\le f_a'[\alpha|(1-t_1)z+t_1w|+\beta|(1-t_2)z+t_2w|]\\&\le \alpha f_a'(|(1-t_1)z+t_1w|)+\beta f_a'(|(1-t_2)z+t_2w|),\end{aligned}$$

which shows that $h_{z,w}$ is convex on $[0,1]$.

If we write the Hermite-Hadamard inequality (2.2) for $h_{z,w}$ on $[0,1]$ then we get

$$\begin{aligned}\int_0^1 f_a'(|(1-t)z+tw|)\,dt&\le\frac{1}{2}\left[\left|f_a'\left(\frac{z+w}{2}\right)\right|+\frac{|f_a'(z)|+|f_a'(w)|}{2}\right]\\&\le\frac{|f_a'(z)|+|f_a'(w)|}{2}\end{aligned}$$

for any $z,w\in D$, which proves (2.19).

We also have

$$f_a'(|(1-t)z+tw|)\le f_a'((1-t)|z|+t|w|)$$

for any $z,w\in D$ and $t\in[0,1]$ and since the function $\ell_{z,w}(t):=f_a'((1-t)|z|+t|w|)$ is convex, then by the inequality (2.2) we have

$$\begin{aligned}\int_0^1 f_a'(|(1-t)z+tw|)\,dt&\le\int_0^1 f_a'((1-t)|z|+t|w|)\,dt\\&\le\frac{1}{2}\left[f_a'\left(\frac{|z|+|w|}{2}\right)+\frac{f_a'(|z|)+f_a'(|w|)}{2}\right]\\&\le\frac{f_a'(|z|)+f_a'(|w|)}{2}.\end{aligned}$$

For $|z|\ne|w|$ we have

$$\int_0^1 f_a'((1-t)|z|+t|w|)\,dt=\frac{f(|z|)-f(|w|)}{|z|-|w|},$$

while for $|z|=|w|$ we have

$$\int_0^1 f_a'((1-t)|z|+t|w|)\,dt=f_a'(|z|).$$

This proves (2.20). □

Corollary 1. *With the assumptions of Theorem 2 and if $R > 1$, then we have*

$$\begin{aligned}(2.22)\qquad &\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right| \\ &\leq 2\left|\sin\left(\frac{s-t}{2}\right)\right|\int_0^1\left|f'\left((1-\lambda)e^{is}+\lambda e^{it}\right)\right|d\lambda \\ &\leq 2\left|\sin\left(\frac{s-t}{2}\right)\right|\int_0^1 f'_a\left(\left|(1-\lambda)e^{is}+\lambda e^{it}\right|\right)d\lambda \\ &\leq \left|\sin\left(\frac{s-t}{2}\right)\right|\left[\left|f'_a\left(\frac{e^{is}+e^{it}}{2}\right)\right|+\frac{\left|f'_a\left(e^{is}\right)\right|+\left|f'_a\left(e^{it}\right)\right|}{2}\right] \\ &\leq \left|\sin\left(\frac{s-t}{2}\right)\right|\left[\left|f'_a\left(e^{is}\right)\right|+\left|f'_a\left(e^{it}\right)\right|\right]\leq 2\left|\sin\left(\frac{s-t}{2}\right)\right|f'_a(1)\end{aligned}$$

for any $t, s \in \mathbb{R}$.

The proof is obvious by Theorem 2 on chosing $z = e^{is}$ and $w = e^{it}$.

Remark 3. *We observe that the integral $\int_0^1 f'_a(|(1-t)z+tw|)\,dt$, which might be difficult to compute in various examples of functions f, has got the simpler bounds*

$$B_1(z,w) := \frac{1}{2}\left[f'_a\left(\left|\frac{z+w}{2}\right|\right)+\frac{f'_a(|z|)+f'_a(|w|)}{2}\right]$$

and

$$B_2(z,w) := \begin{cases}\frac{f_a(|z|)-f_a(|w|)}{|z|-|w|} & \text{if } |z| \neq |w|, \\ \\ f'_a(|w|) & \text{if } |z| = |w|.\end{cases}$$

It is natural then to ask which of these bounds is better?

Let us consider the simple examples of powers, namely $f(z) = z^m$ with $m \geq 1$. Then

$$B_1(z,w) = \frac{1}{2}m\left[\left|\frac{z+w}{2}\right|^{m-1}+\frac{|z|^{m-1}+|w|^{m-1}}{2}\right]$$

and

$$B_2(z,w) := \begin{cases}|w|^{m-1}+|w|^{m-2}|z|+...+|z|^{m-1} & \text{if } |w| \neq |z|, \\ \\ m|z|^{m-1} & \text{if } |w| = |z|.\end{cases}$$

If we take $w = tz$ with $|z| = 1$ and $|t| \neq 1$ then we get

$$B_1(t) = \frac{1}{2}m\left[\left|\frac{1+t}{2}\right|^{m-1}+\frac{1+|t|^{m-1}}{2}\right]$$

and

$$B_2(t) = |t|^{m-1}+...+|t|+1.$$

If we take $m = 4$ and plot the difference

$$d(t) := 2\left(\left|\frac{t+1}{2}\right|^3+\frac{1+|t|^3}{2}\right)-\left(|t|^3+|t|^2+|t|+1\right)$$

on the interval $[-8, 8]$, then we can conclude that some time the first bound is better than the second, while other time the conclusion is the other way around. The details are left to the interested reader.

Remark 4. *If the power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ has nonnegative coefficients and is convergent on the open disck $D(0,R)$, then $f'=f'_a$ and we have the inequalities*

$$|f(w)-f(z)|\le|w-z|\int_0^1|f'((1-t)z+tw)|\,dt \tag{2.23}$$
$$\le|w-z|\int_0^1 f'(|(1-t)z+tw|)\,dt$$
$$\le|w-z|\begin{cases}\frac{1}{2}\left[\left|f'\left(\frac{z+w}{2}\right)\right|+\frac{|f'(z)|+|f'(w)|}{2}\right]\\ \frac{f(|z|)-f(|w|)}{|z|-|w|} & \text{if } |z|\neq|w|\\ f'(|z|) & \text{if } |z|=|w|\end{cases}$$

for any $z,w\in D$.

Important examples of functions as power series representations with nonnegative coefficients in addition to the ones from (2.17), are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C}; \tag{2.24}$$
$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1);$$
$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\ z\in D(0,1);$$
$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1);$$
$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is Gamma function.

If we write the inequality (2.23) for $f(z)=(1-z)^{-1}$ with $z\in D(0,1)$, then we have

$$\left|(1-w)^{-1}-(1-z)^{-1}\right| \tag{2.25}$$
$$\le|w-z|\int_0^1|1-[(1-t)z+tw]|^{-2}\,dt$$
$$\le|w-z|\int_0^1(1-|(1-t)z+tw|)^{-2}\,dt$$
$$\le|w-z|\begin{cases}\frac{1}{2}\left[\left(1-\left|\frac{z+w}{2}\right|\right)^{-2}+\frac{(1-|z|)^{-2}+(1-|w|)^{-2}}{2}\right],\\ (1-|w|)^{-1}(1-|z|)^{-1}\ \text{if } |w|\neq|z|,\\ (1-|z|)^{-2}\ \text{if } |w|=|z|.\end{cases}$$

If we write the inequality (2.23) for $f(z)=\ln(1-z)^{-1}$ *with* $z\in D(0,1)$, *then we have*

$$\begin{aligned}
&\left|\ln(1-w)^{-1}-\ln(1-z)^{-1}\right| \\
&\le |w-z|\int_0^1 \left|1-[(1-t)z+tw]\right|^{-1}dt \\
&\le |w-z|\int_0^1 \left(1-|(1-t)z+tw|\right)^{-1}dt \\
&\le |w-z|\begin{cases}\frac{1}{2}\left[\left(1-\left|\frac{z+w}{2}\right|\right)^{-1}+\frac{(1-|z|)^{-1}+(1-|w|)^{-1}}{2}\right],\\[2ex] \frac{\ln(1-|w|)^{-1}-\ln(1-|z|)^{-1}}{|w|-|z|} \text{ if } |w|\ne|z|,\\[2ex] (1-|z|)^{-1} \text{ if } |w|=|z|,\end{cases}
\end{aligned} \tag{2.26}$$

for any $z,w\in D(0,1)$.

Remark 5. *If we write the inequality (2.22) for* $f(z)=\sinh(z)$, *then we have*

$$\begin{aligned}
&\left|\sinh\left(e^{is}\right)-\sinh\left(e^{it}\right)\right| \\
&\le 2\left|\sin\left(\frac{s-t}{2}\right)\right|\int_0^1\left|\cosh\left((1-\lambda)e^{is}+\lambda e^{it}\right)\right|d\lambda \\
&\le 2\left|\sin\left(\frac{s-t}{2}\right)\right|\int_0^1\cosh\left(\left|(1-\lambda)e^{is}+\lambda e^{it}\right|\right)d\lambda \\
&\le \left|\sin\left(\frac{s-t}{2}\right)\right|\left[\left|\cosh\left(\frac{e^{is}+e^{it}}{2}\right)\right|+\frac{\left|\cosh\left(e^{is}\right)\right|+\left|\cosh\left(e^{it}\right)\right|}{2}\right] \\
&\le \left|\sin\left(\frac{s-t}{2}\right)\right|\left[\left|\cosh\left(e^{is}\right)\right|+\left|\cosh\left(e^{it}\right)\right|\right]\le\left|\sin\left(\frac{s-t}{2}\right)\right|\frac{e^2+1}{e} \\
&\le \frac{e^2+1}{2e}|s-t|
\end{aligned} \tag{2.27}$$

for any $t,s\in\mathbb{R}$.

3. Other Bounds for Special Convexities

We say that the function $f:I\subset\mathbb{R}\to(0,\infty)$ is *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x,y\in I$ and $t\in[0,1]$ one has the inequality

$$f(tx+(1-t)y)\le[f(x)]^t[f(y)]^{1-t}. \tag{3.1}$$

The concept can be generalised for functions defined on convex subsets of linear spaces.

We note that if f and g are convex and g is increasing, then $g\circ f$ is convex; moreover, since $f=\exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (3.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t[f(y)]^{1-t}\le tf(x)+(1-t)f(y)$$

for any $x, y \in I$ and $t \in [0,1]$.

It is known that, see [10, p. 199] if $f : I \subset \mathbb{R} \to (0, \infty)$ is *log-convex* then for any $a, b \in I$ with $a < b$, we have

$$\frac{1}{b-a} \int_a^b f(x)\, dx \le L(f(a), f(b)), \tag{3.2}$$

where $L(p, q)$ is the *logarithmic mean* of the strictly positive real numbers p, q, i.e.,

$$L(p,q) := \begin{cases} \frac{p-q}{\ln p - \ln q} & \text{if } p \ne q \\ \\ p & \text{if } p = q. \end{cases}$$

The inequality (3.2) can be written in an equivalent form as

$$\int_0^1 f((1-\lambda) a + \lambda b)\, d\lambda \le L(f(a), f(b)). \tag{3.3}$$

This inequality is obviously true for the more general case of log-convex functions on convex subsets C of linear spaces X and $a, b \in C$.

Proposition 1. *If the function $|f'|^{\alpha}$ with $\alpha \ge 1$ is log-convex on D, then we have the inequalities*

$$\begin{aligned} |f(w) - f(z)| &\le |w - z| \left(\int_0^1 |f'((1-t) z + tw)|^{\alpha}\, dt \right)^{1/\alpha} \\ &\le |w - z| \left[L\left(|f'(z)|^{\alpha}, |f'(w)|^{\alpha} \right) \right]^{1/\alpha} \end{aligned} \tag{3.4}$$

for any $z, w \in D$.

The proof follows by (2.1) and (3.3). The details are omitted.

According to Hudzik & Maligranda [12], a function $f : [0, \infty) \to \mathbb{R}$ is *s-convex in the second sense*, with s fixed in $(0, 1]$ if

$$f((1-t) u + tv) \le (1-t)^s f(u) + t^s f(v) \tag{3.5}$$

for any $u, v \in [0, \infty)$ and $t \in [0,1]$. For various properties of this class of functions see [12] and [10, p. 286].

It is clear that, the above definition can be extended for functions defined on convex subsets C on linear spaces X. Now, if $f : C \to \mathbb{R}$ is s-convex in the second sense on C and for $a, b \in C$ the function $f((1-\cdot) a + \cdot b)$ is Lebesgue integrable on $[0,1]$, then by integrating on (3.5) we get [10, p. 288]

$$\int_0^1 f((1-t) a + tb)\, dt \le \frac{f(a) + f(b)}{s+1}. \tag{3.6}$$

Utilisig this inequality we can state:

Proposition 2. *If the function $|f'|^{\alpha}$ with $\alpha \ge 1$ is s-convex in the second sense on D, then we have the inequalities*

$$\begin{aligned} |f(w) - f(z)| &\le |w - z| \left(\int_0^1 |f'((1-t) z + tw)|^{\alpha}\, dt \right)^{1/\alpha} \\ &\le |w - z| \left[\frac{|f'(z)|^{\alpha} + |f'(w)|^{\alpha}}{s+1} \right]^{1/\alpha} \end{aligned} \tag{3.7}$$

for any $z, w \in D$.

4. Some Discrete Inequalities

In some applications, one can be interested to estimate the quantity

$$\frac{1}{n}\sum_{k=1}^{n} f\left(z_k\right)-f\left(\frac{1}{n}\sum_{j=1}^{n} z_j\right)$$

for various complelx functions $f: D\subset\mathbb{C}\to\mathbb{C}$.

The weighted case is as follows.

Theorem 3. *Let $f: D\subseteq\mathbb{C}\to\mathbb{C}$ be a differentiable function on the convex domain D. Then for any $z_k\in D$ and $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$ we have the inequality*

$$\begin{aligned}
(4.1)\qquad &\left|\sum_{k=1}^{n} p_k f\left(z_k\right)-f\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
&\leq \sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right|\int_0^1\left|f'\left((1-t)\,z_k+t\sum_{j=1}^{n} p_j z_j\right)\right| dt \\
&\leq \begin{cases}
\max_{k\in\{1,...,n\}}\left|z_k-\sum_{j=1}^{n} p_j z_j\right| \\
\times\sum_{k=1}^{n} p_k\int_0^1\left|f'\left((1-t)\,z_k+t\sum_{j=1}^{n} p_j z_j\right)\right| dt; \\
\\
\left(\sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right|^{\beta}\right)^{1/\beta} \\
\times\left[\sum_{k=1}^{n} p_k\int_0^1\left|f'\left((1-t)\,z_k+t\sum_{j=1}^{n} p_j z_j\right)\right|^{\alpha} dt\right]^{1/\alpha}, \\
\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\
\\
\sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right| \\
\times\int_0^1\max_{k\in\{1,...,n\}}\left|f'\left((1-t)\,z_k+t\sum_{j=1}^{n} p_j z_j\right)\right| dt.
\end{cases}
\end{aligned}$$

Proof. We have from (2.1) that

$$\begin{aligned}
&\left|f\left(z_k\right)-f\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
&\leq\left|z_k-\sum_{j=1}^{n} p_j z_j\right|\int_0^1\left|f'\left((1-t)\,z_k+t\sum_{j=1}^{n} p_j z_j\right)\right| dt
\end{aligned}$$

for any $k\in\{1,...,n\}$.

If we multiply by $p_k\geq 0$ and sum over k from 1 to n we get

$$\begin{aligned}
&\sum_{k=1}^{n} p_k\left|f\left(z_k\right)-f\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
&\leq\sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right|\int_0^1\left|f'\left((1-t)\,z_k+t\sum_{j=1}^{n} p_j z_j\right)\right| dt.
\end{aligned}$$

By the generalized triangle inequality for modulus we also have

$$\left|\sum_{k=1}^{n} p_k f(z_k) - f\left(\sum_{j=1}^{n} p_j z_j\right)\right| \leq \sum_{k=1}^{n} p_k \left|f(z_k) - f\left(\sum_{j=1}^{n} p_j z_j\right)\right|$$

and the first ineqality in (4.1) is proved.

Now we use the Hölder weighted inequality

$$\sum_{k=1}^{n} p_k a_k b_k \leq \begin{cases} \max_{k\in\{1,\dots,n\}} \{a_k\} \sum_{k=1}^{n} p_k b_k \\ \\ \left(\sum_{k=1}^{n} p_k a_k^{\beta}\right)^{1/\beta} \left(\sum_{k=1}^{n} p_k b_k^{\alpha}\right)^{1/\alpha}; \quad \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1 \end{cases}$$

where $p_k, a_k, b_k \geq 0$, to get

$$\sum_{k=1}^{n} p_k \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right| dt$$
$$\leq \begin{cases} \max_{k\in\{1,\dots,n\}} \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \\ \times \sum_{k=1}^{n} p_k \int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right| dt; \\ \\ \left(\sum_{k=1}^{n} p_k \left|z_k - \sum_{j=1}^{n} p_j z_j\right|^{\beta}\right)^{1/\beta} \\ \times \left[\sum_{k=1}^{n} p_k \left(\int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right| dt\right)^{\alpha}\right]^{1/\alpha}, \\ \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \\ \sum_{k=1}^{n} p_k \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \\ \times \max_{k\in\{1,\dots,n\}} \left[\int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right| dt\right]. \end{cases}$$

Since

$$\left(\int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right| dt\right)^{\alpha} \leq \int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right|^{\alpha} dt$$

and

$$\max_{k\in\{1,\dots,n\}} \left[\int_0^1 \left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right| dt\right]$$
$$\leq \int_0^1 \max_{k\in\{1,\dots,n\}} \left[\left|f'\left((1-t) z_k + t \sum_{j=1}^{n} p_j z_j\right)\right|\right] dt$$

the last part of (4.1) is thus proved. □

Corollary 2. *With the assumptions in Theorem 3, we have*

$$
\begin{aligned}
(4.2)\qquad & \left|\frac{1}{n}\sum_{k=1}^{n} f(z_k) - f\left(\frac{1}{n}\sum_{j=1}^{n} z_j\right)\right| \\
&\le \frac{1}{n}\sum_{k=1}^{n}\left|z_k - \frac{1}{n}\sum_{j=1}^{n} z_j\right|\int_0^1\left|f'\left((1-t)z_k + t\frac{1}{n}\sum_{j=1}^{n} z_j\right)\right|dt \\
&\le \begin{cases}
\frac{1}{n}\max_{k\in\{1,\dots,n\}}\left|z_k - \frac{1}{n}\sum_{j=1}^{n} z_j\right| \\
\times\sum_{k=1}^{n}\int_0^1\left|f'\left((1-t)z_k + t\frac{1}{n}\sum_{j=1}^{n} z_j\right)\right|dt; \\
\\
\frac{1}{n}\left(\sum_{k=1}^{n}\left|z_k - \frac{1}{n}\sum_{j=1}^{n} z_j\right|^{\beta}\right)^{1/\beta} \\
\times\left[\sum_{k=1}^{n}\int_0^1\left|f'\left((1-t)z_k + t\frac{1}{n}\sum_{j=1}^{n} z_j\right)\right|^{\alpha}dt\right]^{1/\alpha}, \\
\alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\
\\
\frac{1}{n}\sum_{k=1}^{n}\left|z_k - \frac{1}{n}\sum_{j=1}^{n} z_j\right| \\
\times\int_0^1\max_{k\in\{1,\dots,n\}}\left|f'\left((1-t)z_k + t\frac{1}{n}\sum_{j=1}^{n} z_j\right)\right|dt.
\end{cases}
\end{aligned}
$$

Remark 6. *If the function $|f'|$ is convex, then*

$$
\begin{aligned}
&\int_0^1\sum_{k=1}^{n} p_k\left|f'\left((1-t)z_k + t\sum_{j=1}^{n} p_j z_j\right)\right|dt \\
&\le \int_0^1\sum_{k=1}^{n} p_k\left[(1-t)\left|f'(z_k)\right| + t\left|f'\left(\sum_{j=1}^{n} p_j z_j\right)\right|\right]dt \\
&= \frac{1}{2}\left[\sum_{k=1}^{n} p_k\left|f'(z_k)\right| + \left|f'\left(\sum_{j=1}^{n} p_j z_j\right)\right|\right] \le \sum_{k=1}^{n} p_k\left|f'(z_k)\right|
\end{aligned}
$$

and from (4.1) we have the simpler inequality

$$
\begin{aligned}
(4.3)\qquad & \left|\sum_{k=1}^{n} p_k f(z_k) - f\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
&\le \frac{1}{2}\max_{k\in\{1,\dots,n\}}\left|z_k - \sum_{j=1}^{n} p_j z_j\right|\left[\sum_{k=1}^{n} p_k\left|f'(z_k)\right| + \left|f'\left(\sum_{j=1}^{n} p_j z_j\right)\right|\right] \\
&\le \max_{k\in\{1,\dots,n\}}\left|z_k - \sum_{j=1}^{n} p_j z_j\right|\sum_{k=1}^{n} p_k\left|f'(z_k)\right|.
\end{aligned}
$$

Also, with the same assumption for $|f'|$ *we have*

$$\begin{aligned}
(4.4)\qquad & \int_0^1 \max_{k\in\{1,\dots,n\}} \left| f'\left((1-t)\, z_k + t\sum_{j=1}^n p_j z_j \right) \right| dt \\
& \le \int_0^1 \max_{k\in\{1,\dots,n\}} \left\{ (1-t)\, |f'(z_k)| + t \left| f'\left(\sum_{j=1}^n p_j z_j \right) \right| \right\} dt \\
& = \frac{1}{2} \left[\max_{k\in\{1,\dots,n\}} \{|f'(z_k)|\} + \left| f'\left(\sum_{j=1}^n p_j z_j \right) \right| \right] \\
& \le \frac{1}{2} \left[\max_{k\in\{1,\dots,n\}} \{|f'(z_k)|\} + \sum_{j=1}^n p_j\, |f'(z_j)| \right] \le \max_{k\in\{1,\dots,n\}} \{|f'(z_k)|\}
\end{aligned}$$

and from (4.1) we have the simpler inequality

$$\begin{aligned}
(4.5)\qquad & \left| \sum_{k=1}^n p_k f(z_k) - f\left(\sum_{j=1}^n p_j z_j \right) \right| \\
& \le \frac{1}{2} \sum_{k=1}^n p_k \left| z_k - \sum_{j=1}^n p_j z_j \right| \left[\max_{k\in\{1,\dots,n\}} \{|f'(z_k)|\} + \left| f'\left(\sum_{j=1}^n p_j z_j \right) \right| \right] \\
& \le \frac{1}{2} \sum_{k=1}^n p_k \left| z_k - \sum_{j=1}^n p_j z_j \right| \left[\max_{k\in\{1,\dots,n\}} \{|f'(z_k)|\} + \sum_{j=1}^n p_j\, |f'(z_j)| \right] \\
& \le \max_{k\in\{1,\dots,n\}} \{|f'(z_k)|\} \sum_{k=1}^n p_k \left| z_k - \sum_{j=1}^n p_j z_j \right| .
\end{aligned}$$

If for some $\alpha > 1$, *the function* $|f'|^\alpha$ *is convex on* D, *then*

$$\begin{aligned}
(4.6)\qquad & \int_0^1 \sum_{k=1}^n p_k \left| f'\left((1-t)\, z_k + t\sum_{j=1}^n p_j z_j \right) \right|^\alpha dt \\
& \le \int_0^1 \sum_{k=1}^n p_k \left[(1-t)\, |f'(z_k)|^\alpha + t \left| f'\left(\sum_{j=1}^n p_j z_j \right) \right|^\alpha \right] dt \\
& = \frac{1}{2} \left[\sum_{k=1}^n p_k\, |f'(z_k)|^\alpha + \left| f'\left(\sum_{j=1}^n p_j z_j \right) \right|^\alpha \right]
\end{aligned}$$

and from (4.1) we have the simpler inequality

$$\left|\sum_{k=1}^{n}p_{k}f\left(z_{k}\right)-f\left(\sum_{j=1}^{n}p_{j}z_{j}\right)\right| \tag{4.7}$$
$$\leq\frac{1}{2^{1/\alpha}}\left(\sum_{k=1}^{n}p_{k}\left|z_{k}-\sum_{j=1}^{n}p_{j}z_{j}\right|^{\beta}\right)^{1/\beta}$$
$$\times\left[\sum_{k=1}^{n}p_{k}\left|f'\left(z_{k}\right)\right|^{\alpha}+\left|f'\left(\sum_{j=1}^{n}p_{j}z_{j}\right)\right|^{\alpha}\right]^{1/\alpha}$$
$$\leq\left(\sum_{k=1}^{n}p_{k}\left|z_{k}-\sum_{j=1}^{n}p_{j}z_{j}\right|^{\beta}\right)^{1/\beta}\left(\sum_{k=1}^{n}p_{k}\left|f'\left(z_{k}\right)\right|^{\alpha}\right)^{1/\alpha}$$

where $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

The particular case $\alpha=\beta=2$ *is of interest, since*

$$\sum_{k=1}^{n}p_{k}\left|z_{k}-\sum_{j=1}^{n}p_{j}z_{j}\right|^{2}=\sum_{k=1}^{n}p_{k}\left|z_{k}\right|^{2}-\left|\sum_{j=1}^{n}p_{j}z_{j}\right|^{2}$$

and (4.7) becomes

$$\left|\sum_{k=1}^{n}p_{k}f\left(z_{k}\right)-f\left(\sum_{j=1}^{n}p_{j}z_{j}\right)\right| \tag{4.8}$$
$$\leq\frac{\sqrt{2}}{2}\left(\sum_{k=1}^{n}p_{k}\left|z_{k}\right|^{2}-\left|\sum_{j=1}^{n}p_{j}z_{j}\right|^{2}\right)^{1/2}$$
$$\times\left[\sum_{k=1}^{n}p_{k}\left|f'\left(z_{k}\right)\right|^{2}+\left|f'\left(\sum_{j=1}^{n}p_{j}z_{j}\right)\right|^{2}\right]^{1/2}$$
$$\leq\left(\sum_{k=1}^{n}p_{k}\left|z_{k}\right|^{2}-\left|\sum_{j=1}^{n}p_{j}z_{j}\right|^{2}\right)^{1/2}\left(\sum_{k=1}^{n}p_{k}\left|f'\left(z_{k}\right)\right|^{2}\right)^{1/2}$$

provided $\left|f'\right|^{2}$ *is convex on* D.

For the exponential function $f(z) = \exp z$, the inequality (4.1) becomes

$$
\begin{aligned}
(4.9)\quad & \left|\sum_{k=1}^{n} p_k \exp(z_k) - \exp\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
& \leq \sum_{k=1}^{n} p_k \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \int_0^1 \exp\left((1-t)\operatorname{Re} z_k + t\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right) dt \\
& \leq \begin{cases}
\max_{k\in\{1,\dots,n\}} \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \\
\times \sum_{k=1}^{n} p_k \int_0^1 \exp\left((1-t)\operatorname{Re} z_k + t\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right) dt; \\
\\
\left(\sum_{k=1}^{n} p_k \left|z_k - \sum_{j=1}^{n} p_j z_j\right|^{\beta}\right)^{1/\beta} \\
\times \left[\sum_{k=1}^{n} p_k \int_0^1 \exp\left[\alpha\left((1-t)\operatorname{Re} z_k + t\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right)\right] dt\right]^{1/\alpha}, \\
\alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\
\\
\sum_{k=1}^{n} p_k \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \\
\times \int_0^1 \exp\left((1-t)\max_{k\in\{1,\dots,n\}} \operatorname{Re} z_k + t\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right) dt,
\end{cases}
\end{aligned}
$$

for any $z_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$.

Since $|\exp|^{\alpha}$, $\alpha \geq 1$ is convex, then the simpler inequalities can be stated as well

$$
\begin{aligned}
(4.10)\quad & \left|\sum_{k=1}^{n} p_k \exp(z_k) - \exp\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
& \leq \frac{1}{2} \max_{k\in\{1,\dots,n\}} \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \left[\sum_{k=1}^{n} p_k \exp \operatorname{Re}(z_k) + \exp\left(\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right)\right] \\
& \leq \max_{k\in\{1,\dots,n\}} \left|z_k - \sum_{j=1}^{n} p_j z_j\right| \sum_{k=1}^{n} p_k \exp \operatorname{Re}(z_k),
\end{aligned}
$$

$$
\begin{aligned}
(4.11)\qquad & \left|\sum_{k=1}^{n} p_k \exp\left(z_k\right)-\exp\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
& \leq \frac{1}{2} \sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right| \\
& \times\left[\exp \left\{\max _{k \in\{1, \ldots, n\}} \operatorname{Re}\left(z_k\right)\right\}+\exp \left(\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right)\right] \\
& \leq \frac{1}{2} \sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right| \\
& \times\left[\exp \left\{\max _{k \in\{1, \ldots, n\}} \operatorname{Re}\left(z_k\right)\right\}+\sum_{j=1}^{n} p_j \exp \operatorname{Re}\left(z_j\right)\right] \\
& \leq \exp \left\{\max _{k \in\{1, \ldots, n\}} \operatorname{Re}\left(z_k\right)\right\} \sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right| .
\end{aligned}
$$

and

$$
\begin{aligned}
(4.12)\qquad & \left|\sum_{k=1}^{n} p_k \exp\left(z_k\right)-\exp\left(\sum_{j=1}^{n} p_j z_j\right)\right| \\
& \leq \frac{1}{2^{1/\alpha}}\left(\sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right|^{\beta}\right)^{1/\beta} \\
& \times\left[\sum_{k=1}^{n} p_k \exp\left[\alpha \operatorname{Re}\left(z_k\right)\right]+\exp\left[\alpha\left(\sum_{j=1}^{n} p_j \operatorname{Re} z_j\right)\right]\right]^{1/\alpha} \\
& \leq\left(\sum_{k=1}^{n} p_k\left|z_k-\sum_{j=1}^{n} p_j z_j\right|^{\beta}\right)^{1/\beta}\left(\sum_{k=1}^{n} p_k \exp\left[\alpha \operatorname{Re}\left(z_k\right)\right]\right)^{1/\alpha},
\end{aligned}
$$

for any $z_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, \ldots, n\}$ with $\sum_{k=1}^{n} p_k=1$, where $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

If we consider the power function $f(z)=z^m, m\geq 1$. It is clear that $|f'|$ is convex and then by (4.3) we get

$$\begin{aligned}
(4.13)\qquad & \left|\sum_{k=1}^{n}p_k z_k^m-\left(\sum_{j=1}^{n}p_j z_j\right)^m\right| \\
&\leq \frac{1}{2}m\max_{k\in\{1,...,n\}}\left|z_k-\sum_{j=1}^{n}p_j z_j\right|\left[\sum_{k=1}^{n}p_k|z_k|^{m-1}+\left|\sum_{j=1}^{n}p_j z_j\right|^{m-1}\right] \\
&\leq m\max_{k\in\{1,...,n\}}\left|z_k-\sum_{j=1}^{n}p_j z_j\right|\sum_{k=1}^{n}p_k|z_k|^{m-1},
\end{aligned}$$

for any $z_k\in\mathbb{C}$ and $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$.

From (4.5) we also get

$$\begin{aligned}
(4.14)\qquad & \left|\sum_{k=1}^{n}p_k z_k^m-\left(\sum_{j=1}^{n}p_j z_j\right)^m\right| \\
&\leq \frac{1}{2}m\sum_{k=1}^{n}p_k\left|z_k-\sum_{j=1}^{n}p_j z_j\right|\left[\left\{\max_{k\in\{1,...,n\}}|z_k|\right\}^{m-1}+\left|\sum_{j=1}^{n}p_j z_j\right|^{m-1}\right] \\
&\leq \frac{1}{2}m\sum_{k=1}^{n}p_k\left|z_k-\sum_{j=1}^{n}p_j z_j\right|\left[\left\{\max_{k\in\{1,...,n\}}|z_k|\right\}^{m-1}+\sum_{k=1}^{n}p_k|z_k|^{m-1}\right] \\
&\leq m\left\{\max_{k\in\{1,...,n\}}|z_k|\right\}^{m-1}\sum_{k=1}^{n}p_k\left|z_k-\sum_{j=1}^{n}p_j z_j\right|,
\end{aligned}$$

for any $z_k\in\mathbb{C}$ and $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$.

Finally, from (4.8) we have

$$\begin{aligned}
(4.15)\qquad & \left|\sum_{k=1}^{n}p_k z_k^m-\left(\sum_{j=1}^{n}p_j z_j\right)^m\right| \\
&\leq m\frac{\sqrt{2}}{2}\left(\sum_{k=1}^{n}p_k|z_k|^2-\left|\sum_{j=1}^{n}p_j z_j\right|^2\right)^{1/2} \\
&\times\left[\sum_{k=1}^{n}p_k|z_k|^{2(m-1)}+\left|\sum_{j=1}^{n}p_j z_j\right|^{2(m-1)}\right]^{1/2} \\
&\leq m\left(\sum_{k=1}^{n}p_k|z_k|^2-\left|\sum_{j=1}^{n}p_j z_j\right|^2\right)^{1/2}\left(\sum_{k=1}^{n}p_k|z_k|^{2(m-1)}\right)^{1/2}
\end{aligned}$$

for any $z_k\in\mathbb{C}$ and $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^{n}p_k=1$.

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