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ON INVARIANCE EQUATION FOR MEANS OF POWER GROWTH

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Abstract. We discuss properties of the solutions of the invariance equation

$$M(N(x,y),K(x,y)) = M(x,y)$$

for homogeneous, symmetric means M, N, K of power growth.

By a mean we understand a function $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfying for all $x, y \in \mathbb{R}_+$ the conditions

$$\min(x,y) \le M(x,y) \le \max(x,y).$$

A mean is symmetric if for all x, y holds $M(x,y) = M(y,x)$ and homogeneous if $M(tx,ty) = tM(x,y)$ for all positive t. Given two means N, K, finding another mean M satisfying for all x, y the equation

$$M(N(x,y),K(x,y)) = M(x,y) \tag{1}$$

is called the invariance problem, and the equation (1) is called invariance equation. There is a vaste literature on the subject. The book "Pi and the AGM" ([3]) gives many examples and discusses probably the best known case of the arithmetic-geometric mean, while the historical overview and information on recent developments can be found in [1] and in [2].

The solution to the invariance equation is known to exists in most cases, so it is quite natural to ask whether the solution shares some properties of means N nad K. Sometimes the answer is immediate: if both K and N are symmetric, then obviously M is symmetric too. Sometimes it is not obvious and surprising: if K and N are homogeneous, then M need not be homogeneous.

Ádám Besenyei proved in [1] that in the class of Heinz means

$$H_p(x,y) = \frac{x^p y^{1-p} + x^{1-p} y^p}{2}, \quad 0 \le p \le \frac{1}{2} \tag{2}$$

the invariance equation

$$H_p(H_q(x,y), H_r(x,y)) = H_p(x,y) \tag{3}$$

has only trivial solutions $p = q = r$. The aim of this note is to extend this result to a much broader class of means.

Definition 1. We say that a homogeneous, symmetric mean M is of power growth if there exist a real number $\operatorname{ord}(M)$ and a positive number C_M such that

$$\lim_{x \to 0} \frac{M(x,1)}{x^{\operatorname{ord}(M)}} = C_M.$$

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We shall call $\operatorname{ord}(M)$ the order of M.

Observe that for $0 < x < 1$ the inequality $x \le M(x,1) \le 1$ yields $x^{1-m} \le \frac{M(x,1)}{x^m} \le x^{-m}$. For $m < 0$ the right-hand side tends to 0 and for $m > 1$ the left-hand side tends to infinity. Thus we conclude that $0 \le \operatorname{ord}(M) \le 1$.

Theorem 1. *Let $M,\ N,\ K$ be symmetric, homogeneous means on $\mathbf{R}^2_+$ of power growth. Assume additionally* $\operatorname{ord}(N) \ge \operatorname{ord}(K)$ *and*

$$C_N^{\operatorname{ord}(M)} C_K^{1-\operatorname{ord}(M)} \ne 1 \quad or \quad \operatorname{ord}(M)(1-\operatorname{ord}(N)+\operatorname{ord}(K)) \ne \operatorname{ord}(K). \tag{4}$$

If

$$M(N(x,y),K(x,y)) = M(x,y),$$

then $\operatorname{ord}(M) = \operatorname{ord}(N) = \operatorname{ord}(K)$.

Proof. Denote the orders of $M,\ N,\ K$ by $m,\ n,\ k$ respectively.
Suppose first that $n > k$. We have

$$\begin{aligned}
\frac{M(x,1)}{x^m} &= \frac{M(N(x,1),K(x,1))}{x^m} = x^{-m}K(x,1)M\left(\frac{N(x,1)}{K(x,1)},1\right) \\
&= x^{-m}K(x,1)\left(\frac{N(x,1)}{K(x,1)}\right)^m \frac{M\left(\frac{N(x,1)}{K(x,1)},1\right)}{\left(\frac{N(x,1)}{K(x,1)}\right)^m} \\
&= x^{-m+mn+(1-m)k}\left(\frac{N(x,1)}{x^n}\right)^m\left(\frac{K(x,1)}{x^k}\right)^{1-m}\frac{M\left(\frac{N(x,1)}{K(x,1)},1\right)}{\left(\frac{N(x,1)}{K(x,1)}\right)^m}.
\end{aligned}$$

Since $\frac{N(x,1)}{K(x,1)}$ tends to 0 as x tends to 0, we obtain a contradiction: the limit of the left-hand side equals C_M while the right-hand side tends to 0 or infinity (in case $\operatorname{ord}(M)(1-\operatorname{ord}(N)+\operatorname{ord}(K)) \ne \operatorname{ord}(K)$), or to $C_N^m C_K^{1-m} C_M \ne C_M$.
Therefore we conclude that $n = k$. But this implies

$$M\left(\frac{N(x,1)}{x^n},\frac{K(x,1)}{x^n}\right) = \frac{M(N(x,1),K(x,1))}{x^n} = \frac{M(x,1)}{x^n}$$

and since the left-hand side remains bounded and separated from 0 for small x we conclude that $n = m$.

□

It is worth observing that the Heinz means are linked to the arithmetic mean by the formula $H_\alpha(x,y) = A(x^\alpha y^{1-\alpha}, x^{1-\alpha}y^\alpha)$. Clearly, we can apply the same method to an arbitary homogeneous, symmetric mean M thus obtaining a one-parameter family of means interpolating between M and the geometric mean. The following theorem deals with one-parameter families created this way.

Theorem 2. *Let M be a symmetric, homogeneous mean of order* $\operatorname{ord}(M) \ne \frac{1}{2}$ *with $C_M \ne 1$ and let*

$$M_\alpha(x,y) = M(x^\alpha y^{1-\alpha}, x^{1-\alpha}y^\alpha) \text{ for } 0 \le \alpha \le \frac{1}{2}.$$

Then the invariance equation

$$M_\alpha(M_\beta(x,y),M_\gamma(x,y)) = M_\alpha(x,y) \tag{5}$$

admits only trivial solutions $\alpha = \beta = \gamma$.

Proof. The identity

$$M_\alpha(x,1) = M(x^\alpha, x^{1-\alpha}) = x^\alpha M(1, x^{1-2\alpha}) = x^{\alpha+\mathrm{ord}(M)(1-2\alpha)} \frac{M(1,x^{1-2\alpha})}{x^{\mathrm{ord}(M)(1-2\alpha)}},$$

implies that

$$C_{M_\alpha} = \begin{cases} C_M & \alpha < \frac{1}{2}, \\ 1 & \alpha = \frac{1}{2}, \end{cases} \quad \text{and} \quad \mathrm{ord}(M_\alpha) = \alpha + \mathrm{ord}(M)(1-2\alpha)$$

thus the means in the family are of different order and the result would follow from Theorem 1 once we verify the condition (4). To this end assume $\beta \le \gamma$.
Consider two cases:
Case 1: $\mathrm{ord}(M) < \frac{1}{2}$
The function $\delta \to \mathrm{ord}(M_\delta)$ increases from $\mathrm{ord}(M)$ to $\frac{1}{2}$, so $\mathrm{ord}(M_\beta) \le \mathrm{ord}(M_\gamma)$ and $C_{M_\beta}^{1-\mathrm{ord}(M_\alpha)} C_{M_\gamma}^{\mathrm{ord}(M_\alpha)} = 1$ is possible only if $C_{M_\beta} = C_{M_\gamma} = 1$ (which is equivalent to $\beta = \gamma = \frac{1}{2}$) or $C_{M_\gamma} = 1$ and $1 - \mathrm{ord}(M_\alpha) = 0$. The first case gives immediately $\alpha = \frac{1}{2}$, while the second case is impossible, as $1 - \mathrm{ord}(M_\alpha) \ge \frac{1}{2}$.
Case 2: $\mathrm{ord}(M) > \frac{1}{2}$
Now $\mathrm{ord}(M_\delta)$ decreases from $\mathrm{ord}(M)$ to $\frac{1}{2}$, so $\mathrm{ord}(M_\beta) \ge \mathrm{ord}(M_\gamma)$ and the equality $C_{M_\beta}^{\mathrm{ord}(M_\alpha)} C_{M_\gamma}^{1-\mathrm{ord}(M_\alpha)} = 1$ can hold only if $C_{M_\beta} = C_{M_\gamma} = 1$ or $C_{M_\gamma} = 1$ and $\mathrm{ord}(M_\alpha) = 0$. Again, the first case leads to $\alpha = \frac{1}{2}$, while the second case is impossible, as $\mathrm{ord}(M_\alpha) \ge \frac{1}{2}$. □

Applying Theorem 2 to the arithmetic mean we obtain the result of Besenyei.

Corollary 1 ([1], Theorem 4)**.** *In the class of Heinz means* (2) *the identity* (3) *holds if and only if* $p = q = r$.

As an application of Theorem 1 consider the following families of means:

$$Q_s(x,y) = G^{\frac{2}{s}}(x,y) E^{\frac{s-2}{s}}(s-1,1;x,y) \tag{6}$$

and

$$H_s^1(x,y) = G^{\frac{2}{s}}(x,y) E^{\frac{s-2}{s}}(1-1/s,1/s;x,y), \tag{7}$$

where G is the geometric mean, $E(p,q;x,y) = \left(\frac{q}{p}\frac{x^p-y^p}{x^q-y^q}\right)^{1/(p-q)}$ is the Stolarsky mean and $s \ge 2$. Note that

$$Q_n(x,y) = \left(\frac{x^{n-1}y + x^{n-2}y^2 + \cdots + xy^{n-1}}{n-1}\right)^{1/n}$$

and

$$H_n^1(x,y) = \frac{x^{\frac{n-1}{n}}y^{\frac{1}{n}} + \cdots + x^{\frac{1}{n}}y^{\frac{n-1}{n}}}{n-1}.$$

We see that $\mathrm{ord}(Q_s) = \mathrm{ord}(H_s^1) = 1-1/s$, $C_{Q_s} = (s-1)^{-1/s}$ and $C_{H_s^1} = (s-1)^{-1}$. By Theorem 1 the invariance equations admit only trivial solutions in the two families. (The assumption (4) does not hold if $N = K = Q_2$ or $N = K = H_2^1$, but in this case triviality of the solution of the invariance equation follows immediately).

References

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