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SCHUR-CONVEXITY, SCHUR-GEOMETRIC AND HARMONIC CONVEXITIES OF DUAL FORM OF A CLASS SYMMETRIC FUNCTIONS

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Abstract. By properties of Schur-convex function, Schur-geometrically convex function and Schur-harmonically convex function, Schur-convexity, Schur-geometric and harmonic convexities of the dual form for a class of symmetric functions are simply proved. As an application, several inequalities are obtained, some of which extend the known ones.
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1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \dots, x_n) : x_i \in \mathbb{R}, i = 1, \dots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \dots, x_n) : x_i > 0, i = 1, \dots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$ respectively.

For convenience, we introduce some definitions as follows.

Definition 1. [1, 2] Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$.

(ii) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. [1, 2] Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3. [1, 2] Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$.

(i) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \dots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

(*iii*) Let $\Omega \subset \mathbb{R}^n$. A function φ: $\Omega \to \mathbb{R}$ is said to be a log-convex function on Ω if function $\ln \varphi$ is convex.

Theorem A. *(Schur-Convex Function Decision Theorem)*[1, p. 5]*: Let $\Omega \subset \mathbb{R}^n$ is symmetric and has a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur-convex (Schur-concave) function, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

Definition 4. [3] Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(*i*) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0,1]$ such that $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Theorem B. *(Schur-Geometrically Convex Function Decision Theorem)*[3]*: Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{2}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

Definition 5. [4] Let $\Omega \subset \mathbb{R}^n_+$.

(1) A set Ω is said to be harmonically convex if $\frac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.

(2) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$.

Theorem C. *(Schur-Harmonically Convex Function Decision Theorem)*[4]*: Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω°. Then φ is Schur-harmonically convex(Schur-harmonically concave) on Ω if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (\leq 0), \quad \boldsymbol{x} \in \Omega^\circ. \tag{3}$$

Let interval $I \subset \mathbb{R}$ and let $\varphi : I \to \mathbb{R}_+$ be a log-convex function. Define the symmetric function F_k by

$$F_k(\boldsymbol{x}) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod\nolimits_{j=1}^k f(x_{i_j}), \ k = 1, \ldots, n. \tag{4}$$

In 2010, for 1,2 and $n-1$, I. Roventa [5]proved that $F_k(\boldsymbol{x})$ is a Schur-convex function on I^n, but without discuss the case of $2 < k < n-1$. In 2011, Shu-hong Wang et al.[6] studied completely Schur convexity, Schur geometric and harmonic convexities of $F_k(\boldsymbol{x})$ on I^n, using the above decision theorems, i.e. Theorem A,Theorem B and Theorem C respectively to prove the following three theorems.

Theorem D. *Let $I \subset \mathbb{R}$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}$ be continuous on I and differentiable in the interior of I. If f is a log-convex function, then for any $k = 1, 2, ..., n$, $F_k(\boldsymbol{x})$ is a Schur-convex function on I^n*

Theorem E. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I. If f is an increasing log-convex function, then for any $k = 1, 2, ..., n$, $F_k(\boldsymbol{x})$ is a Schur-geometrically convex function on I^n.*

Theorem F. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I. If f is an increasing log-convex function, then for any $k = 1, 2, ..., n$, $F_k(\boldsymbol{x})$ is a Schur-harmonically convex function on I^n.*

In this paper, we study the dual form of $F_k(\boldsymbol{x})$:

$$F_k^*(\boldsymbol{x}) = \prod_{1 \le i_1 < \ldots < i_k \le n} \sum_{j=1}^{k} f(x_{i_j}),\ k = 1, \ldots, n. \tag{5}$$

By properties of Schur-convex function, Schur-geometrically convex function and Schur-harmonically convex function, we obtained the following results:

Theorem 1. *Let $I \subset \mathbb{R}$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}$ be continuous on I and differentiable in the interior of I. If f is a log-convex function, then for any $k = 1, 2, ..., n$, $F_k^*(\boldsymbol{x})$ is a Schur-convex function on I^n*

Theorem 2. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I. If f is an increasing log-convex function, then for any $k = 1, 2, ..., n$, $F_k^*(\boldsymbol{x})$ is a Schur-geometrically convex function on I^n.*

Theorem 3. *Let $I \subset \mathbb{R}_+$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}_+$ be continuous on I and differentiable in the interior of I. If f is an increasing log-convex function, then for any $k = 1, 2, ..., n$, $F_k^*(\boldsymbol{x})$ is a Schur-harmonically convex function on I^n.*

2. Lemmas

To prove the above three theorems, we need the following lemmas.

Lemma 1. [1, p. 67],[2] *If φ is symmetric and convex (concave) on symmetric convex set Ω, then φ is Schur-convex (Schur-concave) on Ω.*

Lemma 2. [1, p. 73],[2] *Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}_+$. Then $\log \varphi$ is Schur-convex (Schur-concave) if and only if φ is Schur-convex (Schur-concave).*

Lemma 3. [1, p. 642],[2] *Let $\Omega \subset \mathbb{R}^n$ be open convex set, $\varphi : \Omega \to \mathbb{R}$. For $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, defined one variable function $g(t) = \varphi(t\boldsymbol{x} + (1-t)\boldsymbol{y})$ on interval $(0,1)$. Then φ is convex (concave) on Ω if and only if g is convex (concave) on $[0,1]$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.*

Lemma 4. *Let $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$. If f is a log-convex function, then the functions $p(t) = \log g(t)$ is convex on $[0,1]$, where*

$$g(t) = \sum_{j=1}^{m} f(tx_j + (1-t)y_j).$$

Proof.

$$p'(t) = \frac{g'(t)}{g(t)}.$$

where

$$g'(t) = \sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1-t)y_j).$$

$$p''(t) = \frac{g''(t)g(t) - (g'(t))^2}{g^2(t)},$$

where

$$g''(t) = \sum_{j=1}^{m} (x_j - y_j)^2 f''(tx_j + (1-t)y_j),$$

by the Cauchy inequality, we have

$$\begin{aligned}
& g''(t)g(t) - (g'(t))^2 \\
&= \left(\sum_{j=1}^{m} (x_j - y_j)^2 f''(tx_j + (1-t)y_j) \right) \left(\sum_{j=1}^{m} f(tx_j + (1-t)y_j) \right) \\
&- \left(\sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1-t)y_j) \right)^2 \\
&\geq \left(\sum_{j=1}^{m} |x_j - y_j| \sqrt{f''(tx_j + (1-t)y_j)} \cdot \sqrt{f(tx_j + (1-t)y_j)} \right)^2 \\
&- \left(\sum_{j=1}^{m} (x_j - y_j) f'(tx_j + (1-t)y_j) \right)^2
\end{aligned}$$

From the log-convexity of f it follows that $(logf(u))'' = \frac{f''(u)f(u) - (f'(u))^2}{f^2(u)} \geq 0$, hence

$$\sqrt{f''(tx_j + (1-t)y_j)} \cdot \sqrt{f(tx_j + (1-t)y_j)} \geq f'(tx_j + (1-t)y_j),$$

and then $g''(t)g(t) - (g'(t))^2 \geq 0$, i.e. $p''(t) \geq 0$, that is $p(t) = \log g(t)$ is convex on $[0,1]$.

The proof of Lemma 4 is completed. $\square$

Lemma 5. *Let*

$$f(t) = \frac{x^t - 1}{t}.$$

If $x > 1$, then $f(t)$ is a log-convex function on $\mathbb{R}_+$.

Proof. By computing, we have

$$\left(\log f(t)\right)^{''} = -\frac{x^t (\log x)^2}{(x^t - 1)^2} + \frac{1}{t^2}.$$

We need only prove $\left(\log f(t)\right)^{''} \geq 0$. It equivalent to

$$t^2 x^t (\log x)^2 \leq (x^t - 1)^2. \tag{6}$$

In both sides the inequality (6), extracting the square root and dividing by x^t, then the inequality (6) equivalent to

$$g(t) := x^{\frac{t}{2}} - x^{-\frac{t}{2}} - t \log x \geq 0.$$

When $x > 1$, $g^{'}(x) = \frac{1}{2} \log x \left(x^{\frac{t}{2}} - x^{-\frac{t}{2}} - 2\right) \geq 0$, hence $g(t)$ is increasing on $\mathbb{R}_+$, and then $g(t) \geq g(0) = 0$, that is $\left(\log f(t)\right)^{''} \geq 0$.

The proof of Lemma 5 is completed. □

3. Proof of Main Results

Proof of Theorem 1: For any $1 \leq i_1 < \cdots < i_k \leq n$, by Lemma 3 and Lemma 4, it follows that $\ln \sum_{j=1}^{k} f(x_{i_j})$ is convex on I^k. Obviously, $\ln \sum_{j=1}^{k} f(x_{i_j})$ is also convex on I^n, and then $\log F_k^*(\boldsymbol{x}) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \log \sum_{j=1}^{k} f(x_{i_j})$ is convex on I^n. Furthermore, it is clear that $\log F_k^*(\boldsymbol{x})$ is symmetric on I^n, by Lemma 1, it follows that $\log F_k^*(\boldsymbol{x})$ is Schur-convex on I^n, and then from Lemma 2 we conclude that $F_k^*(\boldsymbol{x})$ is also Schur-convex on I^n.

The proof of Theorem 1 is completed.

Proof of Theorem 2: For $\boldsymbol{x} \in I \subset \mathbb{R}_+$ and $x_1 \neq x_2$, we have

$$\begin{aligned}
\Delta &= (\log x_1 - \log x_2)\left(x_1 \frac{\partial F_k^*}{\partial x_1} - x_2 \frac{\partial F_k^*}{\partial x_2}\right) \\
&= (\log x_1 - \log x_2)\left(x_1 \frac{\partial F_k^*}{\partial x_1} - x_1 \frac{\partial F_k^*}{\partial x_2} + x_1 \frac{\partial F_k^*}{\partial x_2} - x_2 \frac{\partial F_k^*}{\partial x_2}\right) \\
&= x_1 \frac{\log x_1 - \log x_2}{x_1 - x_2} (x_1 - x_2)\left(\frac{\partial F_k^*}{\partial x_1} - \frac{\partial F_k^*}{\partial x_2}\right) + \frac{\partial F_k^*}{\partial x_2} (x_1 - x_2)(\log x_1 - \log x_2).
\end{aligned}$$

Since $F_k^*(\boldsymbol{x})$ is Schur-convex on I^n, by Theorem A, we have

$$(x_1 - x_2)\left(\frac{\partial F_k^*}{\partial x_1} - \frac{\partial F_k^*}{\partial x_2}\right) \geq 0.$$

Notice that f and $\log t$ is increasing, we have $\frac{\partial F_k^*}{\partial x_2} \geq 0$, $\frac{\log x_1 - \log x_2}{x_1 - x_2} \geq 0$ and $(x_1 - x_2)(\log x_1 - \log x_2) \geq 0$, so that $\Delta \geq 0$, by Theorem B, it follows that $F_k^*(\boldsymbol{x})$ is Schur-geometric convex on I^n.

Proof of Theorem 3: The proof of Theorem 3 similar to Theorem 2, the detailed proof is left to the reader.

Remark 1. If using the decision theorems, i.e. Theorem A, Theorem B and Theorem C respectively direct to prove Theorem 1, Theorem 2 and Theorem 3, I am afraid not above proofs are simple, interested readers may wish to try.

4. Applications

Theorem 4. *The symmetric function*

$$Q_k(\boldsymbol{x}) = \prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} \frac{1+x_{i_j}}{1-x_{i_j}},\ \ k=1,\ldots,n. \tag{7}$$

is Schur-convex function, Schur-geometrically and harmonically convex function on $(0,1)^n$. *And for* $\boldsymbol{x}\in(0,1)^n$, *we have*

$$\prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} \frac{1+x_{i_j}}{1-x_{i_j}} \ge \left(\frac{k(n+s)}{n-s}\right)^{C_n^k},\ \ k=1,\ldots,n. \tag{8}$$

where $s=\sum_{i=1}^n x_i$ *and* $C_n^k=\frac{n!}{k!(n-k)!}$.

Proof. Let $f(x)=\frac{1+x}{1-x}, x\in(0,1)$. By computing, we have $f^{'}(x)=\frac{2}{(1-x)^2}>0$ and $\log(f(x))^{''}=\frac{4x}{(1+x)^2(1-x)^2}\ge 0$, that is f is an increasing log-convex function. By Theorem 1, Theorem 2 and Theorem 3, it follows that $Q_k(\boldsymbol{x})$ is respectively Schur-convex function, Schur-geometrically and harmonically convex function on $(0,1)^n$.

Since $\boldsymbol{y}=\left(\frac{s}{n},\frac{s}{n},...,\frac{s}{n}\right)\prec\boldsymbol{x}=(x_1,x_2,...,x_n)$, from Schur-convexity of $G_k(\boldsymbol{x})$, it follows that $Q_k(\boldsymbol{y})\le Q_k(\boldsymbol{x})$, i.e. inequality (7) holds.

The proof of Theorem 4 is completed. □

Specially, taking $k=1, s=1$, from the inequality (8) we can get the known Klamkin inequality:

$$\prod_{i=1}^{n}\frac{1+x_i}{1-x_i}\ge\left(\frac{n+1}{n-1}\right)^n. \tag{9}$$

By analogous proof with Theorem 4, we can obtain the following theorem.

Theorem 5. *The symmetric function*

$$R_k(\boldsymbol{x}) = \prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} \frac{x_{i_j}}{1-x_{i_j}},\ \ k=1,\ldots,n. \tag{10}$$

is Schur-convex function, Schur-geometrically and harmonically convex function on $[\frac{1}{2},1)^n$. *And for* $\boldsymbol{x}\in[\frac{1}{2},1)^n$, *we have*

$$\prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} \frac{x_{i_j}}{1-x_{i_j}} \ge \left(\frac{ks}{n-s}\right)^{C_n^k},\ \ k=1,\ldots,n. \tag{11}$$

where $s=\sum_{i=1}^n x_i$ *and* $C_n^k=\frac{n!}{k!(n-k)!}$.

Theorem 6. *The symmetric function*

$$D_k(\boldsymbol{x}) = \prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} x_{i_j}^{x_{i_j}},\ \ k=1,\ldots,n. \tag{12}$$

is Schur-convex on $\mathbb{R}_+^n$ *and Schur-geometric and harmonic convex on* $[\mathrm{e}^{-1},\infty)^n$. *And for* $\boldsymbol{x}\in\mathbb{R}_+^n$, *we have*

$$\prod_{1\le i_1<\ldots<i_k\le n} \sum_{j=1}^{k} x_{i_j}^{x_{i_j}} \ge \left(k[A(\boldsymbol{x})]^{A(\boldsymbol{x})}\right)^{C_n^k},\ \ k=1,\ldots,n. \tag{13}$$

where $A(\boldsymbol{x})=\frac{1}{n}\sum_{i=1}^n x_i$ *and* $C_n^k=\frac{n!}{k!(n-k)!}$.

Proof. It is not difficult to verify that x^x is log-convex function on $(0,\infty)$ and increasing on $[e^{-1},\infty)$. By Theorem 1, Theorem 2 and Theorem 3, it follows that $D_k(\boldsymbol{x})$ is Schur-convex on $\mathbb{R}_+^n$ and Schur-geometric and harmonic convex on $[e^{-1},\infty)^n$.

Since $\boldsymbol{y}=(A(\boldsymbol{x}),A(\boldsymbol{x}),...,A(\boldsymbol{x}))\prec\boldsymbol{x}=(x_1,x_2,...,x_n)$, from Schur-convexity of $D_k(\boldsymbol{x})$, it follows that $D_k(\boldsymbol{y})\le D_k(\boldsymbol{x})$, i.e. inequality (11) holds.

The proof of Theorem 6 is completed. □

From Lemma 5 and Theorem 1,we can obtain the following Theorem 2.

Theorem 7. *Let* $x>1$.

$$P_k(\boldsymbol{t})=\prod_{1\le i_1<\ldots<i_k\le n}\sum_{j=1}^{k}\frac{x^{t_{i_j}}-1}{t_{i_j}},\ k=1,\ldots,n. \tag{14}$$

is Schur-convex on $\mathbb{R}_+^n$. *And for* $\boldsymbol{t}\in\mathbb{R}_+^n$, *we have*

$$\prod_{1\le i_1<\ldots<i_k\le n}\sum_{j=1}^{k}\frac{x^{t_{i_j}}-1}{t_{i_j}}\ge\left(\frac{k(x^{A(\boldsymbol{t})}-1)}{A(\boldsymbol{t})}\right)^{C_n^k},\ k=1,\ldots,n. \tag{15}$$

where $A(\boldsymbol{t})=\frac{1}{n}\sum_{i=1}^n t_i$ *and* $C_n^k=\frac{n!}{k!(n-k)!}$.

Specially, taking $n=2,k=1$ and $\boldsymbol{t}=(m+r,m-r)$, from the inequality (15) we can get the known inequality:

$$(x^{m-r}-1)(x^{m+r}-1)\ge\left(1-\frac{r^2}{m^2}\right)(x^m-1)^2, \tag{16}$$

where $r\in\mathbb{N},m\ge 2,r<m$.

Theorem 8. *Let* $0<\mu(E)<\infty,1\le p<\infty$ *and let*

$$N_p(f)=\left(\frac{1}{\mu(E)}\int_E|f|^p d\mu\right)^{\frac{1}{p}}. \tag{17}$$

Then

$$B_k(\boldsymbol{p})=\prod_{1\le i_1<\ldots<i_k\le n}\sum_{j=1}^{k}(N_{p_i}(f))^{p_i},\ k=1,\ldots,n. \tag{18}$$

is Schur-convex function, Schur-geometrically and harmonically convex function on $[1,\infty)^n$.

Proof. Since $(N_p(f))^p$ is an increasing log-convex function (see[7], p.36), from Theorem 1, Theorem 2 and Theorem 3, it follows that Theorem 7 holds. □

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