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A GENERALIZED ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL WITH APPLICATIONS FOR SELFADJOINT AND UNITARY OPERATORS

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Abstract. Sharp bounds for a generalised Čebyšev functional for the Riemann-Stieltjes integral are given. Applications for continuous functions of selfadjoint operators and unitary operators on Hilbert spaces are provided as well.

1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [25] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m,M,n,N such that

$$m\le f(t)\le M \quad\text{and}\quad n\le g(t)\le N \quad\text{for a.e. } t\in[a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [5], states that

$$|C(f,g)|\le\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2, \tag{1.4}$$

provided that f',g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f,g:[a,b]\to\mathbb{R}$ are assumed to be absolutely continuous and $f',g'\in L_\infty[a,b]$ while $\|f'\|_\infty=ess\sup_{t\in[a,b]}|f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [34]:

$$|C(f,g)|\le\frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g'\in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

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The case of *euclidean norms* of the derivative was considered by A. Lupaş in [28] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [3] have proved the following results:

$$|C(f,g)| \leq \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \tag{1.7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \leq \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} ess \sup_{t \in [a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|, \tag{1.8}$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.7) we obtain

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \\ &\leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \end{aligned} \tag{1.9}$$

and if g satisfies (1.3), then

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \\ &\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \\ &\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt. \end{aligned} \tag{1.10}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [26], [31] and [35] and the references therein.

In [11], in order to extend the Grüss inequality to *Riemann-Stieltjes integral*, S.S. Dragomir introduced the following Čebyšev functional

$$\begin{aligned} T(f,g;u) &:= \frac{1}{u(b)-u(a)} \int_a^b f(t)\,g(t)\,du(t) \\ &\quad - \frac{1}{u(b)-u(a)} \int_a^b f(t)\,du(t) \cdot \frac{1}{u(b)-u(a)} \int_a^b g(t)\,du(t), \end{aligned} \tag{1.11}$$

where f, g are continuous on $[a, b]$ and u is of *bounded variation* on $[a, b]$ with $u(b) \neq u(a)$.

The following result that provides sharp bounds for the Čebyšev functional defined above was obtained in [11].

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $u : [a, b] \to \mathbb{R}$ with $u(a) \neq u(b)$. Assume also that there exists the real constants γ, Γ such that*

$$\gamma \leq f(t) \leq \Gamma \quad \text{for each } t \in [a, b]. \tag{1.12}$$

a) *If u is of bounded variation on $[a, b]$, then we have the inequality*

$$\begin{aligned} &|T(f, g; u)| \\ &\leq \frac{1}{2} \cdot \frac{\Gamma - \gamma}{|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, du(s) \right\|_\infty \bigvee_a^b (u), \end{aligned} \tag{1.13}$$

where $\bigvee_a^b (u)$ denotes the total variation of u in $[a, b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller quantity.

b) *If $u : [a, b] \to \mathbb{R}$ is monotonic nondecreasing on $[a, b]$, then one has the inequality:*

$$\begin{aligned} &|T(f, g; u)| \\ &\leq \frac{1}{2} \cdot \frac{\Gamma - \gamma}{u(b) - u(a)} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, du(s) \right| du(t). \end{aligned} \tag{1.14}$$

The constant $\frac{1}{2}$ is sharp.

c) *Assume that $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable functions on $[a, b]$ and f satisfies the condition (1.12). If $u : [a, b] \to \mathbb{R}$ is Lipschitzian with the constant L, then we have the inequality*

$$\begin{aligned} &|T(f, g; u)| \\ &\leq \frac{1}{2} \cdot \frac{L(\Gamma - \gamma)}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s)\, du(s) \right| dt. \end{aligned} \tag{1.15}$$

The constant $\frac{1}{2}$ is best possible in (1.15).

We observe that if $u(t) = t$, then from (1.13) we get

$$|C(f, g)| \leq \frac{1}{2} (\Gamma - \gamma) \left\| g - \frac{1}{b - a} \int_a^b g(s)\, ds \right\|_\infty$$

while from (1.14) and (1.15) we recapture the inequality between the first and last term in (1.10).

For some recent inequalities for Riemann-Stieltjes integral see [7]-[12] and [27].

Motivated by the above results we consider here a more general Čebyšev functional depending on four functions and defined as

$$\begin{aligned} &T(f, g, h; u) \\ &:= \frac{1}{u(b) - u(a)} \int_a^b h(t)\, du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b f(t)\, g(t)\, du(t) \\ &- \frac{1}{u(b) - u(a)} \int_a^b g(t)\, du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b f(t)\, h(t)\, du(t), \end{aligned} \tag{1.16}$$

provided that all the Riemann-Stieltjes integrals incorporated in (1.16) exist and $u(b) \neq u(a)$. That happens, for instance, when $u:[a,b] \to \mathbb{C}$ is of bounded variation and $f,g,h:[a,b] \to \mathbb{C}$ are continuous on $[a,b]$.

The functional $T(f,g,h;u)$ can be written in a determinant form as

$$T(f,g,h;u) = \det \begin{bmatrix} \frac{1}{u(b)-u(a)} \int_a^b h(t)\,du(t) & \frac{1}{u(b)-u(a)} \int_a^b g(t)\,du(t) \\ \frac{1}{u(b)-u(a)} \int_a^b f(t)\,h(t)\,du(t) & \frac{1}{u(b)-u(a)} \int_a^b f(t)\,g(t)\,du(t) \end{bmatrix}. \tag{1.17}$$

We remark that if $h(t) = \mathbf{1}(t) = 1$ for all $t \in [a,b]$ then we get $T(f,g,\mathbf{1};u) = T(f,g;u)$. By (1.17) we then have the determinant form of $T(f,g;u)$ as

$$T(f,g;u) = \det \begin{bmatrix} 1 & \frac{1}{u(b)-u(a)} \int_a^b g(t)\,du(t) \\ \frac{1}{u(b)-u(a)} \int_a^b f(t)\,du(t) & \frac{1}{u(b)-u(a)} \int_a^b f(t)\,g(t)\,du(t) \end{bmatrix}. \tag{1.18}$$

We also observe that if e denotes the identity mapping on $[a,b]$, i.e. $e(t) = t, t \in [a,b]$ then by choosing $u = e$ in (1.18) we have $T(f,g;e) = C(f,g)$.

In this paper we establish some sharp bounds for the magnitude of the functional $T(f,g,h;u)$ under various assumptions for the functions involved and apply them to obtain new inequalities for continuous functions of selfadjoint operators as well as of unitary operators on Hilbert spaces.

2. The results

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f:[a,b] \to \mathbb{C} | \operatorname{Re}\left[(\Gamma - f(t)) \left(\overline{f(t)} - \overline{\gamma} \right) \right] \geq 0 \text{ for each } t \in [a,b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f:[a,b] \to \mathbb{C} | \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for each } t \in [a,b] \right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{2.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma+\Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.1) is thus a simple consequence of this fact. $\square$

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a,b] \to \mathbb{C} \mid\ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for each } t \in [a,b] \}. \end{aligned} \tag{2.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a,b] \to \mathbb{C} \mid\ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b] \}. \end{aligned} \tag{2.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \tag{2.4}$$

The following result can be stated.

Theorem 2. *Assume that $u : [a,b] \to \mathbb{C}$ with $u(b) \neq u(a)$ and $f, g, h : [a,b] \to \mathbb{C}$ are such that the Riemann-Stieltjes integrals in the definition of $T(f,g,h;u)$ exist. Assume also that there exists the complex numbers $\gamma, \Gamma, \gamma \neq \Gamma$, such that*

$$f \in \bar{U}_{[a,b]}(\gamma, \Gamma). \tag{2.5}$$

i) *If u is of bounded variation on $[a,b]$, then we have the inequality*

$$\begin{aligned} |T(f,g,h;u)| &\leq \frac{1}{2} |\Gamma - \gamma| \bigvee_a^b (u) \frac{1}{|u(b) - u(a)|^2} \\ &\quad \times \sup_{t \in [a,b]} \left| \det \begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s) \end{bmatrix} \right| \\ &\leq \frac{1}{2} |\Gamma - \gamma| \left[\bigvee_a^b (u) \right]^2 \frac{1}{|u(b) - u(a)|^2} \\ &\quad \times \sup_{(t,s)^2 \in [a,b]^2} \left| \det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix} \right|. \end{aligned} \tag{2.6}$$

The constant $\frac{1}{2}$ is sharp in the first inequality.

ii) *If $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

(2.7)

$$\begin{aligned}
|T(f,g,h;u)| &\leq \frac{1}{2}|\Gamma-\gamma|\frac{1}{[u(b)-u(a)]^2} \\
&\times\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|du(t) \\
&\leq \frac{1}{2}|\Gamma-\gamma|\frac{1}{[u(b)-u(a)]^2} \\
&\times\int_a^b\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|du(s)\,du(t) \\
&\leq \frac{\sqrt{2}}{2}|\Gamma-\gamma|\frac{1}{[u(b)-u(a)]^2} \\
&\times\left(\det\begin{bmatrix} \int_a^b |g(t)|^2\,du(t) & \int_a^b\left(g(t)\overline{h(t)}\right)du(t) \\ \int_a^b\left(\overline{g(t)}h(t)\right)du(t) & \int_a^b|h(t)|^2\,du(t)\end{bmatrix}\right)^{1/2}.
\end{aligned}$$

The multiplicative constant $\frac{1}{2}$ in the first inequality is best possible.

iii) *If u is Lipschitzian with the constant $L>0$, then we have the inequality*

$$\begin{aligned}
|T(f,g,h;u)| &\leq \frac{1}{2}|\Gamma-\gamma|\frac{L}{|u(b)-u(a)|^2} \\
&\times\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(t)\,du(t) & \int_a^b h(t)\,du(t)\end{bmatrix}\right|dt \\
&\leq \frac{1}{2}|\Gamma-\gamma|\frac{L^2}{[u(b)-u(a)]^2} \\
&\times\int_a^b\int_a^b\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s)\end{bmatrix}\right|dsdt \\
&\leq \frac{\sqrt{2}}{2}(\Gamma-\gamma)\frac{L^2}{[u(b)-u(a)]^2} \\
&\times\left(\det\begin{bmatrix} \int_a^b |g(t)|^2\,dt & \int_a^b\left(g(t)\overline{h(t)}\right)dt \\ \int_a^b\left(\overline{g(t)}h(t)\right)dt & \int_a^b|h(t)|^2\,dt\end{bmatrix}\right)^{1/2}.
\end{aligned} \tag{2.8}$$

The constant $\frac{1}{2}$ in the first inequality is sharp.

Remark 1. *We notice that the above Theorem 2 not only provides a generalization of Theorem 2 but also an extension of that result to the complex valued functions.*

Perhaps simpler, however coarser bounds for the magnitude of $T(f,g,h;u)$ can be provided if some connection between the other two functions g and h are known.

Corollary 2. *Assume that $u:[a,b]\to\mathbb{C}$ with $u(b)\neq u(a)$ and $f,g,h:[a,b]\to\mathbb{C}$ are such that the Riemann-Stieltjes integrals in the definition of $T(f,g,h;u)$ exist. Assume also that there exists the complex numbers γ,Γ such that (2.5) holds true.*

a) *Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exists the complex constants φ and Φ such that either*

$$\operatorname{Re}\left[\left(\Phi h(t)-g(t)\right)\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right]\geq 0 \text{ for any } t\in[a,b] \tag{2.9}$$

or, equivalently,

$$\left|g(t)-\frac{\varphi+\Phi}{2}h(t)\right|\leq\frac{1}{2}\left|\Phi-\varphi\right|\left|h(t)\right| \text{ for any } t\in[a,b], \tag{2.10}$$

holds, then we have

$$\left|T(f,g,h;u)\right|\leq\frac{\sqrt{2}}{4}\left|\Gamma-\gamma\right|\left|\Phi-\varphi\right|\frac{1}{\left[u(b)-u(a)\right]^{2}}\int_{a}^{b}\left|h(t)\right|^{2}du(t). \tag{2.11}$$

aa) *Let $u:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exists the complex constants φ and Φ such that either (2.9) or (2.10) hold true, then*

$$\left|T(f,g,h;u)\right|\leq\frac{\sqrt{2}}{4}\left|\Gamma-\gamma\right|\left|\Phi-\varphi\right|\frac{L^{2}}{\left[u(b)-u(a)\right]^{2}}\int_{a}^{b}\left|h(t)\right|^{2}dt. \tag{2.12}$$

Remark 2. *The above Corollary 2 can be then used to provide simpler bounds for the functional $T(\cdot,\cdot;\cdot)$ as follows:*

b) *Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exists the complex constants λ and Λ such that either*

$$\operatorname{Re}\left[\left(\Lambda-g(t)\right)\left(\overline{g(t)}-\overline{\lambda}\right)\right]\geq 0 \text{ for any } t\in[a,b] \tag{2.13}$$

or, equivalently,

$$\left|g(t)-\frac{\lambda+\Lambda}{2}h(t)\right|\leq\frac{1}{2}\left|\Lambda-\lambda\right| \text{ for any } t\in[a,b], \tag{2.14}$$

holds, then we have

$$\left|T(f,g;u)\right|\leq\frac{\sqrt{2}}{4}\frac{\left|\Gamma-\gamma\right|\left|\Lambda-\lambda\right|}{u(b)-u(a)}. \tag{2.15}$$

bb) *Let $u:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exists the complex constants λ and Λ such that either (2.13) or (2.14) hold true, then*

$$\left|T(f,g;u)\right|\leq\frac{\sqrt{2}}{4}\left|\Gamma-\gamma\right|\left|\Lambda-\lambda\right|\frac{L^{2}(b-a)}{\left[u(b)-u(a)\right]^{2}}. \tag{2.16}$$

3. Proofs

We observe that the following identity of interest holds:

$$\begin{aligned}
&[u(b)-u(a)]^2 T(f,g,h;u) \\
&= \int_a^b h(t)\,du(t)\cdot\int_a^b f(t)g(t)\,du(t) - \int_a^b g(t)\,du(t)\cdot\int_a^b f(t)h(t)\,du(t) \\
&= \int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix} du(t)
\end{aligned} \tag{3.1}$$

for each $\eta\in\mathbb{C}$.

i) It is well known that if the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists, $p:[a,b]\to\mathbb{C}$ is bounded and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then we have the inequality [2]

$$\left|\int_a^b p(t)\,dv(t)\right| \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v). \tag{3.2}$$

Taking the modulus in (3.1) and utilizing (3.2) we get

$$\begin{aligned}
&|u(b)-u(a)|^2|T(f,g,h;u)| \\
&= \left|\int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix} du(t)\right| \\
&\le \sup_{t\in[a,b]}\left|(f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right| \\
&\le \sup_{t\in[a,b]}|f(t)-\eta|\sup_{t\in[a,b]}\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right|.
\end{aligned} \tag{3.3}$$

Since $f\in\bar{U}_{[a,b]}(\gamma,\Gamma)$, then

$$\left|f(t)-\frac{\Gamma+\gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \quad \text{for each } t\in[a,b]. \tag{3.4}$$

Utilising (3.3) for $\eta=\frac{\gamma+\Gamma}{2}$ and (3.4) we deduce the first inequality in (2.6). The rest is obvious.

We observe that for $h(t)=1, t\in[a,b]$ and f a real function bounded below by γ and above by Γ, we recapture from the first part of (2.6) the inequality (1.13) obtained in [11] whose multiplicative constant $\frac{1}{2}$ is best possible. This fact implies that the constant $\frac{1}{2}$ in the first inequality is also best possible.

ii) It is well known that if the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$, $\int_a^b |p(t)|\,dv(t)$ exist and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then we have the inequality

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t). \tag{3.5}$$

For instance, when p is continuous and v is monotonic nondecreasing then (3.5) holds true.

Taking the modulus in (3.1) and utilizing (3.5) we get

$$\begin{aligned} &(u(b)-u(a))^2 \left|T(f,g,h;u)\right| \\ &= \left| \int_a^b (f(t)-\eta) \det \begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s) \end{bmatrix} du(t) \right| \\ &\leq \int_a^b \left| (f(t)-\eta) \det \begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s) \end{bmatrix} \right| du(t) \\ &= \int_a^b |f(t)-\eta| \left| \det \begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s) \end{bmatrix} \right| du(t), \end{aligned} \tag{3.6}$$

which, by (3.4), produces the first inequality in (2.7).

The sharpness of this inequality follows from (1.14) which is a particular case of (2.7).

Further on, observe that

$$\begin{aligned} \left| \det \begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s) \end{bmatrix} \right| &= \left| \int_a^b \det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix} du(s) \right| \\ &\leq \int_a^b \left| \det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix} \right| du(s), \end{aligned}$$

which, by integration on $[a,b]$ over the monotonic nondecreasing integrator u, produces the second part of (2.7).

Now, on utilizing the Cauchy-Bunyakovsky-Schwarz's double integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators,namely

$$\begin{aligned} &\left| \int_a^b \int_a^b h(t,s)\, m(t,s)\, du(t)\, du(s) \right|^2 \\ &\leq \int_a^b \int_a^b |h(t,s)|^2\, du(t)\, du(s) \int_a^b \int_a^b |m(t,s)|^2\, du(t)\, du(s) \end{aligned} \tag{3.7}$$

where $h, m : [a,b]^2 \to \mathbb{C}$ are continuous, then we have

$$\begin{aligned} &\frac{1}{[u(b)-u(a)]^2} \int_a^b \int_a^b \left| \det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix} \right| du(s)\, du(t) \\ &\leq \left[\frac{1}{[u(b)-u(a)]^2} \int_a^b \int_a^b \left| \det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix} \right|^2 du(s)\, du(t) \right]^{1/2}. \end{aligned} \tag{3.8}$$

Since

$$\int_a^b \int_a^b \left|\det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix}\right|^2 du(s)\, du(t)$$
$$= \int_a^b \int_a^b \Big\{ |g(t)|^2 |h(s)|^2 + |g(s)|^2 |h(t)|^2$$
$$-2\operatorname{Re}\left[(g(t) h(s)) \overline{(g(s) h(t))} \right] \Big\} du(s)\, du(t)$$

and

$$\int_a^b \int_a^b |g(t)|^2 |h(s)|^2 du(s)\, du(t) = \int_a^b \int_a^b |g(s)|^2 |h(t)|^2 du(s)\, du(t)$$
$$= \int_a^b |g(t)|^2 du(t) \int_a^b |h(t)|^2 du(t)$$

and

$$\int_a^b \int_a^b \operatorname{Re}\left[(g(t) h(s)) \overline{(g(s) h(t))} \right] du(s)\, du(t)$$
$$= \int_a^b \int_a^b \operatorname{Re}\left[\left(g(t) \overline{h(t)} \right) \overline{\left(g(s) \overline{h(s)} \right)} \right] du(s)\, du(t)$$
$$= \operatorname{Re}\left[\int_a^b \left(g(t) \overline{h(t)} \right) du(t) \int_a^b \overline{\left(g(s) \overline{h(s)} \right)} du(s) \right]$$
$$= \left| \int_a^b \left(g(t) \overline{h(t)} \right) du(t) \right|^2$$

then

$$\int_a^b \int_a^b \left|\det \begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix}\right|^2 du(s)\, du(t) \tag{3.9}$$
$$= 2\left[\int_a^b |g(t)|^2 du(t) \int_a^b |h(t)|^2 du(t) - \left| \int_a^b \left(g(t) \overline{h(t)} \right) du(t) \right|^2 \right]$$
$$= 2\det \begin{bmatrix} \int_a^b |g(t)|^2 du(t) & \int_a^b \left(g(t) \overline{h(t)} \right) du(t) \\ \int_a^b \left(\overline{g(t)} h(t) \right) du(t) & \int_a^b |h(t)|^2 du(t) \end{bmatrix}.$$

Making use of (3.8) and (3.9) we obtain the last part of (2.7).

iii) It is well known that if $p : [a, b] \to \mathbb{C}$ is Riemann integrable and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and we have the inequality

$$\left| \int_a^b p(t)\, dv(t) \right| \le L \int_a^b |p(t)|\, dt. \tag{3.10}$$

Taking the modulus in (3.1) and utilizing (3.10) we get

$$\begin{aligned} (3.11)\qquad & |u(b)-u(a)|^2\,|T(f,g,h;u)| \\ &= \left|\int_a^b (f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix} du(t)\right| \\ &\le L\int_a^b \left|(f(t)-\eta)\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right| dt \\ &= L\int_a^b |f(t)-\eta|\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_a^b g(s)\,du(s) & \int_a^b h(s)\,du(s)\end{bmatrix}\right| dt. \end{aligned}$$

Utilising (3.4) and (3.11) we deduce the first inequality (2.8).

The second part follows from a similar argument to the one in the second part of the statement ii) by choosing $u(t)=t$ and the details are omitted.

We observe that for $h(t)=1, t\in[a,b]$ we recapture from the first inequality in (2.8) the inequality (1.15) which is sharp.

Now, in order to prove the statements a) and aa) we need the following result that is of interest in its turn.

Lemma 1. *Let $u:[a,b]\to\mathbb{R}$ be a function of bounded variation and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exists the complex constants φ and Φ such that*

$$(3.12)\qquad \int_a^b |h(t)|^2\,du(t)\int_a^b \operatorname{Re}\left[(\Phi h(t)-g(t))\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right] du(t)\ge 0,$$

then we have

$$\begin{aligned} (3.13)\qquad & \det\begin{bmatrix} \int_a^b |g(t)|^2\,du(t) & \int_a^b \left(g(t)\,\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)}h(t)\right) du(t) & \int_a^b |h(t)|^2\,du(t)\end{bmatrix} \\ &\le \frac{1}{4}|\Phi-\varphi|^2\left[\int_a^b |h(t)|^2\,du(t)\right]^2. \end{aligned}$$

Proof. Consider the quantities

$$\begin{aligned} K_1 \;:&= \operatorname{Re}\left\{\left[\Phi\int_a^b |h(t)|^2\,du(t)-\int_a^b\left(g(t)\,\overline{h(t)}\right)du(t)\right]\right. \\ &\quad\left.\times\left[\overline{\int_a^b\left(g(t)\,\overline{h(t)}\right)du(t)}-\overline{\varphi}\int_a^b |h(t)|^2\,du(t)\right]\right\} \end{aligned}$$

and

$$K_2 := \int_a^b |h(t)|^2\,du(t)\int_a^b \operatorname{Re}\left[(\Phi h(t)-g(t))\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right] du(t).$$

We have by simple calculation that

$$\begin{aligned} K_1 &= \int_a^b |h(t)|^2 \, du(t) \operatorname{Re}\left[\Phi \overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} + \overline{\varphi} \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right] \\ &\quad - \left|\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right|^2 - \left[\int_a^b |h(t)|^2 \, du(t)\right]^2 \operatorname{Re}(\Phi\overline{\varphi}) \end{aligned}$$

and

$$\begin{aligned} K_2 &= \int_a^b |h(t)|^2 \operatorname{Re}\left[\Phi \overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} + \overline{\varphi} \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right] \\ &\quad - \int_a^b |g(t)|^2 \, du(t) \int_a^b |h(t)|^2 \, du(t) - \left[\int_a^b |h(t)|^2 \, du(t)\right]^2 \operatorname{Re}(\Phi\overline{\varphi}) \end{aligned}$$

which produces the equality of interest

$$\begin{aligned} K_1 - K_2 &= \int_a^b |g(t)|^2 \, du(t) \int_a^b |h(t)|^2 \, du(t) - \left|\int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right|^2 \\ &= \det \begin{bmatrix} \int_a^b |g(t)|^2 \, du(t) & \int_a^b \left(g(t)\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)}h(t)\right) du(t) & \int_a^b |h(t)|^2 \, du(t) \end{bmatrix}. \end{aligned} \tag{3.14}$$

Since, by (3.12), we have $K_2 \geq 0$, then it follows from (3.14) that

$$\begin{aligned} &\det \begin{bmatrix} \int_a^b |g(t)|^2 \, du(t) & \int_a^b \left(g(t)\overline{h(t)}\right) du(t) \\ \int_a^b \left(\overline{g(t)}h(t)\right) du(t) & \int_a^b |h(t)|^2 \, du(t) \end{bmatrix} \\ &\leq \operatorname{Re}\left\{\left[\Phi \int_a^b |h(t)|^2 \, du(t) - \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right]\right. \\ &\quad \left.\times \left[\overline{\int_a^b \left(g(t)\overline{h(t)}\right) du(t)} - \overline{\varphi} \int_a^b |h(t)|^2 \, du(t)\right]\right\} \\ &= \operatorname{Re}\left\{\left(\Phi \int_a^b |h(t)|^2 \, du(t) - \int_a^b \left(g(t)\overline{h(t)}\right) du(t)\right)\right. \\ &\quad \left.\times \overline{\left(\int_a^b \left(g(t)\overline{h(t)}\right) du(t) - \varphi \int_a^b |h(t)|^2 \, du(t)\right)}\right\}. \end{aligned} \tag{3.15}$$

On utilizing the elementary inequality for complex numbers

$$\operatorname{Re}(u\overline{v}) \leq \frac{1}{4}|u+v|^2, u, v \in \mathbb{C}$$

we have

$$\operatorname{Re}\left\{\left(\Phi\int_a^b |h(t)|^2\,du(t) - \int_a^b \left(g(t)\overline{h(t)}\right)du(t)\right)\right.$$
$$\left.\times\overline{\left(\int_a^b \left(g(t)\overline{h(t)}\right)du(t) - \varphi\int_a^b |h(t)|^2\,du(t)\right)}\right\}$$
$$\leq \frac{1}{4}|\Phi-\varphi|^2\left[\int_a^b |h(t)|^2\,du(t)\right]^2$$

which together with (3.15) produces the desired result (3.13). □

The following particular case is of interest since it provides a reverse inequality for the Cauchy-Bunyakovsky-Schwarz's integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators:

Corollary 3. *Let $u:[a,b]\to\mathbb{R}$ be monotonic nondecreasing and $g,h:[a,b]\to\mathbb{C}$ be continuous on $[a,b]$. If there exists the complex constants φ and Φ such that either*

$$\operatorname{Re}\left[(\Phi h(t)-g(t))\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right]\geq 0 \text{ for any } t\in[a,b] \tag{3.16}$$

or, equivalently,

$$\left|g(t)-\frac{\varphi+\Phi}{2}h(t)\right|\leq\frac{1}{2}|\Phi-\varphi|\,|h(t)| \text{ for any } t\in[a,b], \tag{3.17}$$

holds, then we have

$$0\leq\det\begin{bmatrix}\int_a^b |g(t)|^2\,du(t) & \int_a^b \left(g(t)\overline{h(t)}\right)du(t)\\ \int_a^b \left(\overline{g(t)}h(t)\right)du(t) & \int_a^b |h(t)|^2\,du(t)\end{bmatrix} \tag{3.18}$$
$$\leq\frac{1}{4}|\Phi-\varphi|^2\left[\int_a^b |h(t)|^2\,du(t)\right]^2.$$

4. Applications for Functions of Selfadjoint Operators

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [24, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;
(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;
(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;
(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [24] and the references therein.

For other results see [1], [16]-[20], [29], [32], [33] and [36].

We say that the functions $f, g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \text{ for each } t, s \in [a, b].$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [22] and [23].

For a selfadjoint operator A on the Hilbert space A with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and for $f, g : [m, M] \longrightarrow \mathbb{R}$ that are continuous functions *on* $[m, M]$, we can define the following *Čebyšev functional*

$$C(f, g; A; x) := \langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle$$

where $x \in H$ with $\|x\| = 1$.

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators, see [14]:

Theorem 3. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then*

$$C(f, g; A; x) \geq (\leq)\, 0, \tag{4.1}$$

for any $x \in H$ with $\|x\| = 1$.

The following result of Grüss' type can be stated as well, see [15]:

Theorem 4. *Let A be a selfadjoint operator on the Hilbert space $(H; \langle ., .\rangle)$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} f(t)$ and $\Gamma := \max_{t\in[m,M]} f(t)$ then*

$$|C(f, g; A; x)| \leq \frac{1}{2} \cdot (\Gamma - \gamma)[C(g, g; A; x)]^{1/2} \left(\leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta)\right), \tag{4.2}$$

for each $x \in H$ with $\|x\| = 1$, where $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$.

In order to provide some new vector Grüss' type inequalities for continuous functions of selfadjoint operators in Hilbert spaces, we need the following facts concerning the spectral representation of such functions.

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U)=\int_{m-0}^{M}f(\lambda)\,dE_{\lambda}, \tag{4.3}$$

which in terms of vectors can be written as

$$\langle f(U)x,y\rangle=\int_{m-0}^{M}f(\lambda)\,d\langle E_{\lambda}x,y\rangle, \tag{4.4}$$

for any $x,y\in H$. The function $g_{x,y}(\lambda):=\langle E_{\lambda}x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ and } g_{x,y}(M)=\langle x,y\rangle$$

for any $x,y\in H$. It is also well known that $g_{x}(\lambda):=\langle E_{\lambda}x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

For three continuos functions $f,g,h:[m,M]\to\mathbb{C}$ we define the general *Čebyšev functional* for two vectors $x,y\in H$ by

$$\begin{aligned}
&C(f,g;h,A;x,y)\\
&:=\langle h(A)x,y\rangle\cdot\langle f(A)g(A)x,y\rangle-\langle g(A)x,y\rangle\cdot\langle f(A)h(A)x,y\rangle\\
&=\det\begin{bmatrix}\langle h(A)x,y\rangle & \langle g(A)x,y\rangle\\ \langle f(A)h(A)x,y\rangle & \langle f(A)g(A)x,y\rangle\end{bmatrix}.
\end{aligned} \tag{4.5}$$

We denote $C(f,g;h,A;x,x)$ by $C(f,g;h,A;x)$ for $x\in H$.

In particular, if $h(t)=1, t\in[m,M]$, then we get from (4.5) the functional

$$\begin{aligned}
C(f,g;A;x,y)&:=\langle x,y\rangle\cdot\langle f(A)g(A)x,y\rangle-\langle g(A)x,y\rangle\cdot\langle f(A)x,y\rangle\\
&=\det\begin{bmatrix}\langle x,y\rangle & \langle g(A)x,y\rangle\\ \langle f(A)x,y\rangle & \langle f(A)g(A)x,y\rangle\end{bmatrix}.
\end{aligned} \tag{4.6}$$

where $x,y\in H$.

We denote $C(f,g;A;x,x)$ by $C(f,g;A;x)$ for $x\in H$.

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$, the spectrum of A, $Sp(A)\subseteq[m,M]$ for some scalars $m<M$, $\{E_{\lambda}\}_{\lambda}$ be its spectral family and $g,h:[m,M]\to\mathbb{C}$ continuous functions on $[m,M]$. If $f:[m,M]\to\mathbb{C}$ is continuous and there exists the complex numbers $\gamma,\Gamma,\gamma\neq\Gamma$, such that $f\in\bar{U}_{[a,b]}(\gamma,\Gamma)$ then*

$$\begin{aligned}
&|C(f,g;h,A;x,y)|\\
&\le\frac{1}{2}|\Gamma-\gamma|\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\max_{t\in[m,M]}\left|\det\begin{bmatrix}g(t) & h(t)\\ \langle g(A)x,y\rangle & \langle h(A)x,y\rangle\end{bmatrix}\right|\\
&\le\frac{1}{2}|\Gamma-\gamma|\left[\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\right]^{2}\max_{(t,s)^2\in[a,b]^2}\left|\det\begin{bmatrix}g(t) & h(t)\\ g(s) & h(s)\end{bmatrix}\right|
\end{aligned} \tag{4.7}$$

for any $x,y\in H$.

We also have the simpler inequalities

$$
\begin{aligned}
&|C(f,g;h,A;x,y)| \\
&\leq \frac{1}{2}|\Gamma-\gamma|\,\|x\|\,\|y\| \max_{t\in[m,M]}\left|\det\begin{bmatrix} g(t) & h(t) \\ \langle g(A)x,y\rangle & \langle h(A)x,y\rangle \end{bmatrix}\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma|\,\|x\|^2\,\|y\|^2 \max_{(t,s)^2\in[a,b]^2}\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix}\right|
\end{aligned} \tag{4.8}
$$

for any $x,y\in H$.

Proof. Utilising the inequality (2.6) we can state that

$$
\begin{aligned}
&\left|\det\begin{bmatrix} \int_{m-\varepsilon}^{M} f(t)\,g(t)\,du(t) & \int_{m-\varepsilon}^{M} f(t)\,h(t)\,du(t) \\ \int_{m-\varepsilon}^{M} g(t)\,du(t) & \int_{m-\varepsilon}^{M} h(t)\,du(t) \end{bmatrix}\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma| \bigvee_{m-\varepsilon}^{M}(u) \max_{t\in[m-\varepsilon,M]}\left|\det\begin{bmatrix} g(t) & h(t) \\ \int_{m-\varepsilon}^{M} g(t)\,du(t) & \int_{m-\varepsilon}^{M} h(t)\,du(t) \end{bmatrix}\right| \\
&\leq \frac{1}{2}|\Gamma-\gamma| \left[\bigvee_{m-\varepsilon}^{M}(u)\right]^2 \max_{(t,s)^2\in[m-\varepsilon,M]^2}\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix}\right|
\end{aligned} \tag{4.9}
$$

for any function $u:\mathbb{R}\to\mathbb{C}$ that is of bounded variation on the compact interval $[m-\varepsilon,M]$, where $\varepsilon>0$.

Now, on applying (4.9) for the function $u(t)=\langle E_t x,y\rangle, t\in\mathbb{R}$ which is of bounded variation on $[m-\varepsilon,M]$ and x,y are fixed in H and utilizing the spectral representation theorem (4.4), we deduce the desired inequality (4.7).

We also have the *Total Variation Schwarz's Inequality* for the spectral family $\{E_\lambda\}_\lambda$ (see a detailed proof in [21, p. 11])

$$
\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\leq\|x\|\,\|y\|
$$

any $x,y\in H$.

These prove the inequalities (4.8). □

The following result also holds:

Theorem 6. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$, the spectrum of A, $Sp(A)\subseteq[m,M]$ for some scalars $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family and $g,h:[m,M]\to\mathbb{C}$ continuous functions on $[m,M]$. If $f:[m,M]\to\mathbb{C}$ is continuous and there exists the complex numbers $\gamma,\Gamma,\gamma\neq\Gamma$, such that $f\in$*

$\bar{U}_{[a,b]}(\gamma,\Gamma)$ *then*

$$
\begin{aligned}
(4.10)\quad |C(f,g;h,A;x)| &\le \frac{1}{2}|\Gamma-\gamma| \\
&\times \int_{m-0}^{M}\left|\det\begin{bmatrix} g(t) & h(t) \\ \langle g(A)x,x\rangle & \langle h(A)x,x\rangle \end{bmatrix}\right| d\langle E_t x,x\rangle \\
&\le \frac{1}{2}|\Gamma-\gamma| \\
&\times \int_{m-0}^{M}\int_{m-0}^{M}\left|\det\begin{bmatrix} g(t) & h(t) \\ g(s) & h(s) \end{bmatrix}\right| d\langle E_s x,x\rangle\, d\langle E_t x,x\rangle \\
&\le \frac{\sqrt{2}}{2}|\Gamma-\gamma| \\
&\times \left(\det\begin{bmatrix} \|g(A)x\|^2 & \langle g(A)x,h(A)x\rangle \\ \langle h(A)x,g(A)x\rangle & \|h(A)x\|^2 \end{bmatrix}\right)^{1/2}
\end{aligned}
$$

for any $x\in H$.

Proof. Follows by (2.8) on taking into account that

$$\|g(A)x\|^2=\int_{m-0}^{M}|g(t)|^2\,d\langle E_t x,x\rangle\,,\ \|h(A)x\|^2=\int_{m-0}^{M}|h(t)|^2\,d\langle E_t x,x\rangle$$

and

$$\langle g(A)x,h(A)x\rangle=\int_{m-0}^{M}g(t)\,\overline{h(t)}d\langle E_t x,x\rangle$$

for any $x\in H$.

These equalities follow by the spectral representation (4.4). □

The following particular case which provides a simpler, however a coarser inequality may be more useful for applications.

Corollary 4. *Under the assumptions of Theorem 6 and, in addition, if there exists the complex constants* φ *and* Φ *such that either*

$$\operatorname{Re}\left[(\Phi h(t)-g(t))\left(\overline{g(t)}-\overline{\varphi h(t)}\right)\right]\ge 0 \text{ for any } t\in[a,b] \tag{4.11}$$

or, equivalently,

$$\left|g(t)-\frac{\varphi+\Phi}{2}h(t)\right|\le\frac{1}{2}|\Phi-\varphi|\,|h(t)|\ \text{ for any } t\in[a,b]\,, \tag{4.12}$$

holds, then we have

$$|C(f,g;h,A;x)|\le\frac{\sqrt{2}}{4}|\Gamma-\gamma|\,|\Phi-\varphi|\,\|h(A)x\|^2$$

for any $x\in H$.

5. Applications for Functions of Unitary Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. We recall that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [**?**, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda \in [0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of *Riemann-Stieltjes* type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{5.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f(U)x = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda x, \tag{5.2}$$

$$\langle f(U)x, y \rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y \rangle \tag{5.3}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2, \tag{5.4}$$

for any $x, y \in H$.

Now, for $\gamma, \Gamma \in \mathbb{C}$, define the sets of continuous complex valued functions $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$

$$\begin{aligned}&\bar{U}(\gamma, \Gamma) \\ &:= \left\{ f : \mathcal{C}(0,1) \to \mathbb{C}; \operatorname{Re}\left[\left(\Gamma - f\left(e^{it}\right)\right)\left(\overline{f\left(e^{it}\right)} - \overline{\gamma}\right)\right] \geq 0 \text{ for each } \ t \in [0, 2\pi] \right\}\end{aligned}$$

and

$$\begin{aligned}&\bar{\Delta}(\gamma, \Gamma) \\ &:= \left\{ f : \mathcal{C}(0,1) \to \mathbb{C}; \ \left| f\left(e^{it}\right) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2}|\Gamma - \gamma| \ \text{ for each } \ t \in [a, b] \right\}.\end{aligned}$$

As above, we observe that $\bar{U}(\gamma, \Gamma) = \bar{\Delta}(\gamma, \Gamma)$.

Now, by utilizing the inequality (2.6) for $f \in \bar{U}(\gamma, \Gamma)$ and two continuous functions $g, h : \mathcal{C}(0,1) \to \mathbb{C}$ we can state that

$$\left|\det \begin{bmatrix} \int_0^{2\pi} f\left(e^{it}\right) g\left(e^{it}\right) du(t) & \int_0^{2\pi} f\left(e^{it}\right) h\left(e^{it}\right) du(t) \\ \int_0^{2\pi} g\left(e^{it}\right) du(t) & \int_0^{2\pi} h\left(e^{it}\right) du(t) \end{bmatrix}\right| \tag{5.5}$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \bigvee_0^{2\pi}(u) \max_{t\in[0,2\pi]} \left|\det \begin{bmatrix} g\left(e^{it}\right) & h\left(e^{it}\right) \\ \int_0^{2\pi} g\left(e^{it}\right) du(t) & \int_0^{2\pi} h\left(e^{it}\right) du(t) \end{bmatrix}\right|$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \left[\bigvee_0^{2\pi}(u)\right]^2 \max_{(t,s)^2\in[0,2\pi]^2} \left|\det \begin{bmatrix} g\left(e^{it}\right) & h\left(e^{it}\right) \\ g\left(e^{is}\right) & h\left(e^{is}\right) \end{bmatrix}\right|$$

for any function $u : \mathbb{R} \to \mathbb{C}$ that is of bounded variation on the compact interval $[0, 2\pi]$.

Theorem 7. *Let $f, g, h : \mathcal{C}(0,1) \to \mathbb{C}$ be three continuous functions and $\gamma, \Gamma \in \mathbb{C}$ such that $f \in \bar{U}(\gamma, \Gamma)$. If $U : H \to H$ is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U then*

$$\left|\det \begin{bmatrix} \langle f(U) g(U) x, y\rangle & \langle f(U) h(U) x, y\rangle \\ \langle g(U) x, y\rangle & \langle h(U) x, y\rangle \end{bmatrix}\right| \tag{5.6}$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \max_{t\in[0,2\pi]} \left|\det \begin{bmatrix} g\left(e^{it}\right) & h\left(e^{it}\right) \\ \langle g(U) x, y\rangle & \langle h(U) x, y\rangle \end{bmatrix}\right|$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \max_{(t,s)^2\in[0,2\pi]^2} \left|\det \begin{bmatrix} g\left(e^{it}\right) & h\left(e^{it}\right) \\ g\left(e^{is}\right) & h\left(e^{is}\right) \end{bmatrix}\right|$$

for any $x, y \in H$ and the simpler inequality

$$\left|\det \begin{bmatrix} \langle f(U) g(U) x, y\rangle & \langle f(U) h(U) x, y\rangle \\ \langle g(U) x, y\rangle & \langle h(U) x, y\rangle \end{bmatrix}\right| \tag{5.7}$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \|x\| \|y\| \max_{t\in[0,2\pi]} \left|\det \begin{bmatrix} g\left(e^{it}\right) & h\left(e^{it}\right) \\ \langle g(U) x, y\rangle & \langle h(U) x, y\rangle \end{bmatrix}\right|$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \|x\|^2 \|y\|^2 \max_{(t,s)^2\in[0,2\pi]^2} \left|\det \begin{bmatrix} g\left(e^{it}\right) & h\left(e^{it}\right) \\ g\left(e^{is}\right) & h\left(e^{is}\right) \end{bmatrix}\right|$$

for any $x, y \in H$.

Proof. For given $x, y \in H$, define the function $u(\lambda) := \langle E_\lambda x, y\rangle$, $\lambda \in [0, 2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) =: \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\| \|y\|. \tag{5.8}$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a *generalization of the Schwarz* *inequality* in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle, \tag{5.9}$$

for any $x, y \in H$.

Now, if $d : 0 = t_0 < t_1 < ... < t_{n-1} < t_n = 2\pi$ is an arbitrary partition of the interval $[0, 2\pi]$, then we have by Schwarz's inequality for nonnegative operators (5.9) that

$$\begin{aligned} &\bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, y\rangle\right) \\ &= \sup_d \left\{ \sum_{i=0}^{n-1} \left|\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y\right\rangle\right| \right\} \\ &\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle^{1/2} \right] \right\} := I. \end{aligned} \tag{5.10}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned} I &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle \right]^{1/2} \right\} \\ &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle \right]^{1/2} \right\} \\ &= \left[\bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, x\rangle\right) \right]^{1/2} \left[\bigvee_0^{2\pi} \left(\langle E_{(\cdot)}y, y\rangle\right) \right]^{1/2} = \|x\| \|y\| \end{aligned} \tag{5.11}$$

for any $x, y \in H$.

By making use of (5.10) and (5.11) we get (5.8).

The inequality (5.6) follows by (5.5) on choosing $u(t) := \langle E_t x, y\rangle, t \in [0, 2\pi]$.

The inequality (5.7) follows by (5.6) and by (5.8). □

Remark 3. *The interested reader may obtain other similar results by utilizing the rest of the inequalities for the Riemann-Stieltjes integral established above. However, the details are omitted.*

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