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INEQUALITIES FOR THE RIEMANN-STIELTJES INTEGRAL OF S-DOMINATED INTEGRATORS WITH APPLICATIONS (II)

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Abstract. Assume that $u, v : [a,b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. We say that the complex-valued function $h : [a,b] \to \mathbb{C}$ is *S-dominated* by the pair (u,v) if

$$|h(y) - h(x)|^2 \le [u(y) - u(x)][v(y) - v(x)]$$

for any $x, y \in [a,b]$.

In this paper we show amongst other that

$$\left|\int_a^b f(t) g(t) dh(t)\right|^2 \le \int_a^b |f(t)|^2 du(t) \int_a^b |g(t)|^2 dv(t),$$

for any continuous functions $f, g : [a,b] \to \mathbb{C}$.

Applications for the trapezoidal inequality are given. New inequalities for some Čebyšev and (CBS)-type functionals are presented. Natural applications for continuous functions of selfadjoint and unitary operators on Hilbert spaces are provided as well.

1. Introduction

One of the most important properties of the *Riemann-Stieltjes integral* $\int_a^b f(t) dg(t)$ is the fact that this integral exists if one of the function is of *bounded variation* while the other is *continuous*. The following sharp inequality holds

$$\left|\int_a^b f(t) dg(t)\right| \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (g), \tag{1.1}$$

provided that $f : [a,b] \to \mathbb{C}$ is continuous on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ is of bounded variation on this interval. Here $\bigvee_a^b (g)$ denotes the *total variation* of g on $[a,b]$.

When g is *Lipschitzian* with the constant $L > 0$, i.e.,

$$|g(t) - g(s)| \le L|t - s|$$

for any $t, s \in [a,b]$, then we have

$$\left|\int_a^b f(t) dg(t)\right| \le L \int_a^b |f(t)| dt \tag{1.2}$$

for any *Riemann integrable* function $f : [a,b] \to \mathbb{C}$.

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Moreover, if the *integrator* g is *monotonic nondecreasing* on the interval $[a,b]$ and the *integrand* $f:[a,b]\to\mathbb{C}$ is continuous, then we have the *modulus inequality*

$$\left|\int_a^b f(t)\,dg(t)\right|\le\int_a^b|f(t)|\,dg(t). \tag{1.3}$$

In order to provide other inequalities of this type, we introduced in [29] the following class of functions.

Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. We say that the complex-valued function $h:[a,b]\to\mathbb{C}$ is *S-dominated* by the pair (u,v) if

$$|h(y)-h(x)|^2\le[u(y)-u(x)][v(y)-v(x)] \tag{S}$$

for any $x,y\in[a,b]$.

We observe that by the monotonicity of the functions u and v and by the symmetry of the inequality (S) over x and y we can assume that (S) is satisfied only for $y>x$ with $x,y\in[a,b]$.

We can give numerous examples of such functions.

For instance, if we take $f,g\in L_2[a,b]$ the Hilbert space of all complex-valued functions that are square-Lebesgue integrable and denote

$$h(x):=\int_a^x f(t)\,g(t)\,dt,\ u(x):=\int_a^x|f(t)|^2\,dt \text{ and } v(x):=\int_a^x|g(t)|^2\,dt,$$

then we observe that u and v are monotonic nondecreasing on $[a,b]$ and by Cauchy-Bunyakovsky-Schwarz integral inequality we have for any $y>x$ with $x,y\in[a,b]$ that

$$\begin{aligned}|h(y)-h(x)|^2&=\left|\int_x^y f(t)\,g(t)\,dt\right|^2\le\int_x^y|f(t)|^2\,dt\int_x^y|g(t)|^2\,dt\\&\le[u(y)-u(x)][v(y)-v(x)].\end{aligned}$$

Now, for $p,q>0$ if we consider $f(t):=t^p$ and $g(t):=t^q$ for $t\ge0$, then

$$h_{p,q}(x):=\int_0^x t^{p+q}dt=\frac{1}{p+q+1}x^{p+q+1}$$

and

$$u_p(x):=\int_0^x t^{2p}dt=\frac{1}{2p+1}x^{2p+1},\ v_q(x):=\int_0^x t^{2q}dt=\frac{1}{2q+1}x^{2q+1}.$$

Taking into account the above comments we observe that the function $h_{p,q}$ is *S-dominated* by the pair (u_p,v_q) on any subinterval of $[0,\infty)$.

In the recent paper [29] we proved the following result:

Theorem 1. *Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h:[a,b]\to\mathbb{C}$ is S-dominated by the pair (u,v) and $f:[a,b]\to\mathbb{C}$ is a continuous function on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists and*

$$\left|\int_a^b f(t)\,dh(t)\right|^2\le\int_a^b|f(t)|\,du(t)\int_a^b|f(t)|\,dv(t). \tag{1.4}$$

As some simple applications of this result, we have [29]:

Corollary 1. *Assume that* $u, v : [a, b] \to \mathbb{R}$ *are monotonic nondecreasing on the interval* $[a, b]$. *If* $h : [a, b] \to \mathbb{C}$ *is S-dominated by the pair* (u, v), *then*

$$\begin{aligned} &\left| \frac{h(a) + h(b)}{2}(b-a) - \int_a^b h(t)\,dt \right|^2 \\ &\leq \left[\frac{1}{2}(b-a)[u(b) - u(a)] - \int_a^b sgn\left(t - \frac{a+b}{2}\right) u(t)\,dt \right] \\ &\times \left[\frac{1}{2}(b-a)[v(b) - v(a)] - \int_a^b sgn\left(t - \frac{a+b}{2}\right) v(t)\,dt \right] \\ &\leq \frac{1}{4}(b-a)^2 [u(b) - u(a)][v(b) - v(a)] \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\left| h\left(\frac{a+b}{2}\right)(b-a) - \int_a^b h(t)\,dt \right|^2 \\ &\leq \int_a^b sgn\left(t - \frac{a+b}{2}\right) u(t)\,dt \int_a^b sgn\left(t - \frac{a+b}{2}\right) v(t)\,dt \\ &\leq \frac{1}{4}(b-a)^2 [u(b) - u(a)][v(b) - v(a)]. \end{aligned} \tag{1.6}$$

For related results to the trapezoid inequality, see [11]-[15], [17]-[20], [24]-[25], [30]-[33], [35], [41], [42], [44]-[46] and [54]-[56].

For related results to the midpoint inequality, see [1]-[11], [16]-[17], [21], [23], [25]-[27], [32], [36]-[40], [43], [47]-[53] and [57]-[60].

Motivated by the above results, we establish in this paper a bound for the quantity

$$\left| \int_a^b f(t)\, g(t)\, dh(t) \right|$$

in the case when f and g are continuous while the function of bounded variation h is S-dominated by a pair of monotonic functions. Applications for the trapezoidal type inequalities are given. New inequalities for some Čebyšev and (CBS)-type functionals are presented. Natural applications for continuous functions of selfadjoint and unitary operators on Hilbert spaces are provided as well.

2. Inequalities for S-dominated Functions

We have the following Cauchy-Bunyakovsky-Schwarz type inequality for the Riemann-Stieltjes integral.

Theorem 2. *Let* $f, g : [a, b] \to \mathbb{C}$ *be continuous on* $[a, b]$. *If* $h : [a, b] \to \mathbb{C}$ *is an S-dominated function by the pair* (u, v) *which are monotonic nondecreasing on* $[a, b]$, *then for any continuos nonnegative function* $p : [a, b] \to [0, \infty)$ *we have*

$$\left| \int_a^b pfg\,dh \right|^2 \leq \int_a^b p|f|^2 du \int_a^b p|g|^2 dv. \tag{2.1}$$

Proof. Since the Riemann-Stieltjes integral $\int_a^b pfgdh$ exists, then for any sequence of partitions

$$I_n^{(n)} : a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} = b$$

with the norm

$$v\left(I_n^{(n)}\right) := \max_{i\in\{0,\ldots,n-1\}} \left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$$

as $n \to \infty$, and for any intermediate points $\xi_i^{(n)} \in [t_i^{(n)}, t_{i+1}^{(n)}]$, $i \in \{0, \ldots, n-1\}$ we have:

$$\begin{aligned}
(2.2)\qquad & \left|\int_a^b pfgdh\right| \\
&= \left|\lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) f\left(\xi_i^{(n)}\right) g\left(\xi_i^{(n)}\right) \left[h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right]\right| \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left|f\left(\xi_i^{(n)}\right)\right| \left|g\left(\xi_i^{(n)}\right)\right| \left|h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right| \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left|f\left(\xi_i^{(n)}\right)\right| \left|g\left(\xi_i^{(n)}\right)\right| \\
&\times \left|u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right|^{1/2} \left|u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right|^{1/2} \\
&:= I.
\end{aligned}$$

Utilising the weighted Cauchy-Bunyakovsky-Schwarz discrete inequality

$$\sum_{k=1}^n p_k a_k b_k \le \left(\sum_{k=1}^n p_k a_k^2\right)^{1/2} \left(\sum_{k=1}^n p_k b_k^2\right)^{1/2}$$

where $p_k,\ a_k,\ b_k \ge 0$ for $k \in \{1, ..., n\}$, we have

$$\begin{aligned}
(2.3)\qquad I &\le \left(\lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left|f\left(\xi_i^{(n)}\right)\right|^2 \left[\left|u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right|^{1/2}\right]^2\right)^{1/2} \\
&\times \left(\lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left|g\left(\xi_i^{(n)}\right)\right|^2 \left[\left|v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right|^{1/2}\right]^2\right)^{1/2} \\
&= \left(\lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left|f\left(\xi_i^{(n)}\right)\right|^2 \left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right]\right)^{1/2} \\
&\times \left(\lim_{v\left(I_n^{(n)}\right)\to 0} \sum_{i=0}^{n-1} p\left(\xi_i^{(n)}\right) \left|g\left(\xi_i^{(n)}\right)\right|^2 \left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right]\right)^{1/2} \\
&= \left(\int_a^b p\left|f\right|^2 du\right)^{1/2} \left(\int_a^b p\left|g\right|^2 dv\right)^{1/2}.
\end{aligned}$$

Making use of the inequalities (2.2) and (2.3) we deduce the desired result (2.1). □

Remark 1. *From (2.1) we also have the dual inequality*

$$\left|\int_a^b pfgdh\right|^2 \leq \int_a^b p\left|g\right|^2 du \int_a^b p\left|f\right|^2 dv, \tag{2.4}$$

which together with (2.1) provide

$$\begin{aligned}\left|\int_a^b pfgdh\right|^2 & \\ &\leq \min\left\{\int_a^b p\left|f\right|^2 du \int_a^b p\left|g\right|^2 dv, \int_a^b p\left|g\right|^2 du \int_a^b p\left|f\right|^2 dv\right\}.\end{aligned} \tag{2.5}$$

In particular we have

$$\max\left\{\left|\int_a^b pf^2dh\right|^2, \left|\int_a^b p\left|f\right|^2 dh\right|^2\right\} \leq \int_a^b p\left|f\right|^2 du \int_a^b p\left|f\right|^2 dv. \tag{2.6}$$

We also have the inequality

$$\left|\int_a^b pfdh\right|^2 \leq \min\left\{\int_a^b pdu \int_a^b p\left|f\right|^2 dv, \int_a^b pdv \int_a^b p\left|f\right|^2 du\right\} \tag{2.7}$$

and in particular

$$\begin{aligned}\left|\int_a^b fdh\right|^2 & \\ &\leq \min\left\{\left[u\left(b\right)-u\left(a\right)\right]\int_a^b \left|f\right|^2 dv, \left[v\left(b\right)-v\left(a\right)\right]\int_a^b \left|f\right|^2 du\right\}.\end{aligned} \tag{2.8}$$

3. Applications for the Trapezoid Inequality

In this section we provide some inequalities of trapezoid type by utilizing the above inequalities (2.8) and (2.1).

Theorem 3. *If $f:[a,b]\to\mathbb{C}$ is an S-dominated function by the pair (u,v) that are monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{f\left(a\right)+f\left(b\right)}{2}\left(b-a\right)-\int_a^b f\left(t\right)dt\right|^2 \\ &\leq \min\left\{I\left(u,v\right), I\left(v,u\right)\right\} \\ &\leq \frac{1}{4}\left(b-a\right)^2\left[u\left(b\right)-u\left(a\right)\right]\left[v\left(b\right)-v\left(a\right)\right],\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}I\left(u,v\right) &:= \left[u\left(b\right)-u\left(a\right)\right] \\ &\times\left[\frac{1}{4}\left(b-a\right)^2\left[v\left(b\right)-v\left(a\right)\right]-2\int_a^b\left(t-\frac{a+b}{2}\right)v\left(t\right)dt\right].\end{aligned} \tag{3.2}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have that

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt=\int_a^b\left(t-\frac{a+b}{2}\right)df(t)\,. \tag{3.3}$$

Utilizing the inequality (2.8) we have

$$\begin{aligned}&\left|\int_a^b\left(t-\frac{a+b}{2}\right)df(t)\right|^2 \\ &\leq \min\left\{[u(b)-u(a)]\int_a^b\left(t-\frac{a+b}{2}\right)^2 dv(t)\,,\right. \\ &\left.[v(b)-v(a)]\int_a^b\left(t-\frac{a+b}{2}\right)^2 du(t)\right\}.\end{aligned} \tag{3.4}$$

Integrating by parts in the Riemann-Stieltjes integral we have for v

$$\begin{aligned}&\int_a^b\left(t-\frac{a+b}{2}\right)^2 dv(t) \\ &=\left(t-\frac{a+b}{2}\right)^2 v(t)\Big|_a^b-2\int_a^b\left(t-\frac{a+b}{2}\right)v(t)\,dt \\ &=\frac{1}{4}(b-a)^2[v(b)-v(a)]-2\int_a^b\left(t-\frac{a+b}{2}\right)v(t)\,dt,\end{aligned} \tag{3.5}$$

and a similar equation for u.

Utilizing (3.4) we deduce the first inequality (3.1).

By the Čebyšev inequality for monotonic nondecreasing functions F, G that states that

$$\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt\geq\frac{1}{b-a}\int_a^b F(t)\,dt\cdot\frac{1}{b-a}\int_a^b G(t)\,dt$$

we also have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)v(t)\,dt \\ &\geq\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\cdot\frac{1}{b-a}\int_a^b v(t)\,dt=0\end{aligned}$$

and a similar inequality for u.

This proves the last part of the inequality (3.1). □

We also have:

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on (a,b) and $u,v:[a,b]\to\mathbb{R}$ be differentiable and convex on (a,b). If f' is S-dominated by the pair*

(u', v') *which are monotonic nondecreasing on* (a, b), *then*

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right|^2 \leq \left[\int_a^b u(t)\,dt-(b-a)\,u(b)+\frac{1}{2}(b-a)^2\,u'(b)\right] \times \left[\int_a^b v(t)\,dt-(b-a)\,v(a)-\frac{1}{2}(b-a)^2\,v'(a)\right]. \tag{3.6}$$

Proof. Observe that for f' of bounded variation, the following Riemann-Stieltjes integral exists and integrating by parts twice we have

$$\begin{aligned}\int_a^b (t-a)(b-t)\,df'(t) &= (t-a)(b-t)\,f'(t)\big|_a^b+2\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)\,dt \\ &= 2\left[\left(t-\frac{a+b}{2}\right)f(t)\Big|_a^b-\int_a^b f(t)\,dt\right] \\ &= 2\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right]\end{aligned} \tag{3.7}$$

giving the identity

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt=\frac{1}{2}\int_a^b (t-a)(b-t)\,df'(t). \tag{3.8}$$

Utilising the inequality (2.1) we have

$$\left|\int_a^b (t-a)(b-t)\,df'(t)\right|^2 \leq \int_a^b (t-a)^2\,du'(t)\int_a^b (b-t)^2\,dv'(t). \tag{3.9}$$

Integrating by parts, we have

$$\begin{aligned}\int_a^b (t-a)^2\,du'(t) &= (t-a)^2\,u'(t)\Big|_a^b-2\int_a^b (t-a)\,u'(t)\,dt \\ &= (b-a)^2\,u'(b)-2\left[(t-a)\,u(t)\big|_a^b-\int_a^b u(t)\,dt\right] \\ &= 2\int_a^b u(t)\,dt-2(b-a)\,u(b)+(b-a)^2\,u'(b)\end{aligned}$$

giving that

$$\frac{1}{2}\int_a^b (t-a)^2\,du'(t)=\int_a^b u(t)\,dt-(b-a)\,u(b)+\frac{1}{2}(b-a)^2\,u'(b). \tag{3.10}$$

We also have

$$\begin{aligned}\int_a^b (b-t)^2 dv'(t) &= (b-t)^2 v'(t)\Big|_a^b + 2\int_a^b (b-t) v'(t)\,dt \\ &= -(b-a)^2 v'(a) + 2\left[(b-t) v(t)|_a^b + \int_a^b v(t)\,dt\right] \\ &= 2\int_a^b v(t)\,dt - 2(b-a) v(a) - (b-a)^2 v'(a)\end{aligned}$$

giving that

$$\frac{1}{2}\int_a^b (b-t)^2 dv'(t) = \int_a^b v(t)\,dt - (b-a) v(a) - \frac{1}{2}(b-a)^2 v'(a). \tag{3.11}$$

Making use of (3.8)-(3.11) we deduce the desired inequality (3.6). □

4. Applications for Čebyšev and (CBS)-type Functionals

We can employ the inequality (2.1) to obtain some inequalities for Čebyšev and (CBS)-type functionals as follows:

Theorem 5. *Let $f, g : [a,b] \to \mathbb{C}$ be continuous $[a,b]$. If $h : [a,b] \to \mathbb{C}$ is an S-dominated function by the pair (u,v) which are monotonic nondecreasing on $[a,b]$, with $u(a) < u(b)$, $v(a) < v(b)$ and $h(a) \neq h(b)$, then*

$$|C(f,g;h,u,v)|^2 \leq \frac{[u(b)-u(a)][v(b)-v(a)]}{|h(b)-h(a)|^2} C(f;u)\, C(g;v) \tag{4.1}$$

where

$$\begin{aligned}& C(f,g;h,u,v) \\ &:= \frac{1}{h(b)-h(a)}\int_a^b fg\,dh + \frac{1}{u(b)-u(a)}\int_a^b f\,du \cdot \frac{1}{v(b)-v(a)}\int_a^b g\,dv \\ &- \frac{1}{v(b)-v(a)}\int_a^b g\,dv \cdot \frac{1}{h(b)-h(a)}\int_a^b f\,dh \\ &- \frac{1}{u(b)-u(a)}\int_a^b f\,du \cdot \frac{1}{h(b)-h(a)}\int_a^b g\,dh\end{aligned} \tag{4.2}$$

and

$$C(f;u) := \frac{1}{u(b)-u(a)}\int_a^b |f|^2 du - \left|\frac{1}{u(b)-u(a)}\int_a^b f\,du\right|^2. \tag{4.3}$$

Proof. From the inequality (2.1) we have

$$\begin{aligned}&\left|\int_a^b \left(f - \frac{1}{u(b)-u(a)}\int_a^b f\,du\right)\left(g - \frac{1}{v(b)-v(a)}\int_a^b g\,dv\right) dh\right|^2 \\ &\leq \int_a^b \left|f - \frac{1}{u(b)-u(a)}\int_a^b f\,du\right|^2 du \cdot \int_a^b \left|g - \frac{1}{v(b)-v(a)}\int_a^b g\,dv\right|^2 dv.\end{aligned} \tag{4.4}$$

Observe that

$$\begin{aligned}
(4.5)\qquad & \int_a^b \left(f - \frac{1}{u(b)-u(a)}\int_a^b f du\right)\left(g - \frac{1}{v(b)-v(a)}\int_a^b g dv\right) dh \\
&= \int_a^b fg dh + \frac{h(b)-h(a)}{u(b)-u(a)}\int_a^b f du \cdot \frac{1}{v(b)-v(a)}\int_a^b g dv \\
&\quad - \frac{1}{v(b)-v(a)}\int_a^b g dv \int_a^b f dh - \frac{1}{u(b)-u(a)}\int_a^b f du \int_a^b g dh \\
&= [h(b)-h(a)]\left[\frac{1}{h(b)-h(a)}\int_a^b fg dh \right.\\
&\quad + \frac{1}{u(b)-u(a)}\int_a^b f du \cdot \frac{1}{v(b)-v(a)}\int_a^b g dv \\
&\quad - \frac{1}{v(b)-v(a)}\int_a^b g dv \cdot \frac{1}{h(b)-h(a)}\int_a^b f dh \\
&\quad \left. - \frac{1}{u(b)-u(a)}\int_a^b f du \cdot \frac{1}{h(b)-h(a)}\int_a^b g dh \right] \\
&= [h(b)-h(a)]\, C(f,g;h,u,v),
\end{aligned}$$

$$\begin{aligned}
(4.6)\qquad & \int_a^b \left| f - \frac{1}{u(b)-u(a)}\int_a^b f du \right|^2 du \\
&= \int_a^b |f|^2 du - \frac{1}{u(b)-u(a)}\left|\int_a^b f du\right|^2 \\
&= [u(b)-u(a)] \\
&\quad \times \left[\frac{1}{u(b)-u(a)}\int_a^b |f|^2 du - \left|\frac{1}{u(b)-u(a)}\int_a^b f du\right|^2\right] \\
&= [u(b)-u(a)]\, C(f;u)
\end{aligned}$$

and, similarly,

$$(4.7)\qquad \int_a^b \left| g - \frac{1}{v(b)-v(a)}\int_a^b g dv \right|^2 dv = [v(b)-v(a)]\, C(g;v).$$

Making use of (4.4)-(4.7) we deduce the desired result (4.1). □

Theorem 6. *Let $f, g : [a,b] \to \mathbb{C}$ be continuous on $[a,b]$. If $h : [a,b] \to \mathbb{C}$ is an S-dominated function by the pair (u,v), which are monotonic nondecreasing on $[a,b]$, then*

$$(4.8)\qquad |L(f,g;h)|^2 \le \frac{1}{2}\left[B(f;u)B(f;u,v)B(g;u,v)B(g;v)\right]^{1/2}$$

where

$$L(f,g;h) := [h(b)-h(a)]\int_a^b fg dh - \int_a^b f dh \int_a^b g dh,$$

$$B(f;u):=[u(b)-u(a)]\int_a^b |f|^2\,du-\left|\int_a^b f\,du\right|^2\ (\geq 0) \tag{4.9}$$

and

$$\begin{aligned}B(f;u,v)&:=[v(b)-v(a)]\int_a^b |f|^2\,du+[u(b)-u(a)]\int_a^b |f|^2\,dv\\&-2\operatorname{Re}\left(\int_a^b f\,du\int_a^b \overline{f}\,dv\right)(\geq 0)\,.\end{aligned} \tag{4.10}$$

Proof. Utilising the inequality (2.1) we have

$$\begin{aligned}&\left|\int_a^b (f(x)-f(y))(g(x)-g(y))\,dh(y)\right|\\&\leq\left(\int_a^b |f(x)-f(y)|^2\,du(y)\right)^{1/2}\left(\int_a^b |g(x)-g(y)|^2\,dv(y)\right)^{1/2}\end{aligned} \tag{4.11}$$

for any $x\in[a,b]$.

We know that for any continuous function $\ell:[a,b]\to\mathbb{C}$ we have the inequality (see 1.4)

$$\left|\int_a^b \ell(x)\,dh(x)\right|^2\leq\int_a^b |\ell(x)|\,du(x)\int_a^b |\ell(x)|\,dv(x)\,. \tag{4.12}$$

By this inequality we have

$$\begin{aligned}&\left|\int_a^b\left(\int_a^b (f(x)-f(y))(g(x)-g(y))\,dh(y)\right)dh(x)\right|^2\\&\leq\int_a^b\left(\left|\int_a^b (f(x)-f(y))(g(x)-g(y))\,dh(y)\right|\right)du(x)\\&\times\int_a^b\left(\left|\int_a^b (f(x)-f(y))(g(x)-g(y))\,dh(y)\right|\right)dv(x)\\&\leq\int_a^b\left(\int_a^b |f(x)-f(y)|^2\,du(y)\right)^{1/2}\\&\times\left(\int_a^b |g(x)-g(y)|^2\,dv(y)\right)^{1/2}du(x)\\&\times\int_a^b\left(\int_a^b |f(x)-f(y)|^2\,du(y)\right)^{1/2}\\&\times\left(\int_a^b |g(x)-g(y)|^2\,dv(y)\right)^{1/2}dv(x)\\&:=J,\end{aligned} \tag{4.13}$$

where for the last part we used (4.11).

Further, by the Cauchy-Bunyakovsky-Schwarz inequality for the Riemann-Stieltjes integral of monotonic nondecreasing integrators we have for u

$$\begin{aligned}
&\int_a^b \left(\int_a^b |f(x)-f(y)|^2 du(y)\right)^{1/2} \left(\int_a^b |g(x)-g(y)|^2 dv(y)\right)^{1/2} du(x)\\
&\leq \left[\int_a^b \left(\int_a^b |f(x)-f(y)|^2 du(y)\right) du(x)\right]^{1/2}\\
&\times \left[\int_a^b \left(\int_a^b |g(x)-g(y)|^2 dv(y)\right) du(x)\right]^{1/2}
\end{aligned}$$

and a similar inequality for v.

Then

$$\begin{aligned}
J \leq &\left[\int_a^b \left(\int_a^b |f(x)-f(y)|^2 du(y)\right) du(x)\right]^{1/2}\\
\times &\left[\int_a^b \left(\int_a^b |g(x)-g(y)|^2 dv(y)\right) du(x)\right]^{1/2}\\
\times &\left[\int_a^b \left(\int_a^b |f(x)-f(y)|^2 du(y)\right) dv(x)\right]^{1/2}\\
\times &\left[\int_a^b \left(\int_a^b |g(x)-g(y)|^2 dv(y)\right) dv(x)\right]^{1/2}.
\end{aligned} \tag{4.14}$$

Since

$$\begin{aligned}
&\int_a^b \left(\int_a^b |f(x)-f(y)|^2 du(y)\right) du(x)\\
&= 2\left[[u(b)-u(a)]\int_a^b |f|^2 du - \left|\int_a^b f du\right|^2\right]\\
&= 2B(f;u),
\end{aligned}$$

$$\begin{aligned}
&\int_a^b \left(\int_a^b |g(x)-g(y)|^2 dv(y)\right) du(x)\\
&= [v(b)-v(a)]\int_a^b |g|^2 du + [u(b)-u(a)]\int_a^b |g|^2 dv\\
&- 2\operatorname{Re}\left(\int_a^b g du \int_a^b \overline{g} dv\right)\\
&= B(g;u,v),
\end{aligned}$$

$$\int_a^b \left(\int_a^b |f(x) - f(y)|^2 du(y) \right) dv(x)$$
$$= [v(b) - v(a)] \int_a^b |f|^2 du + [u(b) - u(a)] \int_a^b |f|^2 dv$$
$$- 2\operatorname{Re}\left(\int_a^b f du \int_a^b \overline{f} dv \right)$$
$$= B(f; u, v)$$

and

$$\int_a^b \left(\int_a^b |g(x) - g(y)|^2 dv(y) \right) dv(x)$$
$$= 2\left[[v(b) - v(a)] \int_a^b |g|^2 dv - \left| \int_a^b g dv \right|^2 \right]$$
$$= 2B(g; v).$$

Moreover,

$$\int_a^b \left(\int_a^b (f(x) - f(y))(g(x) - g(y)) dh(y) \right) dh(x)$$
$$= 2\left[(h(b) - h(a)) \int_a^b fg dh - \int_a^b f dh \int_a^b g dh \right]$$
$$= 2L(f, g; h).$$

Making use of (4.13) and (4.14) we deduce the desired result (4.8). □

5. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{5.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [34, p. 256]:

Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda | \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties:

a) $E_\lambda \le E_{\lambda'}$ for $\lambda \le \lambda'$;
b) $E_{m-0} = 0, E_M = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;

We have the representation

$$A=\int_{m-0}^{M}\lambda dE_{\lambda}. \tag{5.2}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left\Vert \varphi\left(A\right)-\sum_{k=1}^{n}\varphi\left(\lambda_{k}^{\prime}\right)\left[E_{\lambda_{k}}-E_{\lambda_{k-1}}\right]\right\Vert \leq\varepsilon \tag{5.3}$$

whenever

$$\begin{cases}\lambda_{0}<m=\lambda_{1}<...<\lambda_{n-1}<\lambda_{n}=M,\\ \\ \lambda_{k}-\lambda_{k-1}\leq\delta\text{ for }1\leq k\leq n,\\ \\ \lambda_{k}^{\prime}\in\left[\lambda_{k-1},\lambda_{k}\right]\text{ for }1\leq k\leq n\end{cases} \tag{5.4}$$

this means that

$$\varphi\left(A\right)=\int_{m-0}^{M}\varphi\left(\lambda\right)dE_{\lambda}, \tag{5.5}$$

where the integral is of Riemann-Stieltjes type.

With the above assumptions for A,E_{λ} and φ we have the representations

$$\varphi\left(A\right)x=\int_{m-0}^{M}\varphi\left(\lambda\right)dE_{\lambda}x\quad\text{for all }x\in H \tag{5.6}$$

and

$$\left\langle \varphi\left(A\right)x,y\right\rangle =\int_{m-0}^{M}\varphi\left(\lambda\right)d\left\langle E_{\lambda}x,y\right\rangle \quad\text{for all }x,y\in H. \tag{5.7}$$

In particular,

$$\left\langle \varphi\left(A\right)x,x\right\rangle =\int_{m-0}^{M}\varphi\left(\lambda\right)d\left\langle E_{\lambda}x,x\right\rangle \quad\text{for all }x\in H. \tag{5.8}$$

Moreover, we have the equality

$$\left\Vert \varphi\left(A\right)x\right\Vert ^{2}=\int_{m-0}^{M}\left\vert \varphi\left(\lambda\right)\right\vert ^{2}d\left\Vert E_{\lambda}x\right\Vert ^{2}\text{ for all }x\in H. \tag{5.9}$$

Utilising Theorem 2 we can prove easily the following Schwarz type inequality:

Proposition 1. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m=\min\left\{\lambda\,|\lambda\in Sp\left(A\right)\right\}=:\min Sp\left(A\right)$ and $M=\max\left\{\lambda\,|\lambda\in Sp\left(A\right)\right\}=:\max Sp\left(A\right).$ If $f,g:\mathbb{R}\rightarrow\mathbb{C}$ are continuous functions on $\left[m,M\right],$ then we have the inequality*

$$\left\vert \left\langle f\left(A\right)g\left(A\right)x,y\right\rangle \right\vert ^{2}\leq\left\langle \left\vert f\left(A\right)\right\vert ^{2}x,x\right\rangle \left\langle \left\vert g\left(A\right)\right\vert ^{2}y,y\right\rangle \tag{5.10}$$

for any $x,y\in H$.

Proof. Let $\varepsilon>0$ and for fixed $x,y\in H$ define the functions $h,u,v:\left[m-\varepsilon,M\right]\rightarrow\mathbb{C}$ given by

$$h\left(t\right):=\left\langle E_{t}x,y\right\rangle ,\;u\left(t\right):=\left\langle E_{t}x,x\right\rangle \text{ and }v\left(t\right):=\left\langle E_{t}y,y\right\rangle$$

where $\left\{E_{\lambda}\right\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A.

For $t,s\in[m-\varepsilon,M]$ with $t>s$ by utilizing the Schwarz inequality for nonnegative operators P

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle,$$

we have

$$\begin{aligned}|h(t)-h(s)|^2 &= |\langle(E_t-E_s)x,y\rangle|^2\leq\langle(E_t-E_s)x,x\rangle\langle(E_t-E_s)y,y\rangle\\ &= (u(t)-u(s))(v(t)-v(s)),\end{aligned}$$

which shows that h is S-dominated by the monotonic nondecreasing functions (u,v) on $[m-\varepsilon,M]$.

Applying Theorem 2 for f, g, h, u and v on $[m-\varepsilon,M]$ we have

$$\begin{aligned}&\left|\int_{m-\varepsilon}^{M}f(t)g(t)\,d(\langle E_tx,y\rangle)\right|^2\\ &\leq\int_{m-\varepsilon}^{M}|f(t)|^2\,d(\langle E_tx,x\rangle)\int_{m-\varepsilon}^{M}|g(t)|^2\,d(\langle E_ty,y\rangle)\end{aligned}\tag{5.11}$$

for any $x,y\in H$.

Letting $\varepsilon\to0+$ in (5.11) and utilizing the representation of continuous functions of selfadjoint operators, we deduce the desired result (5.10). □

Remark 2. *The above inequality can be also proved by using the Schwarz inequality*

$$|\langle u,v\rangle|^2\leq\langle u,u\rangle\langle v,v\rangle$$

for $u=f(U)x$ and $v=g(U)y$ and utilizing the properties of continuous functional calculus. The details are omitted.

For the continuous functions $f,g:\mathbb{R}\to\mathbb{C}$ and the selfadjoint operator A define the functionals

$$\begin{aligned}&C(f,g;A,x,y)\\ &:=\langle f(A)g(A)x,y\rangle+\langle x,y\rangle\cdot\frac{\langle f(A)x,x\rangle}{\|x\|^2}\cdot\frac{\langle g(A)y,y\rangle}{\|y\|^2}\\ &-\frac{\langle g(A)y,y\rangle}{\|y\|^2}\cdot\langle f(A)x,y\rangle-\frac{\langle f(A)x,x\rangle}{\|x\|^2}\cdot\langle f(A)x,y\rangle\end{aligned}\tag{5.12}$$

and

$$C(f;A,x):=\left\langle|f(A)|^2x,x\right\rangle-\frac{|\langle f(A)x,x\rangle|^2}{\|x\|^2}\,(\geq0),\tag{5.13}$$

where $x,y\in H$ and $x,y\neq0$.

Proposition 2. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Assume that $f,g:\mathbb{R}\to\mathbb{C}$ are continuous on $[m,M]$. Then for any $x,y\in H$ with $x,y\neq0$, we have*

$$|C(f,g;A,x,y)|^2\leq C(f;A,x)\,C(g;A,y).\tag{5.14}$$

The proof follows by Theorem 5 by a similar argument to the one from Proposition 1 and we omit the details.

Now we can define for the continuous functions $f, g : [a, b] \to \mathbb{C}$ and the selfadjoint operator A the following functionals as well:

$$L(f, g; A, x, y) := \langle x, y\rangle \langle f(A) g(A) x, y\rangle - \langle f(A) x, y\rangle \langle g(A) x, y\rangle, \tag{5.15}$$

$$B(f; x) := \|x\|^2 \left\langle |f(A)|^2 x, x \right\rangle - |\langle f(A) x, x\rangle|^2 \ (\geq 0) \tag{5.16}$$

and

$$\begin{aligned} B(f; x, y) &:= \|y\|^2 \left\langle |f(A)|^2 x, x \right\rangle + \|x\|^2 \left\langle |f(A)|^2 y, y \right\rangle \\ &\quad - 2 \operatorname{Re} \left(\langle f(A) x, x\rangle \left\langle \overline{f}(A) y, y\right\rangle \right) (\geq 0), \end{aligned} \tag{5.17}$$

for any $x, y \in H$.

Utilising Theorem 5 we can state the following result as well:

Proposition 3. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Assume that $f, g : \mathbb{R} \to \mathbb{C}$ are continuous on $[m, M]$. Then for any $x, y \in H$*

$$|L(f, g; A, x, y)|^2 \leq \frac{1}{2} \left[B(f; x) B(f; x, y) B(g; x, y) B(g; y) \right]. \tag{5.18}$$

6. Applications for Unitary Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. We recall that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [34, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of *Riemann-Stieltjes* type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0, 2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex-valued function $f : \mathcal{C}(0, 1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0, 1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{6.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f(U) x = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda x, \tag{6.2}$$

$$\langle f(U) x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{6.3}$$

and

$$\|f(U) x\|^2 = \int_0^{2\pi} \left| f\left(e^{i\lambda}\right) \right|^2 d\|E_\lambda x\|^2, \tag{6.4}$$

for any $x, y \in H$.

Proposition 4. *Let U be a unitary operator on the Hilbert space H. Then for every continuous complex-valued functions $f, g : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have*

$$|\langle f(U)\, g(U)\, x, y\rangle|^2 \le \left\langle |f(U)|^2 x, x \right\rangle \left\langle |g(U)|^2 y, y \right\rangle \tag{6.5}$$

for any $x, y \in H$.

Proof. Let $\{E_\lambda\}_{\lambda \in [0,2\pi]}$ be the spectral family of the unitary operator U. For fixed $x, y \in H$ define the functions $h, u, v : [0, 2\pi] \to \mathbb{C}$ given by

$$h(t) := \langle E_t x, y\rangle,\ u(t) := \langle E_t x, x\rangle \text{ and } v(t) := \langle E_t y, y\rangle .$$

For $t, s \in [0, 2\pi]$ with $t > s$ by utilizing the Schwarz inequality for nonnegative operators P

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle \langle Py, y\rangle ,$$

we have

$$\begin{aligned} |h(t) - h(s)|^2 &= |\langle (E_t - E_s)\, x, y\rangle|^2 \le \langle (E_t - E_s)\, x, x\rangle \langle (E_t - E_s)\, y, y\rangle \\ &= (u(t) - u(s))\,(v(t) - v(s)), \end{aligned}$$

which shows that h is S-dominated by the monotonic nondecreasing functions (u, v) on $[0, 2\pi]$.

Applying Theorem 2 for $f\left(e^{i\cdot}\right)$, h, u and v on $[0, 2\pi]$ we have

$$\begin{aligned} &\left| \int_0^{2\pi} f\left(e^{it}\right) g\left(e^{it}\right) d\left(\langle E_t x, y\rangle\right) \right|^2 \\ &\le \int_0^{2\pi} \left| f\left(e^{it}\right)\right|^2 d\left(\langle E_t x, x\rangle\right) \int_0^{2\pi} \left| g\left(e^{it}\right)\right|^2 d\left(\langle E_t y, y\rangle\right) \end{aligned}$$

for any $x, y \in H$.

Utilising the representation of continuous functions of unitary operators, we deduce the desired result (6.5). □

Remark 3. *The interested reader may state some inequalities for functions of unitary operators that are similar to those incorporated in Proposition 2 and 3. The details are however omitted.*

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