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INEQUALITIES FOR THE RIEMANN-STIELTJES INTEGRAL OF (p,q)-H-DOMINATED INTEGRATORS WITH APPLICATIONS

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Abstract. Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. For $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we say that the complex-valued function $h:[a,b]\to\mathbb{C}$ is (p,q)-H-*dominated* by the pair (u,v) if

$$|h(y)-h(x)|\le[u(y)-u(x)]^{1/p}[v(y)-v(x)]^{1/q}$$

for any $x,y\in[a,b]$ with $y\ge x$.

In this paper we show amongst other that

$$\left|\int_a^b fdh\right|\le\left(\int_a^b|f|\,du\right)^{1/p}\left(\int_a^b|f|\,dv\right)^{1/q},$$

and

$$\left|\int_a^b fgdh\right|\le\left(\int_a^b|f|^p\,du\right)^{1/p}\left(\int_a^b|g|^q\,dv\right)^{1/q}$$

for any continuous functions $f,g:[a,b]\to\mathbb{C}$.

Applications for the trapezoidal and midpoint inequalities are also given.

1. Introduction

One of the most important properties of the *Riemann-Stieltjes integral* $\int_a^b f(t)\,dg(t)$ is the fact that this integral exists if one of the function is of *bounded variation* while the other is *continuous*. The following sharp inequality holds

$$\left|\int_a^b f(t)\,dg(t)\right|\le\max_{t\in[a,b]}|f(t)|\bigvee_a^b(g),\tag{1.1}$$

provided that $f:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ is of bounded variation on this interval. Here $\bigvee_a^b(g)$ denotes the *total variation* of g on $[a,b]$.

When g is *Lipschitzian* with the constant $L>0$, i.e.,

$$|g(t)-g(s)|\le L|t-s|$$

for any $t,s\in[a,b]$, then we have

$$\left|\int_a^b f(t)\,dg(t)\right|\le L\int_a^b|f(t)|\,dt\tag{1.2}$$

for any *Riemann integrable* function $f:[a,b]\to\mathbb{C}$.

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Moreover, if the integrator g is *monotonic nondecreasing* on the interval $[a,b]$ and $f:[a,b]\to\mathbb{C}$ is continuous, then we have the *modulus inequality*

$$\left|\int_a^b f(t)\,dg(t)\right|\le\int_a^b|f(t)|\,dg(t)\,. \tag{1.3}$$

The above inequalities have been used by many authors to derive various integral inequalities. We provide here some simple examples.

The following *generalized trapezoidal inequality* for the function of bounded variation $f:[a,b]\to\mathbb{C}$ was obtained in 1999 by the author [21, Proposition 1]

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(x-a)f(a)-(b-x)f(b)\right|\\&\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\,,\end{aligned} \tag{1.4}$$

where $x\in[a,b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity. See also [19] for a different proof and other details.

The best inequality one can derive from (1.4) is the *trapezoid inequality*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f)\,. \tag{1.5}$$

Here the constant $\frac{1}{2}$ is also best possible.

For related results, see [11]-[15], [17]-[20], [24]-[25], [29]-[32], [33], [39], [40], [42]-[44] and [52]-[54].

In order to extend the classical *Ostrowski's inequality* for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, the author obtained in 1999 (see [21] or the RGMIA preprint version of [23]) the following result

$$\left|\int_a^b f(t)\,dt-f(x)(b-a)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)\,, \tag{1.6}$$

for any $x\in[a,b]$ and $f:[a,b]\to\mathbb{C}$ a function of bounded variation on $[a,b]$. Here $\bigvee_a^b(f)$ denotes the *total variation* of f on $[a,b]$ and the constant $\frac{1}{2}$ is best possible in (1.6). The best inequality one can obtain from (1.6) is the *midpoint inequality*, namely

$$\left|\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)(b-a)\right|\le\frac{1}{2}(b-a)\bigvee_a^b(f)\,, \tag{1.7}$$

for which the constant $\frac{1}{2}$ is also sharp.

For related results, see [1]-[11], [16]-[17], [21], [23], [25]-[27], [31], [34]-[38], [41], [45]-[51] and [55]-[58].

Motivated by the above results, we establish in this paper bounds for the quantities

$$\left|\int_a^b fdh\right|\text{ and }\left|\int_a^b fgdh\right|$$

in the case when the integrands f,g, are continuous while the function of bounded variation h is (p,q)-H-*dominated* by a pair of monotonic functions in the sense presented at the beginning of the next section. Applications for the trapezoidal and midpoint inequalities are also given.

2. Some General Inequalities

Assume that $u, v : [a, b] \to \mathbb{R}$ are *monotonic nondecreasing* on the interval $[a, b]$. Assume everywhere in what follows that $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that the complex-valued function $h : [a, b] \to \mathbb{C}$ is (p, q)-H-*dominated* by the pair (u, v) if

$$|h(y) - h(x)| \le [u(y) - u(x)]^{1/p} [v(y) - v(x)]^{1/q} \tag{S}$$

for any $x, y \in [a, b]$ with $y \ge x$.

We can give numerous examples of such functions.

For instance, if we take f, g two measurable complex-valued functions such that $|f|^p$ and $|g|^q$ are Lebesgue integrable and denote

$$h(x) := \int_a^x f(t) g(t) \, dt, \; u(x) := \int_a^x |f(t)|^p \, dt \text{ and } v(x) := \int_a^x |g(t)|^q \, dt,$$

then we observe that u and v are monotonic nondecreasing on $[a, b]$ and by *Hölder integral inequality* we have for any $y \ge x$ with $x, y \in [a, b]$ that

$$\begin{aligned} |h(y) - h(x)| &= \left| \int_x^y f(t) g(t) \, dt \right| \le \left(\int_x^y |f(t)|^p \, dt \right)^{1/p} \left(\int_x^y |g(t)|^q \, dt \right)^{1/q} \\ &\le [u(y) - u(x)]^{1/p} [v(y) - v(x)]^{1/q}. \end{aligned}$$

Now, for $m, n > 0$ if we consider $f(t) := t^m$ and $g(t) := t^n$ for $t \ge 0$, then

$$h_{m,n}(x) := \int_0^x t^{m+n} dt = \frac{1}{m+n+1} x^{m+n+1}$$

and

$$u_{m,p}(x) := \int_0^x t^{pm} dt = \frac{1}{2pm+1} x^{2pm+1}, \; v_{n,q}(x) := \int_0^x t^{qn} dt = \frac{1}{2qn+1} x^{2qn+1},$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Taking into account the above comments we observe that the function $h_{m,n}$ is (p, q)-H-*dominated* by the pair $(u_{m,p}, v_{n,q})$ on any subinterval of $[0, \infty)$.

Proposition 1. *If $h : [a, b] \to \mathbb{C}$ is (p, q)-H-dominated by the pair (u, v), then h is of bounded variation on any subinterval $[c, d] \subset [a, b]$ and*

$$\bigvee_c^d (h) \le [u(d) - u(c)]^{1/p} [v(d) - v(c)]^{1/q}. \tag{2.1}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Consider a division δ of the interval $[c, d]$ given by

$$\delta : c = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Since $h : [a, b] \to \mathbb{C}$ is (p, q)-H-*dominated* by the pair (u, v) then we have

$$|h(x_{i+1}) - h(x_i)| \le [u(x_{i+1}) - u(x_i)]^{1/p} [v(x_{i+1}) - v(x_i)]^{1/q}$$

for any $i \in \{0, ..., n-1\}$.

Summing this inequality over i from 0 to $n-1$ and utilizing the Hölder discrete inequality we have

$$\begin{aligned}\sum_{i=1}^{n-1}|h(x_{i+1})-h(x_i)| & \\ \leq \sum_{i=1}^{n-1}[u(x_{i+1})-u(x_i)]^{1/p}[v(x_{i+1})-v(x_i)]^{1/q} & \\ \leq \left(\sum_{i=1}^{n-1}[u(x_{i+1})-u(x_i)]\right)^{1/p}\left(\sum_{i=1}^{n-1}[v(x_{i+1})-v(x_i)]\right)^{1/q} & \\ = [u(d)-u(c)]^{1/p}[v(d)-v(c)]^{1/q}. &\end{aligned} \tag{2.2}$$

Taking the supremum over δ we deduce the desired result (2.1). □

Corollary 1. *If $h:[a,b]\to\mathbb{C}$ is (p,q)-H-dominated by the pair (u,v), then the cumulative variation function $V:[a,b]\to[0,\infty)$ defined by*

$$V(x):=\bigvee_a^x(h)$$

is also (p,q)-H-dominated by the pair (u,v).

The following result is a kind of Hölder integral inequality for the Riemann-Stieltjes integral:

Theorem 1. *Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h:[a,b]\to\mathbb{C}$ is (p,q)-H-dominated by the pair (u,v) and $f:[a,b]\to\mathbb{C}$ is a continuous function on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists and*

$$\left|\int_a^b f(t)\,dh(t)\right|\leq\left(\int_a^b|f(t)|\,du(t)\right)^{1/p}\left(\int_a^b|f(t)|\,dv(t)\right)^{1/q}. \tag{2.3}$$

Proof. Since the Riemann-Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists, then for any sequence of partitions

$$I_n^{(n)}:a=t_0^{(n)}<t_1^{(n)}<\cdots<t_{n-1}^{(n)}<t_n^{(n)}=b$$

with the norm

$$v\left(I_n^{(n)}\right):=\max_{i\in\{0,\ldots,n-1\}}\left(t_{i+1}^{(n)}-t_i^{(n)}\right)\to 0$$

as $n \to \infty$, and for any intermediate points $\xi_i^{(n)} \in [t_i^{(n)}, t_{i+1}^{(n)}]$, $i \in \{0, \dots, n-1\}$ we have:

$$
\begin{aligned}
(2.4)\quad & \left| \int_a^b f(t)\, dh(t) \right| \\
&= \left| \lim_{v\left(I_n^{(n)}\right) \to 0} \sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right) \left[h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right) \right] \right| \\
&\le \lim_{v\left(I_n^{(n)}\right) \to 0} \sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right) \right| \left| h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right) \right| \\
&\le \lim_{v\left(I_n^{(n)}\right) \to 0} \sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right) \right| \left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right) \right]^{1/p} \left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right) \right]^{1/q} \\
&\le \lim_{v\left(I_n^{(n)}\right) \to 0} \left(\sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right) \right| \left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right) \right] \right)^{1/p} \\
&\times \lim_{v\left(I_n^{(n)}\right) \to 0} \left(\sum_{i=0}^{n-1} \left| f\left(\xi_i^{(n)}\right) \right| \left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right) \right] \right)^{1/q} \\
&= \left(\int_a^b |f(t)|\, du(t) \right)^{1/p} \left(\int_a^b |f(t)|\, dv(t) \right)^{1/q},
\end{aligned}
$$

where for the last inequality we employed the Hölder weighted discrete inequality

$$
\sum_{k=1}^{n} m_k a_k b_k \le \left(\sum_{k=1}^{n} m_k a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} m_k b_k^q \right)^{1/q},
$$

where $m_k,\ a_k,\ b_k \ge 0$ for $k \in \{1, ..., n\}$. $\square$

We have the following weighted Hölder type inequality for the Riemann-Stieltjes integral as well.

Theorem 2. *Let $f, g : [a, b] \to \mathbb{C}$ be continuous on $[a, b]$. If $h : [a, b] \to \mathbb{C}$ is (p, q)-H-dominated by the pair (u, v), which are monotonic nondecreasing on $[a, b]$, then for any continuos nonnegative function $\ell : [a, b] \to [0, \infty)$ we have*

$$
(2.5)\qquad \left| \int_a^b \ell f g dh \right| \le \left(\int_a^b \ell |f|^p\, du \right)^{1/p} \left(\int_a^b \ell |g|^q\, dv \right)^{1/q}.
$$

In particular, for $\ell = 1$ we have

$$
(2.6)\qquad \left| \int_a^b f g dh \right| \le \left(\int_a^b |f|^p\, du \right)^{1/p} \left(\int_a^b |g|^q\, dv \right)^{1/q}.
$$

Proof. Since the Riemann-Stieltjes integral $\int_a^b \ell f g dh$ exists, then for any sequence of partitions

$$
I_n^{(n)} : a = t_0^{(n)} < t_1^{(n)} < \cdots < t_{n-1}^{(n)} < t_n^{(n)} = b
$$

with the norm

$$v\left(I_n^{(n)}\right) := \max_{i\in\{0,\dots,n-1\}}\left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$$

as $n\to\infty$, and for any intermediate points $\xi_i^{(n)} \in [t_i^{(n)}, t_{i+1}^{(n)}]$, $i \in \{0,\dots,n-1\}$ we have:

$$\begin{aligned}
(2.7)\qquad & \left|\int_a^b \ell f g dh\right| \\
&= \left|\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right) f\left(\xi_i^{(n)}\right) g\left(\xi_i^{(n)}\right)\left[h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right]\right| \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right)\left|f\left(\xi_i^{(n)}\right)\right|\left|g\left(\xi_i^{(n)}\right)\right|\left|h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right| \\
&\le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right)\left|f\left(\xi_i^{(n)}\right)\right|\left|g\left(\xi_i^{(n)}\right)\right| \\
&\times\left|u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right|^{1/p}\left|u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right|^{1/q} \\
&:= I.
\end{aligned}$$

Utilising the weighted Hölder discrete inequality

$$\sum_{k=1}^n \ell_k a_k b_k \le \left(\sum_{k=1}^n \ell_k a_k^p\right)^{1/p}\left(\sum_{k=1}^n \ell_k b_k^q\right)^{1/q}$$

where $\ell_k,\ a_k,\ b_k \ge 0$ for $k\in\{1,...,n\}$, we have

$$\begin{aligned}
(2.8)\qquad I &\le \left(\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right)\left|f\left(\xi_i^{(n)}\right)\right|^p\left[\left|u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right|^{1/p}\right]^p\right)^{1/p} \\
&\times\left(\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right)\left|g\left(\xi_i^{(n)}\right)\right|^q\left[\left|v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right|^{1/q}\right]^q\right)^{1/q} \\
&= \left(\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right)\left|f\left(\xi_i^{(n)}\right)\right|^p\left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right]\right)^{1/p} \\
&\times\left(\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\ell\left(\xi_i^{(n)}\right)\left|g\left(\xi_i^{(n)}\right)\right|^q\left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right]\right)^{1/q} \\
&= \left(\int_a^b \ell\left|f\right|^p du\right)^{1/p}\left(\int_a^b \ell\left|g\right|^q dv\right)^{1/q}.
\end{aligned}$$

Making use of the inequalities (2.7) and (2.8) we deduce the desired result (2.5). □

Remark 1. *From (2.1) we also have the dual inequality*

$$\left|\int_a^b \ell fgdh\right| \leq \left(\int_a^b \ell |g|^p du\right)^{1/p} \left(\int_a^b \ell |f|^q dv\right)^{1/q}, \tag{2.9}$$

which together with (2.1) provide

$$\begin{aligned} &\left|\int_a^b \ell fgdh\right| \\ &\leq \min\left\{\left(\int_a^b \ell |f|^p du\right)^{1/p} \left(\int_a^b \ell |g|^q dv\right)^{1/q}, \right. \\ &\left. \left(\int_a^b \ell |g|^p du\right)^{1/p} \left(\int_a^b \ell |f|^q dv\right)^{1/q}\right\}. \end{aligned} \tag{2.10}$$

In particular we have

$$\begin{aligned} &\max\left\{\left|\int_a^b \ell f^2 dh\right|, \left|\int_a^b \ell |f|^2 dh\right|\right\} \\ &\leq \left(\int_a^b \ell |f|^p du\right)^{1/p} \left(\int_a^b \ell |f|^q dv\right)^{1/q}. \end{aligned} \tag{2.11}$$

We also have the inequality

$$\begin{aligned} \left|\int_a^b \ell fdh\right| \leq \min&\left\{\left(\int_a^b \ell du\right)^{1/p} \left(\int_a^b \ell |f|^q dv\right)^{1/q}, \right. \\ &\left. \left(\int_a^b \ell dv\right)^{1/p} \left(\int_a^b \ell |f|^q du\right)^{1/q}\right\} \end{aligned} \tag{2.12}$$

and in particular

$$\begin{aligned} \left|\int_a^b fdh\right| \leq \min&\left\{[u(b) - u(a)]^{1/p} \left(\int_a^b |f|^q dv\right)^{1/q}, \right. \\ &\left. [v(b) - v(a)]^{1/p} \left(\int_a^b |f|^q du\right)^{1/q}\right\}. \end{aligned} \tag{2.13}$$

3. Trapezoid and Midpoint Inequalities

The following result holds:

Theorem 3. *Assume that $u, v : [a,b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h : [a,b] \to \mathbb{C}$ is (p,q)-H-dominated by the pair (u,v) for $p, q > 1$*

with $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}
&\left|\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt\right| \\
&\le\left[\frac{1}{2}(b-a)[u(b)-u(a)]-\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]^{1/p} \\
&\times\left[\frac{1}{2}(b-a)[v(b)-v(a)]-\int_a^b sgn\left(t-\frac{a+b}{2}\right)v(t)\,dt\right]^{1/q} \\
&\le\frac{1}{2}(b-a)[u(b)-u(a)]^{1/p}[v(b)-v(a)]^{1/q}.
\end{aligned}\tag{3.1}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have that

$$\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt=\int_a^b\left(t-\frac{a+b}{2}\right)dh(t).\tag{3.2}$$

Applying the inequality (2.3) we have

$$\begin{aligned}
&\left|\int_a^b\left(t-\frac{a+b}{2}\right)dh(t)\right| \\
&\le\left(\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right)^{1/p}\left(\int_a^b\left|t-\frac{a+b}{2}\right|dv(t)\right)^{1/q}.
\end{aligned}\tag{3.3}$$

Integrating by parts in the Riemann-Stieltjes integral we also have

$$\begin{aligned}
&\int_a^b\left|t-\frac{a+b}{2}\right|du(t) \\
&=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)du(t)+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)du(t) \\
&=\left(\frac{a+b}{2}-t\right)u(t)\Big|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}u(t)\,dt \\
&+\left(t-\frac{a+b}{2}\right)u(t)\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b u(t)\,dt \\
&=-\frac{b-a}{2}u(a)+\int_a^{\frac{a+b}{2}}u(t)\,dt+\frac{b-a}{2}u(b)-\int_{\frac{a+b}{2}}^b u(t)\,dt \\
&=\frac{1}{2}(b-a)[u(b)-u(a)]-\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt
\end{aligned}\tag{3.4}$$

and a similar relation for v.

By the Čebyšev inequality for monotonic nondecreasing functions F, G that states that

$$\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt\ge\frac{1}{b-a}\int_a^b F(t)\,dt\cdot\frac{1}{b-a}\int_a^b G(t)\,dt$$

we also have

$$\begin{aligned}(3.5)\qquad &\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt\\ &\geq \frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)dt\int_a^b u(t)\,dt=0\end{aligned}$$

and a similar result for v.

Utilizing (3.2)-(3.2) we deduce the desired result (3.1). □

Theorem 4. *Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h:[a,b]\to\mathbb{C}$ is (p,q)-H-dominated by the pair (u,v) for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}(3.6)\qquad &\left|h\left(\frac{a+b}{2}\right)(b-a)-\int_a^b h(t)\,dt\right|\\ &\leq \left[\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]^{1/p}\\ &\times\left[\int_a^b sgn\left(t-\frac{a+b}{2}\right)v(t)\,dt\right]^{1/q}\\ &\leq \frac{1}{2}(b-a)\left[u(b)-u(a)\right]^{1/p}\left[v(b)-v(a)\right]^{1/q}.\end{aligned}$$

Proof. Integrating by parts on the Riemann-Stieltjes integral we have

$$\begin{aligned}(3.7)\qquad &h\left(\frac{a+b}{2}\right)(b-a)-\int_a^b h(t)\,dt\\ &=\int_a^{\frac{a+b}{2}}(t-a)\,dh(t)+\int_{\frac{a+b}{2}}^b(b-t)\,dh(t).\end{aligned}$$

Taking the modulus in (3.7) we have

$$\begin{aligned}(3.8)\qquad &\left|h\left(\frac{a+b}{2}\right)(b-a)-\int_a^b h(t)\,dt\right|\\ &\leq\left|\int_a^{\frac{a+b}{2}}(t-a)\,dh(t)\right|+\left|\int_{\frac{a+b}{2}}^b(b-t)\,dh(t)\right|.\end{aligned}$$

Applying the inequality (2.3) twice, we have

$$\left|\int_a^{\frac{a+b}{2}}(t-a)\,dh(t)\right|\leq\left(\int_a^{\frac{a+b}{2}}(t-a)\,du(t)\right)^{1/p}\left(\int_a^{\frac{a+b}{2}}(t-a)\,dv(t)\right)^{1/q}$$

and

$$\left|\int_{\frac{a+b}{2}}^b(b-t)\,dh(t)\right|\leq\left(\int_{\frac{a+b}{2}}^b(b-t)\,du(t)\right)^{1/p}\left(\int_{\frac{a+b}{2}}^b(b-t)\,dv(t)\right)^{1/q}.$$

Summing these inequalities and utilizing the elementary result

$$\alpha\beta+\lambda\delta\leq(\alpha^p+\lambda^p)^{1/p}(\beta^q+\delta^q)^{1/q}$$

for $\alpha,\beta,\lambda,\delta\geq 0$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
&\left|\int_a^{\frac{a+b}{2}}(t-a)\,dh(t)\right|+\left|\int_{\frac{a+b}{2}}^b(b-t)\,dh(t)\right| \\
&\leq\left(\int_a^{\frac{a+b}{2}}(t-a)\,du(t)\right)^{1/p}\left(\int_a^{\frac{a+b}{2}}(t-a)\,dv(t)\right)^{1/q} \\
&+\left(\int_{\frac{a+b}{2}}^b(b-t)\,du(t)\right)^{1/p}\left(\int_{\frac{a+b}{2}}^b(b-t)\,dv(t)\right)^{1/q} \\
&\leq\left(\int_a^{\frac{a+b}{2}}(t-a)\,du(t)+\int_{\frac{a+b}{2}}^b(b-t)\,du(t)\right)^{1/p} \\
&+\left(\int_a^{\frac{a+b}{2}}(t-a)\,dv(t)+\int_{\frac{a+b}{2}}^b(b-t)\,dv(t)\right)^{1/q}.
\end{aligned}\tag{3.9}$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}(t-a)\,du(t)+\int_{\frac{a+b}{2}}^b(b-t)\,du(t) \\
&=(t-a)\,u(t)\Big|_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}}u(t)\,dt+(b-t)\,u(t)\Big|_{\frac{a+b}{2}}^b+\int_{\frac{a+b}{2}}^b u(t)\,dt \\
&=\frac{1}{2}(b-a)\,u\left(\frac{a+b}{2}\right)-\int_a^{\frac{a+b}{2}}u(t)\,dt \\
&-\frac{1}{2}(b-a)\,u\left(\frac{a+b}{2}\right)+\int_{\frac{a+b}{2}}^b u(t)\,dt \\
&=\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt
\end{aligned}\tag{3.10}$$

and the last integral is nonnegative as shown in the proof of Theorem 3.

The same equality holds for v as well.

Utilising the Grüss integral inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt-\frac{1}{b-a}\int_a^b F(t)\,dt\cdot\frac{1}{b-a}\int_a^b G(t)\,dt\right| \\
&\leq\frac{1}{4}(M-m)(N-n)
\end{aligned}\tag{3.11}$$

that holds for the Lebesgue integrable functions F and G that satisfy the conditions

$$m\leq F(t)\leq M\text{ and }n\leq G(t)\leq N$$

for almost every $t \in [a,b]$, we have

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt \\
&= \frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt \\
&\quad -\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)dt\cdot\frac{1}{b-a}\int_a^b u(t)\,dt \\
&\leq \frac{1}{2}\left[u(b)-u(a)\right]
\end{aligned}$$

which implies that

$$\int_a^b sgn\left(t-\frac{a+b}{2}\right)u(t)\,dt \leq \frac{1}{2}(b-a)\left[u(b)-u(a)\right]. \tag{3.12}$$

A similar result holds for v.

Making use of the inequalities (3.8), (3.9) and (3.12) we deduce the desired result (3.6). □

In this section we provide some inequalities of trapezoid type by utilizing the above inequalities (2.13) and (2.1).

Theorem 5. *If $f:[a,b]\to\mathbb{C}$ is (p,q)-H-dominated by the pair (u,v) for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and (u,v) are monotonic nondecreasing on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \\
&\leq I_{p,q}(u,v) \\
&\leq \frac{1}{2}(b-a)\left[u(b)-u(a)\right]^{1/p}\left[v(b)-v(a)\right]^{1/q},
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
I_{p,q}(u,v) &:= \left[u(b)-u(a)\right]^{1/p} \\
&\quad\times\left\{\frac{1}{2^q}(b-a)^q\left[v(b)-v(a)\right]\right. \\
&\quad\left.-q\int_a^b sgn\left(t-\frac{a+b}{2}\right)\left|t-\frac{a+b}{2}\right|^{q-1}v(t)\,dt\right\}^{1/q}.
\end{aligned} \tag{3.14}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have that

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt = \int_a^b\left(t-\frac{a+b}{2}\right)df(t). \tag{3.15}$$

Utilizing the inequality (2.13) we have

$$\begin{aligned}
&\left|\int_a^b\left(t-\frac{a+b}{2}\right)df(t)\right| \\
&\leq \left[u(b)-u(a)\right]^{1/p}\left(\int_a^b\left|t-\frac{a+b}{2}\right|^q dv(t)\right)^{1/q}.
\end{aligned} \tag{3.16}$$

Integrating by parts in the Riemann-Stieltjes integral we also have

$$\begin{aligned}
&\int_a^b \left|t-\frac{a+b}{2}\right|^q dv(t) \\
&= \left|t-\frac{a+b}{2}\right|^q v(t)\Big|_a^b \\
&- p\int_a^b sgn\left(t-\frac{a+b}{2}\right)\left|t-\frac{a+b}{2}\right|^{q-1} v(t)\,dt \\
&= \frac{1}{2^q}(b-a)^q[v(b)-v(a)] \\
&- q\int_a^b sgn\left(t-\frac{a+b}{2}\right)\left|t-\frac{a+b}{2}\right|^{q-1} v(t)\,dt.
\end{aligned}$$

Utilizing (3.3) we deduce the first inequality (3.13).

By the Čebyšev inequality for monotonic nondecreasing functions F, G that states that

$$\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt \geq \frac{1}{b-a}\int_a^b F(t)\,dt\cdot\frac{1}{b-a}\int_a^b G(t)\,dt$$

we also have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) v(t)\,dt \\
&\geq \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) dt\cdot\frac{1}{b-a}\int_a^b v(t)\,dt = 0.
\end{aligned}$$

This proves the last part of the inequality (3.13). □

We also have another trapezoid type inequality as follows:

Theorem 6. *Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on (a,b) and $u,v:[a,b]\to\mathbb{R}$ be differentiable and convex on (a,b). If f' is (p,q)-H-dominated by the pair (u',v') for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ on (a,b), then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \\
&\leq \left[\frac{1}{2}p(p-1)\int_a^b (t-a)^{p-2}u(t)\,dt\right. \\
&\left.-\frac{1}{2}p(b-a)^{p-1}u(b)+\frac{1}{2}(b-a)^p u'(b)\right]^{1/p} \\
&\times\left[\frac{1}{2}q(q-1)\int_a^b (b-t)^{q-2}v(t)\,dt\right. \\
&\left.-\frac{1}{2}q(b-t)^{q-1}v(a)-\frac{1}{2}(b-a)^q v'(a)\right]^{1/q}.
\end{aligned} \tag{3.17}$$

Proof. Observe that for f' of bounded variation, the following Riemann-Stieltjes integral exists and integrating by parts twice we have

$$\begin{aligned} &\int_a^b (t-a)(b-t)\,df'(t) \\ &= (t-a)(b-t) f'(t)\Big|_a^b + 2\int_a^b \left(t-\frac{a+b}{2}\right) f'(t)\,dt \\ &= 2\left[\left(t-\frac{a+b}{2}\right) f(t)\Big|_a^b - \int_a^b f(t)\,dt\right] \\ &= 2\left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt\right] \end{aligned} \tag{3.18}$$

giving the identity

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt = \frac{1}{2}\int_a^b (t-a)(b-t)\,df'(t). \tag{3.19}$$

Utilising the inequality (2.1) we have

$$\begin{aligned} &\left|\int_a^b (t-a)(b-t)\,df'(t)\right| \\ &\leq \left(\int_a^b (t-a)^p\,du'(t)\right)^{1/p} \left(\int_a^b (b-t)^q\,dv'(t)\right)^{1/q}. \end{aligned} \tag{3.20}$$

Integrating by parts, we have

$$\begin{aligned} &\int_a^b (t-a)^p\,du'(t) \\ &= (t-a)^p u'(t)\Big|_a^b - p\int_a^b (t-a)^{p-1} u'(t)\,dt \\ &= (b-a)^p u'(b) - p\left[(t-a)^{p-1} u(t)\Big|_a^b - (p-1)\int_a^b (t-a)^{p-2} u(t)\,dt\right] \\ &= p(p-1)\int_a^b (t-a)^{p-2} u(t)\,dt - p(b-a)^{p-1} u(b) + (b-a)^p u'(b) \end{aligned}$$

giving that

$$\begin{aligned} &\frac{1}{2}\int_a^b (t-a)^p\,du'(t) \\ &= \frac{1}{2}p(p-1)\int_a^b (t-a)^{p-2} u(t)\,dt - \frac{1}{2}p(b-a)^{p-1} u(b) \\ &+ \frac{1}{2}(b-a)^{2p} u'(b). \end{aligned} \tag{3.21}$$

We also have

$$\begin{aligned}
&\int_a^b (b-t)^q \, dv'(t) \\
&= (b-t)^q v'(t)\Big|_a^b + q\int_a^b (b-t)^{q-1} v'(t)\, dt \\
&= -(b-a)^q v'(a) + q\left[(b-t)^{q-1} v(t)\Big|_a^b + (q-1)\int_a^b (b-t)^{q-2} v(t)\, dt\right] \\
&= q(q-1)\int_a^b (b-t)^{q-2} v(t)\, dt - q(b-t)^{q-1} v(a) - (b-a)^q v'(a)
\end{aligned}$$

giving that

$$\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)^2 \, dv'(t) \\
&= \frac{1}{2} q(q-1)\int_a^b (b-t)^{q-2} v(t)\, dt - \frac{1}{2} q(b-t)^{q-1} v(a) \\
&- \frac{1}{2}(b-a)^q v'(a).
\end{aligned} \tag{3.22}$$

Making use of (3.19)-(3.22) we deduce the desired inequality (3.17). □

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