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INEQUALITIES FOR THE RIEMANN-STIELTJES INTEGRAL OF UNDER THE CHORD FUNCTIONS WITH APPLICATIONS

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Abstract. We say that the function $f:[a,b]\to\mathbb{R}$ is under the chord if

$$\frac{(b-t)\,f(a)+(t-a)\,f(b)}{b-a}\geq f(t)$$

for any $t\in[a,b]$.

In this paper we proved amongst other that

$$\int_a^b u(t)\,df(t)\geq\frac{f(b)-f(a)}{b-a}\int_a^b u(t)\,dt$$

provided that $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and $f:[a,b]\to\mathbb{R}$ is continuous and under the chord.

Some particular cases for the weighted integrals in connection with the Fejér inequalities are provided. Applications for continuous functions of selfadjoint operators on Hilbert spaces are also given.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)<\int_a^b f(x)dx<(b-a)\frac{f(a)+f(b)}{2}, \tag{1.1}$$

$a,b\in\mathbb{R}$, $a<b$. It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [21]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [24].

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [21]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [24].

For related results, see for instance the research papers [1], [3]-[14], [16], [18], [19], [23], [22], [25], [26], [27], the monograph online [13] and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(a+t)=w(b-t),\ \ 0\leq t\leq\frac{1}{2}(b-a),$$

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i.e., $y=w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \le \int_a^b h(x)\,w(x)\,dx \le \frac{h(a)+h(b)}{2}\int_a^b w(x)\,dx. \tag{1.2}$$

If h is concave on (a,b), then the inequalities reverse in (1.2).

Clearly, for $w(x)\equiv 1$ on $[a,b]$ we get 1.1.

Motivated by these classical results and their impact in the literature, it is natural to ask when inequalities for the Riemann-Stieltjes integral of the following types

$$f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \le \int_a^b f(t)\,du(t) \tag{1.3}$$

and

$$\int_a^b f(t)\,du(t) \le [u(b)-u(a)]\,\frac{f(a)+f(b)}{2} \tag{1.4}$$

hold.

In order to address this question, we have introduced in this paper the concept of *under the chord function* on a closed interval $[a,b]$, which generalizes the concept of convex function on $[a,b]$ and established some fundamental inequalities for the Riemann-Stieltjes integral for various classes of integrands and integrators. Some particular cases for the weighted integrals in connection with the Fejér inequalities are provided. Applications for continuous functions of selfadjoint operators on Hilbert spaces are also given.

2. Some Classes of Real Functions

We can introduce the following concept generalizing the notion of convex function.

Definition 1. *We say that the function $f:[a,b]\to\mathbb{R}$ is under the chord if*

$$\frac{(b-t)f(a)+(t-a)f(b)}{b-a} \ge f(t) \tag{2.1}$$

for any $t\in[a,b]$. For simplicity, we denote this by $f\in\mathcal{U}_{Ch}[a,b]$.

It is easy to see that if $f,g\in\mathcal{U}_{Ch}[a,b]$ and $\alpha,\beta\ge 0$ then also $\alpha f+\beta g\in\mathcal{U}_{Ch}[a,b]$ which shows that $\mathcal{U}_{Ch}[a,b]$ is a *convex cone* in the *linear space* of all real-valued functions defined on $[a,b]$. Also, if $f_n\to f$ uniformly on $[a,b]$ and $f_n\in\mathcal{U}_{Ch}[a,b]$ then also $f\in\mathcal{U}_{Ch}[a,b]$ showing that $\mathcal{U}_{Ch}[a,b]$ is also closed in the uniform convergence topology.

Definition 2. *We say that the Lebesgue integrable function $f:[a,b]\to\mathbb{R}$ is sub-trapezoidal if*

$$\frac{f(a)+f(b)}{2}(b-a) \ge \int_a^b f(t)\,dt. \tag{2.2}$$

We denote this by $f\in\mathcal{T}_{Sub}[a,b]$.

As above, we observe that $\mathcal{T}_{Sub}[a,b]$ is a closed convex cone in the uniform convergence topology of the space of all Lebesgue integrable functions defined on $[a,b]$ denoted, as usual, by $\mathcal{L}[a,b]$.

As in the case of convex-concave functions, we can say that f is *above the chord* if $-f \in \mathcal{U}_{Ch}[a,b]$, and f is *super-trapezoidal* if $-f \in \mathcal{T}_{Sub}[a,b]$. Moreover, we say that f is *trapezoidal* if f and $-f \in \mathcal{T}_{Sub}[a,b]$, i.e.

$$\frac{f(a)+f(b)}{2}(b-a)=\int_a^b f(t)\,dt. \tag{2.3}$$

We denote this by $f \in \mathcal{T}[a,b]$. We observe that $\mathcal{T}[a,b]$ is a closed linear subspace of $\mathcal{L}[a,b]$ with the uniform convergence topology.

If we denote by $\mathcal{C}_v[a,b]$ the closed convex cone of all convex functions defined on $[a,b]$, then we can state the following result:

Proposition 1. *We have the strict inclusions*

$$\mathcal{C}_v[a,b] \subsetneqq \mathcal{U}_{Ch}[a,b] \cap \mathcal{L}[a,b] \subsetneqq \mathcal{T}_{Sub}[a,b]. \tag{2.4}$$

Proof. If f is convex on $[a,b]$ then for any $\lambda \in [0,1]$ and $x,y \in [a,b]$ we have

$$\lambda f(x)+(1-\lambda) f(y) \geq f(\lambda x+(1-\lambda) y). \tag{2.5}$$

If we chose $\lambda=\frac{b-t}{b-a}$, $x=a$ and $y=b$ then by (2.5) we have

$$\frac{(b-t) f(a)+(t-a) f(b)}{b-a} \geq f\left(\frac{b-t}{b-a} \cdot a+\frac{t-a}{b-a} \cdot b\right)=f(t)$$

for any $t \in [a,b]$, which shows that $f \in \mathcal{U}_{Ch}[a,b]$. The fact that f is integrable on $[a,b]$ is well known.

Now, if we take

$$f_0:[0,2\pi] \to \mathbb{R},\ f_0(t)=\cos t,$$

then we observe that $f_0 \in \mathcal{U}_{Ch}[0,2\pi] \cap \mathcal{L}[a,b]$ but f_0 is not convex on the whole interval $[0,2\pi]$.

Now, if $f \in \mathcal{U}_{Ch}[a,b] \cap \mathcal{L}[a,b]$, then by integrating (2.1) we have

$$\int_a^b \frac{(b-t) f(a)+(t-a) f(b)}{b-a} dt \geq \int_a^b f(t)\,dt$$

and since

$$\int_a^b \frac{(b-t) f(a)+(t-a) f(b)}{b-a} dt=\frac{f(a)+f(b)}{2}(b-a)$$

we get that $f \in \mathcal{T}_{Sub}[a,b]$.

Consider the function

$$f_1:[0,2\pi] \to \mathbb{R},\ f_1(t)=\sin t$$

then we observe that $f_1 \in \mathcal{T}[0,2\pi]$ and *a fortiori* $f_1 \in \mathcal{T}_{Sub}[0,2\pi]$, but it is easy to see that f_1 is not under the chord on the interval $[0,2\pi]$. □

Proposition 2. *For a function $f:[a,b] \to \mathbb{R}$, the following statements are equivalent:*

(i) $f \in \mathcal{U}_{Ch}[a,b]$;

(ii) *We have the inequality*

$$\frac{f(b)-f(t)}{b-t} \geq \frac{f(t)-f(a)}{t-a} \tag{2.6}$$

for any $t \in (a,b)$.

Proof. We observe that, for $t \in (a,b)$ we have

$$\begin{aligned} & \frac{(b-t)f(a)+(t-a)f(b)}{b-a} - f(t) \\ & = \frac{(b-t)[f(a)-f(t)]+(t-a)[f(b)-f(t)]}{b-a} \\ & = \frac{(b-t)(t-a)}{b-a}\left[\frac{f(b)-f(t)}{b-t} - \frac{f(t)-f(a)}{t-a}\right], \end{aligned}$$

which proves the desired result. □

Corollary 1. *Let* $w : [a,b] \to \mathbb{R}$ *be a Lebesgue integrable function. Define* $f : [a,b] \to \mathbb{R}$ *by* $f(t) = \int_a^t w(s)\,ds$. *Then* $f \in \mathcal{U}_{Ch}[a,b]$ *if and only if*

$$\frac{1}{b-t}\int_t^b w(s)\,ds \geq \frac{1}{t-a}\int_a^t w(s)\,ds \tag{2.7}$$

for any $t \in (a,b)$.

Definition 3. *We say that the function* $f : [a,b] \to \mathbb{R}$ *is symmetric (or anti-symmetric) on the interval* $[a,b]$ *if*

$$f(t) = f(a+b-t)\,(or - f(a+b-t))$$

for any $t \in [a,b]$. *We denote this by* $f \in \mathcal{S}_y[a,b]$ *(or* $f \in \mathcal{A}_s[a,b]$*).*

The following result holds:

Proposition 3. *We have the strict inclusion:*

$$\mathcal{A}_s[a,b] \cap \mathcal{L}[a,b] \subsetneqq \mathcal{T}[a,b]. \tag{2.8}$$

Proof. If $f \in \mathcal{A}_s[a,b] \cap \mathcal{L}[a,b]$ then obviously $f(a) = -f(b)$ and $\int_a^b f(t)\,dt = 0$ and the equality (2.3) is trivially satisfied.

Now, if we consider the function $f_0 : [-2\pi, 2\pi] \to \mathbb{R}$ defined by

$$f_0(t) = \begin{cases} 0 & \text{if } t \in [-2\pi, 0] \\ \sin t & \text{if } t \in (0, 2\pi], \end{cases}$$

then we observe that $f_0 \in \mathcal{T}[-2\pi, 2\pi]$ but f_0 is not anti-symmetric on $[-2\pi, 2\pi]$. □

Proposition 4. *Let* $w : [a,b] \to \mathbb{R}$ *be a Lebesgue integrable function. Define* $f : [a,b] \to \mathbb{R}$ *by*

$$f(t) = \int_a^t w(s)\,ds - \frac{1}{2}\int_a^b w(s)\,ds = \frac{1}{2}\left(\int_a^t w(s)\,ds - \int_t^b w(s)\,ds\right). \tag{2.9}$$

If $w \in \mathcal{S}_y[a,b]$ *then* $f \in \mathcal{A}_s[a,b]$.

Proof. Let $t \in [a,b]$. We have by the definition of f that

$$f(a+b-t) = \int_a^{a+b-t} w(s)\,ds - \frac{1}{2}\int_a^b w(s)\,ds. \tag{2.10}$$

If we make the change of variable $u = a+b-s$, then we have

$$\int_a^{a+b-t} w(s)\,ds = -\int_b^t w(a+b-u)\,du = \int_t^b w(a+b-u)\,du. \tag{2.11}$$

Since $w \in \mathcal{S}_y[a,b]$ then

$$\int_t^b w(a+b-u)\,du = \int_t^b w(u)\,du \tag{2.12}$$

for any $t \in [a,b]$.

On making use of (2.10)-(2.12) we have

$$\begin{aligned} f(a+b-t) &= \int_t^b w(u)\,du - \frac{1}{2}\int_a^b w(s)\,ds \\ &= \frac{1}{2}\left(\int_t^b w(s)\,ds - \int_a^t w(s)\,ds\right) = -f(t) \end{aligned}$$

for any $t \in [a,b]$.

The proof is complete. □

The following result also holds:

Proposition 5. *Let $w : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function. Define $f : [a,b] \to \mathbb{R}$ by*

$$f(t) = \int_a^t w(s)\,ds. \tag{2.13}$$

The following statements are equivalent:

(i) $f\,(or\ -f) \in \mathcal{T}_{Sub}[a,b]$;

(ii) *We have the inequality:*

$$\int_a^b tw(t)\,dt \geq (or\ \leq) \frac{a+b}{2}\int_a^b w(t)\,dt. \tag{2.14}$$

Proof. Utilising the integration by parts for the Riemann integral we have:

$$\begin{aligned} &\frac{f(b)+f(a)}{2}(b-a) - \int_a^b f(t)\,dt \\ &= \frac{1}{2}(b-a)\int_a^b w(t)\,dt - \int_a^b \left(\int_a^t w(s)\,ds\right)dt \\ &= \frac{1}{2}(b-a)\int_a^b w(t)\,dt - \left[\left(\int_a^t w(s)\,ds\right)t\Big|_a^b - \int_a^b tw(t)\,dt\right] \\ &= \frac{1}{2}(b-a)\int_a^b w(t)\,dt - \left[\left(\int_a^b w(s)\,ds\right)b - \int_a^b tw(t)\,dt\right] \\ &= \int_a^b tw(t)\,dt - \frac{a+b}{2}\int_a^b w(t)\,dt, \end{aligned}$$

which proves the desired statement. □

Remark 1. *We observe that by Proposition 5 we have $f \in \mathcal{T}[a,b]$, where f is defined by (2.13), if and only if*

$$\int_a^b tw(t)\,dt = \frac{a+b}{2}\int_a^b w(t)\,dt. \tag{2.15}$$

We denote in the following the closed convex cone of monotonic nondecreasing functions defined on $[a,b]$ by $\mathcal{M}^{\nearrow}[a,b]$ and by $\mathcal{C}[a,b]$ the Banach space of continuous functions on the interval $[a,b]$.

We have the following result:

Corollary 2. *If w (or $-w$) $\in \mathcal{M}^{\nearrow}[a,b]$, then the function f $(-f)$ defined by (2.13) belongs to $\mathcal{T}_{Sub}[a,b]$.*

Proof. We use the Čebyšev inequality that state that

$$\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt \geq (\leq)\,\frac{1}{b-a}\int_a^b F(t)\,dt\frac{1}{b-a}\int_a^b G(t)\,dt$$

provided F and G have the same (opposite) monotonicity on $[a,b]$.

Writing this inequality for $F(t)=t$ and $G(t)=w(t)$ we obtain the desired result. □

Definition 4. *We say that the Lebesgue integrable function $f:[a,b]\to\mathbb{R}$ is of sub(supper)-midpoint type if*

$$\int_a^b f(t)\,dt \geq (\leq)\, f\left(\frac{a+b}{2}\right)(b-a). \tag{2.16}$$

We denote this by $f \in \mathcal{M}_{Sub(Sup)}[a,b]$.

Moreover, we say that f is of *midpoint* type if $f \in \mathcal{M}_{Sub}[a,b] \cap \mathcal{M}_{Sup}[a,b]$, i.e.

$$\int_a^b f(t)\,dt = f\left(\frac{a+b}{2}\right)(b-a). \tag{2.17}$$

We denote this by $f \in \mathcal{M}[a,b]$. We observe that if $f \in \mathcal{A}_s[a,b]$ then obviously $f \in \mathcal{M}[a,b]$ and there are functions which are of midpoint type but not anti-symmetric. Indeed, if we consider the function $f_0:[-2\pi,2\pi]\to\mathbb{R}$ defined by

$$f_0(t) = \begin{cases} 0 & \text{if } t \in [-2\pi, 0] \\ \\ \sin t & \text{if } t \in (0, 2\pi], \end{cases}$$

then we observe that $f_0 \in \mathcal{M}[-2\pi,2\pi]$ but f_0 is not anti-symmetric on $[-2\pi,2\pi]$.

It is obvious that $\mathcal{M}_{Sub}[a,b]$ is a closed convex cone and it contains strictly the convex cone of convex functions defined on $[a,b]$, i.e.

$$\mathcal{C}_v[a,b] \subsetneq \mathcal{M}_{Sub}[a,b].$$

Proposition 6. *Let $w:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function. Define $f:[a,b]\to\mathbb{R}$ by*

$$f(t) = \int_a^t w(s)\,ds.$$

The following statements are equivalent:

(i) f *(or $-f$)* $\in \mathcal{M}_{Sub}[a,b]$;

(ii) *We have the inequality:*

$$\int_a^b tw(t)\,dt \le (or\ \ \ge) a\int_a^{\frac{a+b}{2}} w(s)\,ds + b\int_{\frac{a+b}{2}}^b w(s)\,ds. \tag{2.18}$$

Proof. Utilising the integration by parts for the Riemann integral we have:

$$\begin{aligned}
&\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a) \\
&= \int_a^b \left(\int_a^t w(s)\,ds\right) dt - (b-a)\int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= \left[\left(\int_a^t w(s)\,ds\right) t\Big|_a^b - \int_a^b tw(t)\,dt\right] - (b-a)\int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= \left(\int_a^b w(s)\,ds\right) b - \int_a^b tw(t)\,dt - (b-a)\int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= b\int_a^{\frac{a+b}{2}} w(s)\,ds + b\int_{\frac{a+b}{2}}^b w(s)\,ds - \int_a^b tw(t)\,dt - (b-a)\int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= a\int_a^{\frac{a+b}{2}} w(s)\,ds + b\int_{\frac{a+b}{2}}^b w(s)\,ds - \int_a^b tw(t)\,dt,
\end{aligned}$$

which proves the desired result. □

3. Trapezoidal Inequalities for the Riemann-Stieltjes Integral

We have the following result for the Riemann-Stieltjes integral.

Theorem 2. *Let* $f \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ *and* $u \in \mathcal{M}^{\nearrow}[a,b]$. *Then we have the inequality*

$$\begin{aligned}
f(b)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] + f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] \\
\ge \int_a^b f(t)\,du(t)
\end{aligned} \tag{3.1}$$

or, equivalently, the inequality

$$\int_a^b u(t)\,df(t) \ge \frac{f(b)-f(a)}{b-a}\int_a^b u(t)\,dt. \tag{3.2}$$

Proof. Since $f \in \mathcal{C}[a,b]$ and $u \in \mathcal{M}^{\nearrow}[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and integrating (2.1) over the monotonic nondecreasing integrator u we have

$$\int_a^b \frac{(b-t)f(a) + (t-a)f(b)}{b-a}\,du(t) \ge \int_a^b f(t)\,du(t). \tag{3.3}$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned}
(3.4)\qquad & \int_a^b \frac{(b-t)\,f(a)+(t-a)\,f(b)}{b-a}du(t) \\
&= \frac{(b-t)\,f(a)+(t-a)\,f(b)}{b-a}u(t)\Big|_a^b - \frac{f(b)-f(a)}{b-a}\int_a^b u(t)\,dt \\
&= f(b)\,u(b)-f(a)\,u(a)-\frac{f(b)-f(a)}{b-a}\int_a^b u(t)\,dt \\
&= f(b)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right]+f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right].
\end{aligned}$$

Utilising the inequality (3.3) and the last equality in (3.4) we deduce (3.1).

Integrating by parts in the Riemann-Stieltjes integral we also have

$$(3.5)\qquad \int_a^b f(t)\,du(t) = f(b)\,u(b)-f(a)\,u(a)-\int_a^b u(t)\,df(t).$$

Making use of the inequality (3.3), the second equality in (3.4) and the equality (3.5) we deduce the desired result (3.2). □

In the particular case when f is continuous convex on the interval $[a,b]$ we have:

Corollary 3. *Let $f\in\mathcal{C}[a,b]\cap\mathcal{C}_v[a,b]$ and $u\in\mathcal{M}^{\nearrow}[a,b]$. Then we have the inequality (3.1) and the inequality (3.2).*

Remark 2. *The inequality (3.1) for differentiable convex functions was proved in a different way by P. R. Mercer in 2008, see* [20]. *Without differentiability assumption for the convex function f the inequality (3.1) was also proved in* [14]. *We have shown in here that the inequality (3.1) can be naturally extended to the class of under the chord continuous functions, which is a lot larger than the class of convex functions on a given interval $[a,b]$.*

We also observe that the inequality (3.2) for the case of continuous convex functions was first obtained in 2004 by the author [8] *(see also* [10]*).*

The case when the function u is of trapezoidal type provides the following result:

Corollary 4. *Let $f\in\mathcal{C}[a,b]\cap\mathcal{U}_{Ch}[a,b]$ and $u\in\mathcal{M}^{\nearrow}[a,b]\cap\mathcal{T}[a,b]$. Then we have the inequality*

$$(3.6)\qquad \frac{f(b)+f(a)}{2}[u(b)-u(a)] \geq \int_a^b f(t)\,du(t)$$

or, equivalently, the inequality

$$(3.7)\qquad \int_a^b u(t)\,df(t) \geq \frac{u(b)+u(a)}{2}[f(b)-f(a)].$$

The proof is obvious by Theorem 2 on using the equality

$$\frac{u(a)+u(b)}{2}(b-a)=\int_a^b u(t)\,dt.$$

Remark 3. *We observe that the inequalities (3.6) and (3.7) hold for continuous convex functions f provided $u \in \mathcal{M}^{\nearrow}[a,b] \cap \mathcal{T}[a,b]$, which produce a generalization of the Hermite-Hadamard inequality for convex function, namely*

$$\frac{f(b)+f(a)}{2}(b-a) \geq \int_a^b f(t)\,dt$$

that is obtained from (3.6) when we take $u(t)=t$.

The weighted case is as follows:

Corollary 5. *Let $f \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$, $w \in \mathcal{S}_y[a,b] \cap \mathcal{L}[a,b]$ and $w \geq 0$ on $[a,b]$. Then we have the extension of Fejér inequality*

$$\frac{f(b)+f(a)}{2}\int_a^b w(t)\,dt \geq \int_a^b f(t)\,w(t)\,dt. \tag{3.8}$$

Proof. Consider the function $u:[a,b] \to \mathbb{R}$ defined by

$$u(t) := \int_a^t w(s)\,ds - \frac{1}{2}\int_a^b w(s)\,ds.$$

We observe that $u \in \mathcal{M}^{\nearrow}[a,b]$ and since $w \in \mathcal{S}_y[a,b]$, then by Proposition 4 we deduce that $u \in \mathcal{A}_s[a,b]$.

Applying the inequality (3.6) of Corollary 4 we deduce the desired result (3.8). □

Remark 4. *We observe that for the particular case of f convex function we recapture from (3.8) the classical Féjer inequality* [15] *(see also* [13]*).*

We observe that, by (3.1) for $u=v$, we can state the following equivalent inequality that is of interest for trapezoid type results:

Proposition 7. *Let $f \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $v \in \mathcal{M}^{\nearrow}[a,b]$. Then we have the inequality*

$$\begin{aligned} &\frac{f(b)+f(a)}{2}[v(b)-v(a)] - \int_a^b f(t)\,dv(t) \\ &\geq \frac{f(b)-f(a)}{b-a}\left[\int_a^b v(t)\,dt - \frac{v(a)+v(b)}{2}(b-a)\right]. \end{aligned} \tag{3.9}$$

Remark 5. *We observe that in the case when $v \in \mathcal{M}^{\nearrow}[a,b] \cap \mathcal{T}[a,b]$ or if $f(b) = f(a)$, then (3.9) reduces to (3.6). However, the inequality (3.9) can be also used to provide other sufficient conditions for the inequality (3.6) to hold, as follows.*

Corollary 6. *Let $f \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $v \in \mathcal{M}^{\nearrow}[a,b]$. If either*
(i) $f(b) > f(a)$ and $-v \in \mathcal{T}_{Sub}[a,b]$
or
(ii) $f(b) < f(a)$ and $v \in \mathcal{T}_{Sub}[a,b]$,
then

$$\frac{f(b)+f(a)}{2}[v(b)-v(a)] \geq \int_a^b f(t)\,dv(t). \tag{3.10}$$

The inequality (3.10) obviously holds if f is convex and v is as in Corollary 6.

Remark 6. *Let $f \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $w : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$. If either*

(i) $f(b) > f(a)$ *and*

$$\int_a^b tw(t)\,dt \le \frac{a+b}{2}\int_a^b w(t)\,dt; \tag{3.11}$$

or

(ii) $f(b) < f(a)$ *and*

$$\int_a^b tw(t)\,dt \ge \frac{a+b}{2}\int_a^b w(t)\,dt; \tag{3.12}$$

then

$$\frac{f(b)+f(a)}{2}\int_a^b w(t)\,dt \ge \int_a^b w(t) f(t)\,dt. \tag{3.13}$$

We observe that (3.11) holds if the function $w \in \mathcal{M}^{\searrow}[a,b]$, the closed convex cone of monotonic nonincreasing functions on $[a,b]$. Also, the condition (3.12) is valid if $w \in \mathcal{M}^{\nearrow}[a,b]$.

The following dual result also holds:

Theorem 3. *Let $v \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $g \in \mathcal{M}^{\searrow}[a,b]$. Then we have the inequality*

$$\begin{aligned} &\frac{g(b)+g(a)}{2}[v(b)-v(a)] - \int_a^b g(t)\,dv(t) \\ &\ge \frac{v(b)-v(a)}{b-a}\left[\frac{g(a)+g(b)}{2}(b-a) - \int_a^b g(t)\,dt\right]. \end{aligned} \tag{3.14}$$

The proof is obvious from (3.2) on choosing $u = -g$ and $f = v$, namely

$$\int_a^b g(t)\,dv(t) \le \frac{v(b)-v(a)}{b-a}\int_a^b g(t)\,dt.$$

The inequality (3.14) can be however used to obtain other sufficient conditions for the inequality (3.10) to hold.

Corollary 7. *Let $v \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $g \in \mathcal{M}^{\searrow}[a,b]$. If either*

(i) $g \in \mathcal{T}[a,b]$ *or* $v(b) = v(a)$,

or

(ii) $v(b) > v(a)$ *and* $g \in \mathcal{T}_{Sub}[a,b]$,

or

(iii) $v(b) < v(a)$ *and* $-g \in \mathcal{T}_{Sub}[a,b]$

then

$$\frac{g(b)+g(a)}{2}[v(b)-v(a)] \ge \int_a^b g(t)\,dv(t). \tag{3.15}$$

The following connection with the Féjer inequality can be established.

Remark 7. *Let $w : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ and such that*

$$\frac{1}{b-t}\int_t^b w(s)\,ds \ge \frac{1}{t-a}\int_a^t w(s)\,ds \tag{3.16}$$

for any $t \in (a,b)$. *If* $\int_a^b w(s)\,ds > 0$ *and* $g \in \mathcal{M}^{\searrow}[a,b] \cap \mathcal{T}_{Sub}[a,b]$, *then*

$$\frac{g(b)+g(a)}{2}\int_a^b w(t)\,dt \geq \int_a^b w(t)\,g(t)\,dt. \tag{3.17}$$

4. Midpoint Inequalities for the Riemann-Stieltjes Integral

The following result holds:

Theorem 4. *Let* $f \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}\left[a,\frac{a+b}{2}\right] \cap \mathcal{U}_{Ch}\left[\frac{a+b}{2},a\right]$ *and* $u \in \mathcal{M}^{\nearrow}[a,b]$. *Then we have the inequality*

$$\begin{aligned} &\int_a^b f(t)\,du(t) - [u(b)-u(a)]\,f\left(\frac{a+b}{2}\right) \\ &\leq \left[f\left(\frac{a+b}{2}\right)-f(a)\right]\left[u(a)-\frac{2}{b-a}\int_a^{\frac{a+b}{2}} u(t)\,dt\right] \\ &+\left[f(b)-f\left(\frac{a+b}{2}\right)\right]\left[u(b)-\frac{2}{b-a}\int_{\frac{a+b}{2}}^b u(t)\,dt\right]. \end{aligned} \tag{4.1}$$

Proof. Utilising the integration by parts on the Riemann-Stieltjes integral we have

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} [u(t)-u(a)]\,df(t) + \int_{\frac{a+b}{2}}^b [u(t)-u(b)]\,df(t) \\ &= [u(t)-u(a)]\,f(t)\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} f(t)\,du(t) \\ &+ [u(t)-u(b)]\,f(t)\Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b f(t)\,du(t) \\ &= \left[u\left(\frac{a+b}{2}\right)-u(a)\right] f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} f(t)\,du(t) \\ &+ \left[u(b)-u\left(\frac{a+b}{2}\right)\right] f\left(\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^b f(t)\,du(t) \\ &= [u(b)-u(a)]\,f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t). \end{aligned} \tag{4.2}$$

Consider the function $g:[a,b]\to\mathbb{R}$ defined by

$$g(t) := \begin{cases} u(t)-u(a), & t \in \left[a,\frac{a+b}{2}\right] \\ u(t)-u(b), & t \in \left(a,\frac{a+b}{2}\right]. \end{cases}$$

Then (4.2) can be written as

$$[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,du(t) = \int_a^b g(t)\,df(t). \tag{4.3}$$

Since $f\in\mathcal{C}[a,b]\cap\mathcal{U}_{Ch}\left[a,\frac{a+b}{2}\right]\cap\mathcal{U}_{Ch}\left[\frac{a+b}{2},a\right]$ and $u\in\mathcal{M}^{\nearrow}\left[a,\frac{a+b}{2}\right]\cap\mathcal{M}^{\nearrow}\left[\frac{a+b}{2},b\right]$, then we have from (3.2) that

$$\begin{aligned}\int_a^{\frac{a+b}{2}}&[u(t)-u(a)]\,df(t)\\&\geq\frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}}[u(t)-u(a)]\,dt\\&=\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}}u(t)\,dt-u(a)\right]\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}\int_{\frac{a+b}{2}}^b&[u(t)-u(b)]\,df(t)\\&\geq\frac{f(b)-f\left(\frac{a+b}{2}\right)}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^b[u(t)-u(b)]\,dt\\&=\left[f(b)-f\left(\frac{a+b}{2}\right)\right]\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^b u(t)\,dt-u(b)\right].\end{aligned}\tag{4.5}$$

Adding (4.4) and (4.5) and utilizing (4.3) we deduce the desired inequality (4.1). □

Corollary 8. *Let $f\in\mathcal{C}[a,b]\cap\mathcal{U}_{Ch}\left[a,\frac{a+b}{2}\right]\cap\mathcal{U}_{Ch}\left[\frac{a+b}{2},a\right]$ and $w:[a,b]\to\mathbb{R}$ be a nonnegative Lebesgue integrable function on $[a,b]$. Then we have the inequality*

$$\begin{aligned}\int_a^b&f(t)\,w(t)\,dt-f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt\\&\leq\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}}w(t)\left(t-\frac{a+b}{2}\right)dt\right]\\&+\left[f(b)-f\left(\frac{a+b}{2}\right)\right]\left[\frac{2}{b-a}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)w(t)\,dt\right].\end{aligned}\tag{4.6}$$

Proof. It follows by (4.1) for $u(t):=\int_a^t w(s)\,ds$ and observing that

$$\begin{aligned}u(a)&-\frac{2}{b-a}\int_a^{\frac{a+b}{2}}u(t)\,dt\\&=-\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left(\int_a^t w(s)\,ds\right)dt\\&=-\frac{2}{b-a}\left(\left(\int_a^t w(s)\,ds\right)t\Big|_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}}w(t)\,tdt\right)\\&=-\frac{2}{b-a}\left(\left(\int_a^{\frac{a+b}{2}}w(s)\,ds\right)\frac{a+b}{2}-\int_a^{\frac{a+b}{2}}w(t)\,tdt\right)\\&=\frac{2}{b-a}\int_a^{\frac{a+b}{2}}w(t)\left(t-\frac{a+b}{2}\right)dt\end{aligned}$$

and

$$\begin{aligned}
& u(b)-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}u(t)\,dt \\
& =\int_{a}^{b}w(s)\,ds-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\int_{a}^{t}w(s)\,ds\right)dt \\
& =\int_{a}^{b}w(s)\,ds-\frac{2}{b-a}\left(\left(\int_{a}^{t}w(s)\,ds\right)t\Big|_{\frac{a+b}{2}}^{b}-\int_{\frac{a+b}{2}}^{b}tw(t)\,dt\right) \\
& =\int_{a}^{b}w(s)\,ds-\frac{2}{b-a} \\
& \times\left(\left(\int_{a}^{b}w(s)\,ds\right)b-\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)\frac{a+b}{2}-\int_{\frac{a+b}{2}}^{b}tw(t)\,dt\right) \\
& :=I.
\end{aligned}$$

However

$$\begin{aligned}
& \left(\int_{a}^{b}w(s)\,ds\right)b-\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)\frac{a+b}{2}-\int_{\frac{a+b}{2}}^{b}tw(t)\,dt \\
& =\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)b+\left(\int_{\frac{a+b}{2}}^{b}w(s)\,ds\right)b \\
& -\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)\frac{a+b}{2}-\int_{\frac{a+b}{2}}^{b}tw(t)\,dt \\
& =\frac{b-a}{2}\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)+\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt
\end{aligned}$$

and then

$$\begin{aligned}
I & =\int_{a}^{b}w(s)\,ds-\frac{2}{b-a} \\
& \times\left(\frac{b-a}{2}\left(\int_{a}^{\frac{a+b}{2}}w(s)\,ds\right)+\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt\right) \\
& =\int_{a}^{b}w(s)\,ds-\int_{a}^{\frac{a+b}{2}}w(s)\,ds-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt \\
& =\int_{\frac{a+b}{2}}^{b}w(s)\,ds-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}(b-t)\,w(t)\,dt \\
& =\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\frac{b-a}{2}-b+t\right)w(t)\,dt \\
& =\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)w(t)\,dt,
\end{aligned}$$

which proves the desired inequality (4.6). □

Proposition 8. *Let $u \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $f \in \mathcal{M}^{\nearrow}[a,b]$. Then we have the inequality*

$$\begin{aligned}(4.7)\qquad &\int_a^b f(t)\,du(t) - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \\ &\geq \frac{u(b)-u(a)}{b-a}\left[\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right].\end{aligned}$$

The proof is obvious from (3.2) and the details are omitted.

Corollary 9. *Let $u \in \mathcal{C}[a,b] \cap \mathcal{U}_{Ch}[a,b]$ and $f \in \mathcal{M}^{\nearrow}[a,b]$. If either*
(i) $f \in \mathcal{M}[a,b]$ *or* $u(b) = u(a)$,
or
(ii) $u(b) > u(a)$ *and* $f \in \mathcal{M}_{Sub}[a,b]$,
or
(iii) $u(b) < u(a)$ *and* $f \in \mathcal{M}_{Sup}[a,b]$
then

$$(4.8)\qquad \int_a^b f(t)\,du(t) \geq f\left(\frac{a+b}{2}\right)[u(b)-u(a)].$$

Remark 8. *Let $w : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$ and such that*

$$\frac{1}{b-t}\int_t^b w(s)\,ds \geq \frac{1}{t-a}\int_a^t w(s)\,ds$$

for any $t \in (a,b)$ and $f \in \mathcal{M}^{\nearrow}[a,b]$. If either
(i) $f \in \mathcal{M}[a,b]$ *or* $\int_a^b w(s)\,ds = 0$,
or
(ii) $\int_a^b w(s)\,ds > 0$ *and* $f \in \mathcal{M}_{Sub}[a,b]$,
or
(iii) $\int_a^b w(s)\,ds < 0$ *and* $f \in \mathcal{M}_{Sup}[a,b]$
then

$$(4.9)\qquad \int_a^b f(t)\,w(t)\,dt \geq f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt.$$

5. Applications for Functions of Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$(5.1)\qquad E_\lambda := \varphi_\lambda(A)$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [17, p. 256]:

Theorem 5 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;

b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;

c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi\left(\lambda_k'\right)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{5.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 10. *With the assumptions of Theorem 5 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x, y \in H.$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

The following result holds:

Theorem 6. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. If $f \in \mathcal{C}[m, M] \cap \mathcal{U}_{Ch}[m, M]$, then*

$$\frac{f(m) + f(M)}{2} I - f(A) \ge \frac{f(M) - f(m)}{M - m}\left(\frac{M + m}{2} I - A\right) \tag{5.3}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A.

Making use of the inequality (3.9) for $v(\lambda) := \langle E_\lambda x, x\rangle$, with $x \in H$ we have

$$\begin{aligned}(5.4)\qquad &\frac{f(m)+f(M)}{2}\|x\|^2 - \int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x, x\rangle \\ &\geq \frac{f(M)-f(m)}{M-m}\left[\int_{m-0}^{M}\langle E_\lambda x, x\rangle\,d\lambda - \frac{\|x\|^2}{2}(M-m)\right],\end{aligned}$$

for any $x \in H$.

Integrating by parts we have

$$\begin{aligned}\int_{m-0}^{M}\langle E_\lambda x, x\rangle\,d\lambda &= \langle E_\lambda x, x\rangle\,\lambda\big|_{m-0}^{M} - \int_{m-0}^{M}\lambda d\langle E_\lambda x, x\rangle \\ &= M\|x\|^2 - \int_{m-0}^{M}\lambda d\langle E_\lambda x, x\rangle\end{aligned}$$

and by (5.4) we get

$$\begin{aligned}(5.5)\qquad &\frac{f(m)+f(M)}{2}\|x\|^2 - \int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x, x\rangle \\ &\geq \frac{f(M)-f(m)}{M-m}\left[M\|x\|^2 - \int_{m-0}^{M}\lambda d\langle E_\lambda x, x\rangle - \frac{\|x\|^2}{2}(M-m)\right] \\ &= \frac{f(M)-f(m)}{M-m}\left[\frac{M+m}{2}\|x\|^2 - \int_{m-0}^{M}\lambda d\langle E_\lambda x, x\rangle\right]\end{aligned}$$

for any $x \in H$.

Utilising the spectral representation of functions of selfadjoint operators (5.2) we have from (5.5)

$$\begin{aligned}&\frac{f(m)+f(M)}{2}\|x\|^2 - \langle f(A)x, x\rangle \\ &\geq \frac{f(M)-f(m)}{M-m}\left[\frac{M+m}{2}\|x\|^2 - \langle Ax, x\rangle\right]\end{aligned}$$

for any $x \in H$, which is equivalent with (5.3). $\square$

We also have:

Theorem 7. *Let A be a bounded selfadjoint operator on the Hilbert space H and let* $m = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ *and* $M = \max\{\lambda\,|\lambda \in Sp(A)\}$ $=: \max Sp(A)$. *If* $f \in \mathcal{C}[m,M] \cap \mathcal{U}_{Ch}\left[m, \frac{m+M}{2}\right] \cap \mathcal{U}_{Ch}\left[\frac{m+M}{2}, M\right]$ *and let* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$

be the spectral family of the bounded selfadjoint operator A. Then we have the inequality

$$
\begin{aligned}
(5.6)\qquad & \langle f(A)x,x\rangle - f\left(\frac{m+M}{2}\right)\|x\|^2 \\
& \le \left[f(M) - f\left(\frac{m+M}{2}\right)\right]\left[\frac{2}{M-m}\int_{\frac{m+M}{2}}^{M}\langle (I-E_\lambda)x,x\rangle d\lambda\right] \\
& + \left[f(m) - f\left(\frac{m+M}{2}\right)\right]\left[\frac{2}{M-m}\int_{m-0}^{\frac{m+M}{2}}\langle E_\lambda x,x\rangle d\lambda\right],
\end{aligned}
$$

for any $x \in H$.

The proof follows by (4.7) by a similar argument to the one from the proof of Theorem 6 and the details are omitted.

Remark 9. *If we take in (5.6) $f:\mathbb{R}\to\mathbb{R}$, $f(t) := \left|t - \frac{m+M}{2}\right|^p, p \ge 1$, then we have from (5.6) the following inequality*

$$
\begin{aligned}
(5.7)\qquad & \left\langle \left|A - \frac{m+M}{2}I\right|^p x,x\right\rangle \\
& \le \left(\frac{M-m}{2}\right)^{p-1}\left[\int_{\frac{m+M}{2}}^{M}\langle (I-E_\lambda)x,x\rangle d\lambda + \int_{m-0}^{\frac{m+M}{2}}\langle E_\lambda x,x\rangle d\lambda\right]
\end{aligned}
$$

for any $x \in H$.

The interested reader may state other similar inequalities by choosing various examples of convex functions of interest. The details are omitted.

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