

TWO POINT GAUSS–LEGENDRE QUADRATURE RULE FOR RIEMANN–STIELTJES INTEGRALS

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ABSTRACT. In order to approximate the Riemann–Stieltjes integral $\int_a^b f(t) dg(t)$ by 2–point Gaussian quadrature rule, we introduce the quadrature rule

$$\int_{-1}^1 f(t) dg(t) \approx Af\left(-\frac{\sqrt{3}}{3}\right) + Bf\left(\frac{\sqrt{3}}{3}\right),$$

for suitable choice of A and B . Error estimates for this approximation under various assumptions for the functions involved are provided as well.

1. INTRODUCTION

In numerical analysis, inequalities play a main role in error estimations. A few years ago, by using modern theory of inequalities and Peano kernel approach a number of authors have considered an error analysis of some quadrature rules of Newton–Cotes type. In particular, the Mid–point, Trapezoid and Simpson’s rules have been investigated more recently with the view of obtaining bounds for the quadrature rules in terms of at most first derivative. A comprehensive list of preprints related to this subject may be found at <http://ajmaa.org/RGMIA>

The Newton–Cotes formulas use values of function at equally spaced points. The same practice when the formulas are combined to form the composite rules, but this restriction can significantly decrease the accuracy of the approximation. In fact, these methods are inappropriate when integrating a function on an interval that contains both regions with large functional variation and regions with small functional variation. If the approximation error is to be evenly distributed, a smaller step size is needed for the large-variation regions than for those with less variation.

Among others Gaussian quadrature rules gives the highest possible degree of precision; so that it is recommended to be ‘almost’ the method of choice.

In order to investigate 2–point Gauss–Legendre quadrature rule, Ujević [21] obtained bounds for absolutely continuous functions with derivatives belong to $L_2(a, b)$, as follows:

Theorem 1. *Let $f : [-1, 1] \rightarrow \mathbb{R}$ be an absolutely continuous whose derivative $f' \in L_2(-1, 1)$. Then*

$$(1.1) \quad \left| f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) - \int_{-1}^1 f(t) dt \right| \leq \sqrt{\frac{4-2\sqrt{3}}{3}} \cdot \sigma^{1/2}(f'),$$

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where,

$$\sigma(g) = (b - a) \mathcal{T}(g, g);$$

and

$$\mathcal{T}(g, g) = \frac{1}{b - a} \|g\|_2^2 - \frac{1}{(b - a)^2} \left(\int_a^b g(t) dt \right)^2.$$

Inequality (1.1) is sharp in the sense that the constant $\sqrt{\frac{4-2\sqrt{3}}{3}}$ cannot be replaced by a smaller one.

Some Gaussian and Gaussian-like quadrature rules are considered in [22].

The Riemann–Stieltjes integral $\int_a^b f(t) dg(t)$ is an important concept in Mathematics with multiple applications in several subfields including Probability Theory & Statistics, Complex Analysis, Functional Analysis, Operator Theory and others.

In Numerical Integration, the number of proposed quadrature rules to approximate this type of integrals is very small by comparison with the huge number of methods available to approximate the classical Riemann integral $\int_a^b f(t) dt$.

In recent years, the approximation problem of the Riemann–Stieltjes integral $\int_a^b f dg$ has been studied with the tools of modern inequalities; therefore several error approximation for proposed quadrature rules had been established. The most famous and interesting approximations have been done using Ostrowski and generalized trapezoid type inequalities.

In 2000, Dragomir [13] (see also [12]) has introduced the following significant quadrature rule:

$$\int_a^b f(t) du(t) \cong f(x) [u(b) - u(a)],$$

and several error bounds under various assumptions to the function involved were obtained, the reader may refer to [4], [7]–[10], [12]–[13] and the references therein.

From a different point of view, the authors of [14] considered the problem of approximating the Riemann–Stieltjes integral $\int_a^b f(t) du(t)$ via the generalized trapezoid rule $[u(x) - u(a)] f(a) + [u(b) - u(x)] f(b)$, i.e.,

$$\int_a^b f(t) du(t) \cong [u(x) - u(a)] f(a) + [u(b) - u(x)] f(b), \forall x \in [a, b].$$

For other related results see [7, 8] and [15]–[19].

In 2008, Mercer [20] proved a new version of Hermite–Hadamard inequality for Riemann–Stieltjes integral, and introduced the following Trapezoid type rule:

$$(1.2) \quad \int_a^b f dg \approx [G - g(a)] f(a) + [g(b) - G] f(b)$$

where, $G := \frac{1}{b-a} \int_a^b g(t) dt$.

More recently, Alomari [2] and [3] introduced the quadrature rule:

$$\int_a^b f(t) du(t) \cong \left[u\left(\frac{a+b}{2}\right) - u(a) \right] f(x) + \left[u(b) - u\left(\frac{a+b}{2}\right) \right] f(a+b-x),$$

$\forall x \in [a, \frac{a+b}{2}]$, and therefore several error bounds of this approximation rule under various assumptions for the functions involved are proved. For new result regarding the above quadrature rules the reader may refer to [1], [4, 5, 6].

2. TWO-POINT GAUSSIAN QUADRATURE RULE

To establish a two point Gauss-Legendre quadrature rule for the Riemann-Stieltjes integral $\int_a^b f(t) dg(t)$, let us seek numbers A and B such that

$$(2.1) \quad \int_{-1}^1 f(t) dg(t) \approx Af\left(-\frac{\sqrt{3}}{3}\right) + Bf\left(\frac{\sqrt{3}}{3}\right).$$

To find the scalars A and B , let $f(t) = 1$ and then $f(t) = t$ in (2.1); respectively. By simple calculations we get

$$(2.2) \quad A = \frac{3}{2\sqrt{3}} \left[\int_{-1}^1 g(t) dt - \left(\frac{3-\sqrt{3}}{3}\right)g(1) - \left(\frac{\sqrt{3}+3}{3}\right)g(-1) \right]$$

and

$$(2.3) \quad B = \frac{3}{2\sqrt{3}} \left[\left(\frac{3+\sqrt{3}}{3}\right)g(1) + \left(\frac{3-\sqrt{3}}{3}\right)g(-1) - \int_{-1}^1 g(t) dt \right],$$

and therefore by (2.1), we may write

$$(2.4) \quad \begin{aligned} & \int_{-1}^1 f(t) dg(t) \\ & \approx \frac{3}{2\sqrt{3}} \left[\int_{-1}^1 g(t) dt - \left(\frac{3-\sqrt{3}}{3}\right)g(1) - \left(\frac{\sqrt{3}+3}{3}\right)g(-1) \right] f\left(-\frac{\sqrt{3}}{3}\right) \\ & \quad + \frac{3}{2\sqrt{3}} \left[\left(\frac{3+\sqrt{3}}{3}\right)g(1) + \left(\frac{3-\sqrt{3}}{3}\right)g(-1) - \int_{-1}^1 g(t) dt \right] f\left(\frac{\sqrt{3}}{3}\right). \end{aligned}$$

As special cases, we have

- (1) If g is an odd function, so that we have $g(-x) = -g(x)$, for all $x \in [-1, 1]$ and $\int_{-1}^1 g(t) dt = 0$. Therefore, (2.4), becomes

$$(2.5) \quad \int_{-1}^1 f(t) dg(t) \approx g(1) \left[f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \right].$$

For instance, if $g(t) := t$ for all $t \in [-1, 1]$, then

$$(2.6) \quad \int_{-1}^1 f(t) dt \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right),$$

which reduces to the classical Gauss-Legendre quadrature formula for the Riemann integral $\int_{-1}^1 f(t) dt$.

- (2) If g is an even function, so that we have $g(-x) = g(x)$, for all $x \in [-1, 1]$ and $\int_{-1}^1 g(t) dt = 2 \int_0^1 g(t) dt$. Therefore, (2.4), becomes

$$(2.7) \quad \int_{-1}^1 f(t) dg(t) \approx \frac{3}{\sqrt{3}} \left[g(1) - \int_0^1 g(t) dt \right] \left[f\left(\frac{\sqrt{3}}{3}\right) - f\left(-\frac{\sqrt{3}}{3}\right) \right].$$

Now, we are ready to state our first result.

Theorem 2. Let $f, g : [-1, 1] \rightarrow \mathbb{R}$ be such that f is of r - H_f -Holder type on $[-1, 1]$, where $r > 0$ and $H_f > 0$ are given, and g is of bounded variation on $[-1, 1]$. Then,

$$(2.8) \quad \left| \int_{-1}^1 f(t) dg(t) - Af\left(-\frac{\sqrt{3}}{3}\right) - Bf\left(\frac{\sqrt{3}}{3}\right) \right| \leq H_f \left(\frac{3+\sqrt{3}}{3}\right)^r \cdot \bigvee_{-1}^1(g),$$

where A and B are given in (2.2) and (2.3); respectively.

Proof. From (2.2) and (2.3) it is easy to observe that $A + B = g(1) - g(-1)$. So that,

$$\begin{aligned} \mathcal{ER}(f, g) &:= \int_{-1}^1 f(t) dg(t) - Af\left(-\frac{\sqrt{3}}{3}\right) - Bf\left(\frac{\sqrt{3}}{3}\right) \\ &= \int_{-1}^1 f(t) dg(t) - \frac{1}{g(1) - g(-1)} \int_{-1}^1 \left[Af\left(-\frac{\sqrt{3}}{3}\right) + Bf\left(\frac{\sqrt{3}}{3}\right) \right] dg(t) \\ &= \int_{-1}^1 \left[f(t) - \frac{Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)}{g(1) - g(-1)} \right] dg(t). \end{aligned}$$

It is well-known that for a continuous function $p : [a, b] \rightarrow \mathbb{R}$ and a function $\nu : [a, b] \rightarrow \mathbb{R}$ of bounded variation, one has the inequality

$$(2.9) \quad \left| \int_a^b p(t) d\nu(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(\nu).$$

Using (2.9), we have

$$\begin{aligned} |\mathcal{ER}(f, g)| &= \left| \int_{-1}^1 \left[f(t) - \frac{Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)}{g(1) - g(-1)} \right] dg(t) \right| \\ &\leq \sup \left| f(t) - \frac{Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)}{g(1) - g(-1)} \right| \cdot \bigvee_{-1}^1(g) \\ &= \frac{1}{g(1) - g(-1)} \sup \left| [g(1) - g(-1)] f(t) - [Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)] \right| \cdot \bigvee_{-1}^1(g) \\ &= \frac{1}{g(1) - g(-1)} \sup \left| [A + B] f(t) - [Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)] \right| \cdot \bigvee_{-1}^1(g) \\ &= \frac{1}{g(1) - g(-1)} \sup \left| Af(t) + Bf(t) - Af(-\sqrt{3}/3) - Bf(\sqrt{3}/3) \right| \cdot \bigvee_{-1}^1(g) \\ &\leq \frac{1}{g(1) - g(-1)} \left[A \sup \left| f(t) - f(-\sqrt{3}/3) \right| + B \sup \left| f(t) - f(\sqrt{3}/3) \right| \right] \cdot \bigvee_{-1}^1(g) \\ &\leq \frac{H_f}{g(1) - g(-1)} \left[A \sup_{t \in [-1, 1]} \left| t + \frac{1}{\sqrt{3}} \right|^r + B \sup_{t \in [-1, 1]} \left| t - \frac{1}{\sqrt{3}} \right|^r \right] \cdot \bigvee_{-1}^1(g) \\ &= \frac{H_f}{g(1) - g(-1)} (A + B) \left(\frac{3+\sqrt{3}}{3}\right)^r \cdot \bigvee_{-1}^1(g) \end{aligned}$$

$$= H_f \left(\frac{3 + \sqrt{3}}{3} \right)^r \cdot \bigvee_{-1}^1(g),$$

which gives the required result. ■

Corollary 1. *Let $f : [-1, 1] \rightarrow \mathbb{R}$ be such that f is of r - H_f -Holder type on $[-1, 1]$, where $r > 0$ and $H_f > 0$ are given. Then,*

$$(2.10) \quad \left| \int_{-1}^1 f(t) dt - f\left(-\frac{\sqrt{3}}{3}\right) - f\left(\frac{\sqrt{3}}{3}\right) \right| \leq 2H_f \left(\frac{3 + \sqrt{3}}{3} \right)^r.$$

Theorem 3. *Let $f, g : [-1, 1] \rightarrow \mathbb{R}$ be such that f is of r - H_f -Holder type on $[-1, 1]$, where $r > 0$ and $H_f > 0$ are given, and g is L_g -Lipschitzian on $[-1, 1]$. Then,*

$$(2.11) \quad \left| \int_{-1}^1 f(t) dg(t) - Af\left(-\frac{\sqrt{3}}{3}\right) - Bf\left(\frac{\sqrt{3}}{3}\right) \right| \\ \leq \frac{L_g H_f}{r+1} \left[\left(\frac{3 - \sqrt{3}}{3} \right)^{r+1} + \left(\frac{3 + \sqrt{3}}{3} \right)^{r+1} \right],$$

where A and B are given in (2.2) and (2.3); respectively.

Proof. It is well-known that for a Riemann integrable function $p : [a, b] \rightarrow \mathbb{R}$ and L -Lipschitzian function $\nu : [a, b] \rightarrow \mathbb{R}$, one has the inequality

$$(2.12) \quad \left| \int_a^b p(t) d\nu(t) \right| \leq L \int_a^b |p(t)| dt.$$

Using (2.12), we have

$$\begin{aligned} |\mathcal{ER}(f, g)| &= \left| \int_{-1}^1 \left[f(t) - \frac{Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)}{g(1) - g(-1)} \right] dg(t) \right| \\ &\leq L_g \int_{-1}^1 \left| f(t) - \frac{Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)}{g(1) - g(-1)} \right| dt \\ &= \frac{L_g}{g(1) - g(-1)} \int_{-1}^1 \left| [g(1) - g(-1)] f(t) - [Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)] \right| dt \\ &= \frac{L_g}{g(1) - g(-1)} \int_{-1}^1 \left| [A + B] f(t) - [Af(-\sqrt{3}/3) + Bf(\sqrt{3}/3)] \right| dt \\ &= \frac{L_g}{g(1) - g(-1)} \int_{-1}^1 \left[A \left| f(t) - f(-\sqrt{3}/3) \right| + B \left| f(t) - f(\sqrt{3}/3) \right| \right] dt \\ &= \frac{L_g H_f}{g(1) - g(-1)} \int_{-1}^1 \left[A \left| t + \frac{\sqrt{3}}{3} \right|^r + B \left| t - \frac{\sqrt{3}}{3} \right|^r \right] dt \\ &= \frac{L_g H_f}{r+1} \left[\left(\frac{3 - \sqrt{3}}{3} \right)^{r+1} + \left(\frac{3 + \sqrt{3}}{3} \right)^{r+1} \right], \end{aligned}$$

which gives the required result. ■

Corollary 2. *Let g be as in Theorem 3. If f is L_f -Lipschitzian on $[-1, 1]$. Then,*

$$(2.13) \quad \left| \int_{-1}^1 f(t) dg(t) - Af\left(-\frac{\sqrt{3}}{3}\right) - Bf\left(\frac{\sqrt{3}}{3}\right) \right| \leq \frac{4}{3} L_f L_g,$$

where A and B are given in (2.2) and (2.3); respectively.

Corollary 3. *Let f be as in Theorem 3. Take $g(t) = t$ on $[-1, 1]$. Then,*

$$(2.14) \quad \left| \int_{-1}^1 f(t) dt - f\left(-\frac{\sqrt{3}}{3}\right) - f\left(\frac{\sqrt{3}}{3}\right) \right| \leq \frac{H_f}{r+1} \left[\left(\frac{3-\sqrt{3}}{3}\right)^{r+1} + \left(\frac{3+\sqrt{3}}{3}\right)^{r+1} \right].$$

Remark 1. a) *A result for a monotonic non-decreasing integrator may be stated by using the fact that for a monotonic non-decreasing function $\nu : [a, b] \rightarrow \mathbb{R}$ and continuous function $p : [a, b] \rightarrow \mathbb{R}$, one has the inequality*

$$\left| \int_a^b p(t) d\nu(t) \right| \leq \int_a^b |p(t)| d\nu(t).$$

b) *Another result(s) in terms of L_p norms may be stated by applying the well-known Hölder integral inequality, by noting that*

$$\left| \int_c^d h(s) du(s) \right| \leq \sqrt[q]{u(d) - u(c)} \times \sqrt[p]{\int_c^d |h(s)|^p du(s)}.$$

where, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

A result for absolutely continuous integrand whose derivative belongs to $L_2(-1, 1)$ is given as follows:

Theorem 4. *Let $f, g : [-1, 1] \rightarrow \mathbb{R}$ be two absolutely continuous functions whose derivatives $f', g' \in L_2(-1, 1)$ and $\int_{-1}^1 f(t) g'(t) dt$ exists. Then*

$$(2.15) \quad \left| \int_{-1}^1 f(t) g'(t) dt - Af\left(-\frac{\sqrt{3}}{3}\right) - Bf\left(\frac{\sqrt{3}}{3}\right) \right| \leq \sigma^{1/2}(f) \cdot \sigma^{1/2}(g'),$$

where A and B are given in (2.2) and (2.3); respectively.

$$\sigma(h) = 2\mathcal{T}(h, h);$$

and

$$\mathcal{T}(h, h) = \frac{1}{2} \|h\|_2^2 - \frac{1}{4} \left(\int_{-1}^1 h(t) dt \right)^2.$$

Proof. Since $A + B = g(1) - g(-1)$. For simplicity, we set

$$F = \frac{Af(-1/\sqrt{3}) + Bf(1/\sqrt{3})}{A + B}.$$

Define the mapping $K(t) = f(t) - F$. It is easy to observe that

$$\begin{aligned}
 \mathcal{ER}(f, g) &= \int_{-1}^1 K(t) g'(t) dt \\
 &= \int_{-1}^1 \left[K(t) - \frac{1}{2} \int_{-1}^1 K(s) ds \right] \left[g'(t) - \frac{1}{2} \int_{-1}^1 g'(s) ds \right] dt \\
 (2.16) \quad &= 2\mathcal{T}(K, g').
 \end{aligned}$$

So that, we may write

$$\begin{aligned}
 \mathcal{T}^2(K, g') &= \frac{1}{4} \left\{ \int_{-1}^1 \left[K(t) - \frac{1}{2} \int_{-1}^1 K(s) ds \right] \left[g'(t) - \frac{1}{2} \int_{-1}^1 g'(s) ds \right] dt \right\}^2 \\
 (2.17) \quad &\leq \frac{1}{4} \int_{-1}^1 \left[K(t) - \frac{1}{2} \int_{-1}^1 K(s) ds \right]^2 dt \cdot \int_{-1}^1 \left[g'(t) - \frac{1}{2} \int_{-1}^1 g'(s) ds \right]^2 dt
 \end{aligned}$$

But since

$$\begin{aligned}
 \int_{-1}^1 \left[K(t) - \frac{1}{2} \int_{-1}^1 K(s) ds \right]^2 dt &= \int_{-1}^1 \left[(f(t) - F) - \frac{1}{2} \int_{-1}^1 (f(s) - F) ds \right]^2 dt \\
 &= \int_{-1}^1 \left[\left(f(t) - \frac{Af(-1/\sqrt{3}) + Bf(1/\sqrt{3})}{A+B} \right) \right. \\
 &\quad \left. - \frac{1}{2} \int_{-1}^1 \left(f(s) - \frac{Af(-1/\sqrt{3}) + Bf(1/\sqrt{3})}{A+B} \right) ds \right]^2 dt \\
 &= \int_{-1}^1 \left[f(t) - \frac{1}{2} \int_{-1}^1 f(s) ds \right]^2 dt \\
 &= \int_{-1}^1 f^2(t) dt - \frac{1}{2} \left(\int_{-1}^1 f(t) dt \right)^2 \\
 &= 2\mathcal{T}(f, f),
 \end{aligned}$$

which gives by (2.17) that

$$|\mathcal{T}(K, g')| \leq \mathcal{T}^{1/2}(f, f) \mathcal{T}^{1/2}(g', g').$$

Combining the above inequality with (2.16) we get the required result (2.15). ■

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