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ON INTEGRAL FORMS OF SEVERAL INEQUALITIES

LOREDANA CIURDARIU

Abstract. In this paper we give some integral forms of some refinements and counterparts of Radon's inequality using recent generalizations.

1. Introduction

We will recall the inequality of J. Radon which was published in [6].

For every real numbers $p > 0,\ x_k \geq 0,\ a_k > 0$ for $1 \leq k \leq n$, we have the following inequality:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n})^{p+1}}{(\sum_{k=1}^{n} a_k)^p}, \quad p > 0.$$

In [7], the authors consider two n-tuples $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_1, ..., b_n)$ where $ab = (a_1b_1, a_2b_2, ..., a_nb_n)$ and $a^m = (a_1^m, a_2^m, ..., a_n^m)$, for any real number m. Then $a > 0$ and $b > 0$ if $a_i > 0$ and $b_i > 0$ for every $1 < i < n$. We consider the expression:

$$\Delta_n^{[p]}(a; b) := \sum_{i=1}^{n} \frac{a_i^p}{b_i^{p-1}} - \frac{(\sum_{i=1}^{n} a_i)^p}{(\sum_{i=1}^{n} b_i)^{p-1}}, \tag{1.1}$$

for real number $p > 1$ and for n-tuples $a \geq 0$ and $b \geq 0$.

Then the well-known Radon's inequality can be written as:

$$\Delta_n^{[p]}(a; b) \geq 0. \tag{1.2}$$

Theorem 1. *([7])For every $n \geq 2,\ p \geq 1,\ a_k \geq 0,\ b_k > 0,\ 1 \leq k \leq n$, the following inequality hold:*

$$0 \leq \Delta_n^{[p]}(a; b) \leq p \left(\Delta_n^{[p]}(a; b) - \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \Delta_n^{[p-1]}(a; b) \right) \tag{2.5}$$

and

$$0 \leq \Delta_n^{[p]}(a; b) \leq p(M - m)(M^{p-1} - m^{p-1}) \left(\sum_{i=1}^{n} b_i \right) \tag{2.6,}$$

where $m \leq \frac{a_i}{b_i} \leq M$, for $i = 1, ..., n$.

It is necessary to recall also Theorem 2.9 and Theorem 2.7 from [7].

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Theorem 2. *([7]) There is the inequality:*
(2.19)

$$0 \le \Delta_n^{[p]}(a;b) \le \frac{[(M+m)\sum_{i=1}^n b_i - \sum_{i=1}^n a_i]^p}{(\sum_{i=1}^n b_i)^{p-1}} - \frac{(M+m)^p}{2^{p-1}}\left(\sum_{i=1}^n b_i\right) + \left(\sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}}\right),$$

where $m \le \frac{a_i}{b_i} \le M,\ a_i \ge 0,\ b_i > 0,\ 1 \le i \le n,\ p \ge 1,\ n \ge 2.$

Theorem 3. *([7]) For* $n \ge 2,\ p \ge 1,$ *we have the following inequalities:*

$$\Delta_n^{[p]}(a;b) \ge \max_{1\le i<j\le n}\left[\frac{a_i^p}{b_i^{p-1}} + \frac{a_j^p}{b_j^{p-1}} - \frac{(a_i+a_j)^p}{(b_i+b_j)^{p-1}}\right], \tag{2.16}$$

and

$$0 \le \Delta_n^{[p]}(a;b) \le \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right]\left(\sum_{i=1}^n b_i\right),$$

where $m \le \frac{a_i}{b_i} \le M,\ a_i \ge,\ b_i > 0,\ 1 \le i \le n.$

We need the following result from [7], which will be used also below, in the next section.

Theorem 4. *If* $a = (a_1, a_2, ..., a_n)$ *and* $b = (b_1, b_2, ..., b_n)$ *are n-tuples then we have the inequality:*

$$\frac{p(p-1)m^{p-2}}{2\sum_{i=1}^n b_i}\sum_{1\le i<j\le n}\frac{(a_ib_j - a_jb_i)^2}{b_ib_j} \le \tag{2.13}$$

$$\le \Delta_n^{[p]}(a;b) \le \frac{p(p-1)M^{p-2}}{2\sum_{i=1}^n b_i}\sum_{1\le i<j\le n}\frac{(a_ib_j - a_jb_i)^2}{b_ib_j},$$

where $m \le \frac{a_i}{b_i} \le M,\ p > 1,\ a_i \ge 0,\ b_i > 0,$ *for* $i = 1, ..., n.$

2. Integral forms of several inequalities

Using the same techniques as in [1] we find the following integral form of the inequality (2.5) and (2.6) from Theorem 2.3, see [7].

Theorem 5. *For every* $n \ge 2,\ p \ge 1,\ f(x) \ge 0,\ g(x) > 0$ *and if* $f, g : [a,b] \to \mathbb{R}_+$ *are two continuous functions on* $[a,b]$ *with* $m = \inf_{[a,b]} \frac{f(x)}{g(x)},\ M = \sup_{[a,b]} \frac{f(x)}{g(x)}$ *then we have:*

$$0 \le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \le \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})\int_a^b g(x)dx.$$

If $f, g : [a,b] \to \mathbb{R}_+$ *are two integrable functions on* $[a,b]$ *then*

$$0 \le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \le$$

$$\le p(\int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} - \frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}(\int_a^b \frac{(f(x))^{p-1}}{(g(x))^{p-2}}dx - \frac{(\int_a^b f(x)dx)^{p-1}}{(\int_a^b g(x)dx)^{p-2}})).$$

Proof. Let $n \in \mathbb{N}$ and $x_k = k + \frac{b-a}{n},\ k \in \{0, 1, ..., n\}$. Using Theorem 2.3, see [7] we have

$$0 \leq \sum_{k=1}^{n} \frac{(f(x_k))^p}{(g(x_k))^{p-1}} - \frac{(\sum_{k=1}^{n} f(x_k))^p}{(\sum_{k=1}^{n} g(x_k))^{p-1}} \leq$$

$$\leq p(\sum_{k=1}^{n} \frac{(f(x_k))^p}{(g(x_k))^{p-1}} - \frac{(\sum_{k=1}^{n} f(x_k))^p}{(\sum_{k=1}^{n} g(x_k))^{p-1}} - \frac{\sum_{k=1}^{n} f(x_k)}{\sum_{k=1}^{n} g(x_k)}(\sum_{k=1}^{n} \frac{(f(x_k))^{p-1}}{(g(x_k))^{p-2}} - \frac{(\sum_{k=1}^{n} f(x_k))^{p-1}}{(\sum_{k=1}^{n} g(x_k))^{p-2}})),$$

and

$$0 \leq \sum_{k=1}^{n} \frac{(f(x_k))^p}{(g(x_k))^{p-1}} - \frac{(\sum_{k=1}^{n} f(x_k))^p}{(\sum_{k=1}^{n} g(x_k))^{p-1}} \leq$$

$$\leq \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) \left(\sum_{k=1}^{n} g(x_k) \right),$$

where $m \leq \frac{f(x_k)}{g(x_k)} \leq M$, for $k = 1, ..., n$.

It results that

$$0 \leq \sigma \left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k \right) - \frac{(\sigma(f, \Delta_n, x_k))^p}{(\sigma(g, \Delta_n, x_k))^{p-1}} \leq$$

$$\leq p(\sigma(\frac{f^p}{g^{p-1}}, \Delta_n, x_k) - \frac{(\sigma(f, \Delta_n, x_k))^p}{(\sigma(g, \Delta_n, x_k))^{p-1}} -$$

$$-\frac{\sigma(f, \Delta_n, x_k)}{\sigma(g, \Delta_n, x_k)}(\sigma(\frac{f^{p-1}}{g^{p-2}}, \Delta_n, x_k) - \frac{(\sigma(f, \Delta_n, x_k))^{p-1}}{(\sigma(g, \Delta_n, x_k))^{p-2}}))$$

and

$$0 \leq \sigma \left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k \right) - \frac{(\sigma(f, \Delta_n, x_k))^p}{(\sigma(g, \Delta_n, x_k))^{p-1}} \leq$$

$$\leq \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})\sigma(g, \Delta_n, x_k).$$

We considered here $\sigma\left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k\right)$ is the corresponding Riemann sum of function $\frac{f^p}{g^{p-1}}$, $\Delta_n = (x_0, x_1, ..., x_n)$ division, and the intermediate x_k points. When n tends to infinity, in previous inequality the limits become:

$$0 \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \leq$$

$$\leq p(\int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} - \frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}(\int_a^b \frac{(f(x))^{p-1}}{(g(x))^{p-2}}dx - \frac{(\int_a^b f(x)dx)^{p-1}}{(\int_a^b g(x)dx)^{p-2}}))$$

and

$$0 \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \leq \frac{p}{4}(M-m)(M^{p-1} - m^{p-1}) \int_a^b g(x)dx.$$

∎

The next result is the integral form of the inequality (2.19) of Theorem 2.9, from [7].

Theorem 6. *If $p \geq 1$, f and g are two continuous functions $f, g : [a,b] \to \mathbb{R}_+$ on $[a,b]$, with $m = \inf_{[a,b]} \frac{f(x)}{g(x)}$, $M = \sup_{[a,b]} \frac{f(x)}{g(x)}$ then we have:*

$$0 \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \leq$$

$$\leq \frac{[(M+m)\int_a^b g(x)dx - \int_a^b f(x)dx]^p}{(\int_a^b f(x))^{p-1}} - \frac{(M+m)^p}{2^{p-1}} \int_a^b g(x)dx + \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}.$$

Proof. We will use the same techniques as in previous proof, choosing $x_k = k + \frac{b-a}{n}$, $k \in \{0, 1, ..., n\}$, using Theorem 2.9, Riemann sum of the corresponding functions, $\Delta_n = (x_0, x_1, ..., x_n)$ division, and the intermediate x_k points. Then when n tends to infinity, the limits obtained form the inequality from theorem.

∎

The following integral inequality results from Theorem 3.

Consequence 1. *If $p \geq 1$, and f and g are two continuous functions $f, g : [a,b] \to \mathbb{R}_+$ on $[a,b]$, with $g(x) > 0$, where $m = \inf_{[a,b]} \frac{f(x)}{g(x)}$, $M = \sup_{[a,b]} \frac{f(x)}{g(x)}$ then we have:*

$$0 \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \leq$$

$$\leq \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}} \right] \int_a^b g(x)dx.$$

We will give now the integral form of the inequality (2.13), Theorem 2.5, see [7].

Theorem 7. *Let $f, g : [a,b] \to \mathbb{R}_+$ two integrabile function on $[a,b]$ with $g(x) > 0$, $(\forall)$ $x \in [a,b]$, $p > 1$ and $mg(x) \leq f(x) \leq Mg(x)$, $(\forall)$ $x \in [a,b]$. Then we have the inequality:*

$$\frac{p(p-1)m^{p-2}}{\int_a^b g(x)dx} \int_a^b \int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy \leq$$

$$\leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \leq$$

$$\leq \frac{p(p-1)M^{p-2}}{\int_a^b g(x)dx} \int_a^b \int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy.$$

Proof. Using the definition of double integral and taking $x_k = k + \frac{b-a}{n}$, $y_j = j + \frac{b-a}{n}$, $k \in \{0, 1, ..., n\}$, $j \in \{0, 1, ..., m\}$ we have

$$\int_a^b \int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy =$$

$$= \lim_{n,m \to \infty} \sum_{i=1}^n \sum_{j=1}^m \frac{(f(x_i)g(y_j) - f(y_j)g(x_i))^2}{g(x_i)g(y_j)} (x_{i-1} - x_i)(y_{j-1} - y_j).$$

When $n = m$ tends to infinity

$$\int_a^b \int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy =$$
$$= 2 \lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{(f(x_i)g(y_j) - f(y_j)g(x_i))^2}{g(x_i)g(y_j)} (x_{i-1} - x_i)(y_{j-1} - y_j) =$$
$$= 2 \lim_{n\to\infty} \sum_{1\le i\le j\le n} \frac{(f(x_i)g(x_j) - f(x_j)g(x_i))^2}{g(x_i)g(x_j)} (x_{i-1} - x_i)(x_{j-1} - x_j) =$$
$$= \lim_{n\to\infty} \sum_{1\le i\le j\le n} \frac{(f(x_i)g(x_j) - f(x_j)g(x_i))^2}{g(x_i)g(x_j)} \frac{(b-a)^2}{n^2}$$

and using Theorem 2.5, see [7],

$$\frac{p(p-1)m^{p-1}}{\sum_{i=1}^n g(x_i)\frac{(b-a)}{n}} \sum_{1\le i\le j\le n} \frac{(f(x_i)g(x_j) - f(x_j)g(x_i))^2}{g(x_i)g(x_j)} \frac{(b-a)^2}{n^2} \le$$
$$\le \sum_{i=1}^n \frac{(f(x_i))^p}{(g(x_i))^{p-1}} \frac{b-a}{n} - \frac{(\sum_{i=1}^n f(x_i)\frac{b-a}{n})^p}{(\sum_{i=1}^n g(x_i)\frac{b-a}{n})^{p-1}} \le$$
$$\le \frac{p(p-1)M^{p-1}}{\sum_{i=1}^n g(x_i)\frac{(b-a)}{n}} \sum_{1\le i\le j\le n} \frac{(f(x_i)g(x_j) - f(x_j)g(x_i))^2}{g(x_i)g(x_j)} \frac{(b-a)^2}{n^2}$$

we obtain

$$\frac{p(p-1)m^{p-2}}{\int_a^b g(x)dx} \int_a^b \int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy \le$$
$$\le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \le$$
$$\le \frac{p(p-1)M^{p-2}}{\int_a^b g(x)dx} \int_a^b \int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy.$$

that is the inequality from theorem.

∎

If we compute the double integral from previous theorem we deduce the following inequality:

Consequence 2. *Let $f, g : [a,b] \to \mathbb{R}_+$ two integrabile function on $[a,b]$ with $g(x) > 0$, $(\forall)\ x \in [a,b]$, $p > 1$ and $mg(x) \le f(x) \le Mg(x)$, $(\forall)\ x \in [a,b]$. Then we have the inequality:*

$$p(p-1)m^{p-2}\left(\int_a^b \frac{f^2(x)}{g(x)}dx - \frac{(\int_a^b f(x)dx)^2}{\int_a^b g(x)dx}\right) \le$$
$$\le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{(\int_a^b f(x)dx)^p}{(\int_a^b g(x)dx)^{p-1}} \le$$
$$\le p(p-1)M^{p-2}\left(\int_a^b \frac{f^2(x)}{g(x)}dx - \frac{(\int_a^b f(x)dx)^2}{\int_a^b g(x)dx}\right).$$

Using from [5], the inequality,

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \leq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p},\ p \in (-1, 0)$$

which is the reverse inequality of (1), and the same techniques as in Theorem 4 we obtain below the integral form of previous inequality:

Remark 1. *If* $a, b \in \mathbb{R},\ a < b,\ p \in (-1, 0),\ f, g : [a, b] \to [0, \infty)$ *are integrable function on* $[a, b],\ g(x) \neq 0$ *for any* $x \in [a, b]$, *then*

$$\int_a^b \frac{(f(x))^{p+1}}{(g(x))^p} dx \leq \frac{(\int_a^b f(x)dx)^{p+1}}{(\int_a^b g(x)dx)^p}.$$

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Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara