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Multivariate Generalised Fractional Polya type integral inequalities

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Abstract

Here we present a set of multivariate generalised fractional Polya type integral inequalities on the ball and shell. We treat both the radial and non-radial cases in all possibilities. We give also estimates for the related averages.

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1 Introduction

We mention the following famous Polya's integral inequality, see [9], [10, p, 62], [11] and [12, p. 83].

Theorem 1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a)=f(b)=0$. Then the exists at least one point $\xi\in[a,b]$ such that*

$$\left|f'(\xi)\right| > \frac{4}{(b-a)^2}\int_a^b f(x)\,dx. \tag{1}$$

In [13], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

Theorem 2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a)=f(b)=0$ and $M=\sup_{x\in[a,b]}\left|f'(x)\right|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}M, \tag{2}$$

where $\frac{(b-a)^2}{4}$ *in (2) is the best constant.*

The above motivate the current paper.

In this article we present multivariate fractional Polya type integral inequalities in various cases, similar to (2).

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$a \le x \le b$. We define the subspace $C_{a+}^{\alpha}([a.b])$ of $C^m([a,b])$:

$$C_{a+}^{\alpha}([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{4}$$

For $f \in C_{a+}^{\alpha}([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^{\alpha} f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{5}$$

see [1], p. 24. Canavati first in [5], introduced the above over $[0,1]$.

Notice that $D_{a+}^{\alpha} f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [5] the same over $[0,1]$ that appeared first.

Theorem 3 *Let* $f \in C_{a+}^{\alpha}([a,b])$.

(i) If $\alpha \ge 1$, *then*

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + ... + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^{\alpha} f\right)(t)\, dt, \qquad \text{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, *we have*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^{\alpha} f\right)(t)\, dt, \qquad \text{all } x \in [a,b]. \tag{7}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{8}$$

$x \in [a,b]$, see also [2], [6], [7], [8], [15]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{10}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \text{all } x \in [a,b]. \tag{11}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{12}$$

We need from [3]

Definition 5 *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{13}$$

In [4] we proved the following fractional Polya type integral inequality without any boundary conditions.

Theorem 6 *Let $0 < \alpha < 1$, $f \in C([a,b])$. Assume $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ and $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. Set*

$$M_1(f) = \max \left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}. \tag{14}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq \tag{15}$$

$$\frac{\left(\left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]} + \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2} \right)^{\alpha+1} \leq M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2) 2^{\alpha}}. \tag{16}$$

Inequalities (15), (16) are sharp, namely they are attained by

$$f_{*}(x)=\left\{\begin{array}{ll}(x-a)^{\alpha}, & x\in\left[a,\frac{a+b}{2}\right]\\ (b-x)^{\alpha}, & x\in\left[\frac{a+b}{2},b\right]\end{array}\right\},\quad 0<\alpha<1. \tag{17}$$

Clearly here non zero constant functions f are excluded.

The last result also motivates this work.

Remark 7 *(see [4]) When $\alpha\geq 1$, thus $m=[\alpha]\geq 1$, and by assuming that $f^{(k)}(a)=f^{(k)}(b)=0$, $k=0,1,...,m-1$, we can prove the same statements (15), (16), (17) as in Theorem 6. If we set there $\alpha=1$ we derive exactly Theorem 2. So we have generalized Theorem 2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.*

We present Polya type integral inequalities on the ball and shell.

2 Main Results

We need

Remark 8 *We define the ball $B(0,R)=\{x\in\mathbb{R}^{N}:|x|<R\}\subseteq\mathbb{R}^{N}$, $N\geq 2$, $R>0$, and the sphere*

$$S^{N-1}:=\{x\in\mathbb{R}^{N}:|x|=1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_{N}=\int_{S^{N-1}}d\omega=\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

For $x\in\mathbb{R}^{N}-\{0\}$ we can write uniquely $x=r\omega$, where $r=|x|>0$ and $\omega=\frac{x}{r}\in S^{N-1}$, $|\omega|=1$. Note that $\int_{B(0,R)}dy=\frac{\omega_{N}R^{N}}{N}$ is the Lebesgue measure of the ball.

Following [14, pp. 149-150, exercise 6] and [16, pp. 87-88, Theorem 5.2.2] we can write $F:\overline{B(0,R)}\to\mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)}F(x)\,dx=\int_{S^{N-1}}\left(\int_{0}^{R}F(r\omega)\,r^{N-1}dr\right)d\omega; \tag{18}$$

we use this formula a lot.

Initially the function $f:\overline{B(0,R)}\to\mathbb{R}$ is radial; that is, there exists a function g such that $f(x)=g(r)$, where $r=|x|$, $r\in[0,R]$, $\forall\ x\in\overline{B(0,R)}$, $\alpha>0$, $m=[\alpha]$. Here we assume that $g\in C([0,R])$ with $g\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and

$g\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$. In case of $0<\alpha<1$ then the last boundary conditions are void.

By assumption here and Theorem 3 we have

$$g(s)=\frac{1}{\Gamma(\alpha)}\int_{0}^{s}(s-t)^{\alpha-1}\left(D_{0+}^{\alpha}g\right)(t)\,dt, \tag{19}$$

all $s\in\left[0,\frac{R}{2}\right]$,

also it holds, by assumption and Theorem 4, that

$$g(s)=\frac{1}{\Gamma(\alpha)}\int_{s}^{R}(t-s)^{\alpha-1}\left(D_{R-}^{\alpha}g\right)(t)\,dt, \tag{20}$$

all $s\in\left[\frac{R}{2},R\right]$.

By (19) we get

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_{0}^{s}(s-t)^{\alpha-1}\left|\left(D_{0+}^{\alpha}g\right)(t)\right|dt$$

$$\leq\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}\frac{1}{\Gamma(\alpha)}\int_{0}^{s}(s-t)^{\alpha-1}dt=\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)}s^{\alpha}, \tag{21}$$

for any $s\in\left[0,\frac{R}{2}\right]$.

That is

$$|g(s)|\leq\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)}s^{\alpha}, \tag{22}$$

for any $s\in\left[0,\frac{R}{2}\right]$.

Similarly we obtain

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_{s}^{R}(t-s)^{\alpha-1}\left|\left(D_{R-}^{\alpha}g\right)(t)\right|dt$$

$$\leq\frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha)}\int_{s}^{R}(t-s)^{\alpha-1}dt=\frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha+1)}(R-s)^{\alpha}, \tag{23}$$

for any $s\in\left[\frac{R}{2},R\right]$.

I.e. it holds

$$|g(s)|\leq\frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha+1)}(R-s)^{\alpha}, \tag{24}$$

for any $s\in\left[\frac{R}{2},R\right]$.

Next we observe that

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\overset{(18)}{=}$$

$$\int_{S^{N-1}}\left(\int_0^R |g(s)|\, s^{N-1}ds\right)d\omega=\left(\int_0^R |g(s)|\, s^{N-1}ds\right)\int_{S^{N-1}}d\omega=$$

$$\left(\int_0^R |g(s)|\, s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}= \tag{25}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_0^{\frac{R}{2}}|g(s)|\, s^{N-1}ds+\int_{\frac{R}{2}}^R |g(s)|\, s^{N-1}ds\right\}\overset{(by\ (22)\ and\ (24))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}\int_0^{\frac{R}{2}}s^{\alpha+N-1}ds+\right.$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\int_{\frac{R}{2}}^R (R-s)^{\alpha}\left(\left(s-\frac{R}{2}\right)+\frac{R}{2}\right)^{N-1}ds\right\}= \tag{26}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)}\left(\frac{R}{2}\right)^{\alpha+N}+\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\cdot\right.$$

$$\left.\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\int_{\frac{R}{2}}^R (R-s)^{(\alpha+1)-1}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right]\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)}\left(\frac{R}{2}\right)^{\alpha+N}+\right. \tag{27}$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)}\left(\frac{R}{2}\right)^{\alpha+N-k}\right]\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right.$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}. \tag{28}$$

We have proved that

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right. \tag{29}$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Consider now

$$g_*(s) = \begin{cases} s^\alpha, & s \in \left[0, \frac{R}{2}\right], \\ (R-s)^\alpha, & s \in \left[\frac{R}{2}, R\right], \ \alpha > 0. \end{cases} \tag{30}$$

We have as in [4] that

$$D_{0+}^\alpha s^\alpha = \Gamma(\alpha+1), \quad \textit{all } s \in \left[0, \frac{R}{2}\right], \tag{31}$$

and

$$\left\| D_{0+}^\alpha s^\alpha \right\|_{\infty, \left[0, \frac{R}{2}\right]} = \Gamma(\alpha+1).$$

Similarly as in [] we get

$$D_{R-}^\alpha (R-s)^\alpha = \Gamma(\alpha+1), \quad \textit{all } s \in \left[\frac{R}{2}, R\right], \tag{32}$$

and

$$\left\| D_{R-}^\alpha (R-s)^\alpha \right\|_{\infty, \left[\frac{R}{2}, R\right]} = \Gamma(\alpha+1). \tag{33}$$

That is

$$\left\| D_{0+}^\alpha g_* \right\|_{\infty, \left[0, \frac{R}{2}\right]} = \left\| D_{R-}^\alpha g_* \right\|_{\infty, \left[\frac{R}{2}, R\right]} = \Gamma(\alpha+1). \tag{34}$$

Consequently we find that

$$R.H.S.\ (29) = \frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{1}{(\alpha+N)} + \right.$$

$$\left. (N-1)!\Gamma(\alpha+1) \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)} \right] \right\}. \tag{35}$$

Let $f_* : \overline{B(0,R)} \to \mathbb{R}$ *be radial such that* $f_*(x) = g_*(s)$, $s = |x|$, $s \in [0, R]$, $\forall$ $x \in \overline{B(0,R)}$.

Then we have

$$L.H.S.\ (29) = \int_{B(0,R)} f_*(y)\, dy \overset{(18)}{=}$$

$$\left(\int_0^R g_*(s)\, s^{N-1} ds \right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} s^{\alpha+N-1} ds + \int_{\frac{R}{2}}^R (R-s)^\alpha s^{N-1} ds \right\} = \tag{36}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \left(\frac{R}{2}\right)^{\alpha+N} \frac{1}{(\alpha+N)} + \int_{\frac{R}{2}}^R (R-s)^\alpha \left(\left(s - \frac{R}{2}\right) + \frac{R}{2} \right)^{N-1} ds \right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)} + \right.$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\int_{\frac{R}{2}}^{R}(R-s)^{(\alpha+1)-1}\left(s-\frac{R}{2}\right)^{N-k-1}ds\Bigg\}= \tag{37}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)}+\right.$$

$$\left.\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)}\left(\frac{R}{2}\right)^{\alpha+N-k}\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N)}+\right.$$

$$\left.(N-1)!\Gamma(\alpha+1)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}\overset{(35)}{=}\text{R.H.S. (29)}, \tag{38}$$

proving (29) sharp, infact it is attained.

We have proved the following main result.

Theorem 9 *Let $f:\overline{B(0,R)}\to\mathbb{R}$ be radial; that is, there exists a function g such that $f(x)=g(s)$, $s=|x|$, $s\in[0,R]$, $\forall\ x\in\overline{B(0,R)}$, $\alpha>0$. Assume that $g\in C([0,R])$, with $g\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $g\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$, $m=[\alpha]$. When $0<\alpha<1$ the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right. \tag{39}$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Inequalities (39) are sharp, namely they are attained by a radial function f_* such that $f_*(x)=g_*(s)$, all $s\in[0,R]$, where

$$g_*(s)=\begin{cases}s^{\alpha}, & s\in\left[0,\frac{R}{2}\right],\\ (R-s)^{\alpha}, & s\in\left[\frac{R}{2},R\right].\end{cases} \tag{40}$$

We continue with

Remark 10 *(Continuation of Remark 8) Here we assume that $\alpha \geq 1$. By (19) we get*

$$|g(s)| \leq \frac{s^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}, \tag{41}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Also, by (20), we obtain

$$|g(s)| \leq \frac{(R-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}, \tag{42}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence as in (25) we get

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left(\int_0^R |g(s)|\, s^{N-1} ds \right) = \tag{43}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^{R} |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (41), (42))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left(\int_0^{\frac{R}{2}} s^{N+\alpha-2} ds \right) \left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)} + \right. \tag{44}$$

$$\left. \left(\int_{\frac{R}{2}}^{R} (R-s)^{\alpha-1} s^{N-1} ds \right) \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \right\} =$$

(acting the same as before, see (26)-(28))

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\} \overset{(13)}{=} \tag{45}$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{46}$$

We have proved

Theorem 11 *Here all terms and assumptions as in Theorem 9, however with $\alpha \geq 1$. Then*

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\,\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{47}$$

We continue with

Remark 12 *(Also a continuation of Remark 8) Let here $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. By (19) we have*

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left| \left(D_{0+}^{\alpha} g \right)(t) \right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_0^s (s-t)^{p(\alpha-1)}\, dt \right)^{\frac{1}{p}} \left(\int_0^s \left| \left(D_{0+}^{\alpha} g \right)(t) \right|^q dt \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{s^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{0+}^{\alpha} g \right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}, \tag{48}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Similarly by (20) we obtain

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left| \left(D_{R-}^{\alpha} g \right)(t) \right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_s^R (t-s)^{p(\alpha-1)}\, dt \right)^{\frac{1}{p}} \left(\int_s^R \left| \left(D_{R-}^{\alpha} g \right)(t) \right|^q dt \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(R-s)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{R-}^{\alpha} g \right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}, \tag{49}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence it holds

$$\int_{B(0,R)} |f(y)|\, dy \overset{(25)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds \right\} \overset{(by\ (48),\ (49))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left(\int_0^{\frac{R}{2}} s^{\alpha+N-2+\frac{1}{p}} ds \right) \left\| D_{0+}^{\alpha} g \right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)} + \right. \tag{50}$$

$$\left(\int_{\frac{R}{2}}^{R}(R-s)^{\alpha-1+\frac{1}{p}}s^{N-1}ds\right)\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{\left(\frac{R}{2}\right)^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right.$$

$$\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\left(\int_{\frac{R}{2}}^{R}(R-s)^{\left(\alpha+\frac{1}{p}-1\right)}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right)\right]$$

$$\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{51}$$

$$\left[\sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}\left(\frac{R}{2}\right)^{k}\frac{\Gamma\left(\alpha+\frac{1}{p}\right)\Gamma(N-k)}{\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\left(\frac{R}{2}\right)^{\alpha+\frac{1}{p}+N-k-1}\right]$$

$$\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{52}$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left(\frac{R^{\alpha+N-\frac{1}{q}}}{2^{\alpha+N-\frac{1}{q}}}\right)\cdot$$

$$\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}. \tag{53}$$

We have proved the following

Theorem 13 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. *All other terms and assumptions as in Theorem 9. Then*

$$\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right.$$

$$\left. (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right) \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)} \right] \left\| D_{R-}^{\alpha} g \right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\}. \quad (54)$$

Combining Theorems 9, 11, 13 we derive

Theorem 14 *Let any $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \geq 1$. And let $f : \overline{B(0,R)} \to \mathbb{R}$ be radial; that is, there exists a function g such that $f(x) = g(s)$, $s = |x|$, $s \in [0,R]$, $\forall\ x \in \overline{B(0,R)}$. Assume that $g \in C([0,R])$, with $g \in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $g \in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0) = g^{(k)}(R) = 0$, $k = 0,1,...,m-1$, $m = [\alpha]$. When $0 < \alpha < 1$ the last boundary conditions are void. Then*

$$\left| \int_{B(0,R)} f(y)\,dy \right| \leq \int_{B(0,R)} |f(y)|\,dy \leq$$

$$\min \left\{ \frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)} + \right. \right.$$

$$\left. (N-1)! \left\| D_{R-}^{\alpha} g \right\|_{\infty,\left[\frac{R}{2},R\right]} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)} \right] \right\},$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\},$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right.$$

$$\left. \left. (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right) \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)} \right] \left\| D_{R-}^{\alpha} g \right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\} \right\}. \quad (55)$$

Note 15 *It holds*

$$Vol(B(0,R)) = \frac{2\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}\right) N}. \quad (56)$$

The corresponding estimate on the average follows

Corollary 16 *Let all terms and assumptions as in Theorem 14. Then*

$$\left|\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}f\left(y\right)dy\right|\leq\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq$$

$$\min\left\{\frac{NR^{\alpha}}{2^{\alpha+N}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\right.\right.$$

$$\left.\left.\left(N-1\right)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\},\right.$$

$$\frac{NR^{\alpha-1}}{2^{\alpha+N-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-1\right)\Gamma\left(\alpha\right)}+\right.$$

$$\left.\left(N-1\right)!\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N-k\right)}\right]\right\},$$

$$\frac{NR^{\alpha-\frac{1}{q}}}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.\left.\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}\right\}. \tag{57}$$

We continue with Polya type inequalities on the ball for non-radial functions.

Theorem 17 *Let $f\in C\left(\overline{B\left(0,R\right)}\right)$ that is not necessarily radial, $0<\alpha<2$. Assume for any $\omega\in S^{N-1}$, that $f\left(\cdot\omega\right)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f\left(\cdot\omega\right)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f\left(0\right)=f\left(R\omega\right)=0$. When $0<\alpha<1$ the last boundary conditions are void. We further assume that*

$$\left\|\frac{\partial_{0+}^{\alpha}f\left(r\omega\right)}{\partial r^{\alpha}}\right\|_{\infty,\left(r\in\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f\left(r\omega\right)}{\partial r^{\alpha}}\right\|_{\infty,\left(r\in\left[\frac{R}{2},R\right]\right)}\leq K, \tag{58}$$

for every $\omega\in S^{N-1}$, where $K>0$.

Then

(i)

$$\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq\frac{K\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\cdot \tag{59}$$

$$\left\{\frac{1}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\left(N-1\right)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\},$$

and

(ii)

$$\left|\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}f\left(y\right)dy\right|\leq\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq \tag{60}$$

$$\frac{KNR^{\alpha}}{2^{\alpha+N}}\left\{\frac{1}{(\alpha+N)\,\Gamma\left(\alpha+1\right)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\}.$$

Proof. In Remark 8, see (25), (26), (27), (28), we proved that

$$\int_{0}^{R}\left|g\left(s\right)\right|s^{N-1}ds\leq\left(\frac{R}{2}\right)^{\alpha+N}\cdot$$

$$\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\,\Gamma\left(\alpha+1\right)}+\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\}. \tag{61}$$

In the above (61) we plug in $g\left(\cdot\right)=f\left(\cdot\omega\right)$, for $\omega\in S^{N-1}$ fixed, and we get

$$\int_{0}^{R}\left|f\left(s\omega\right)\right|s^{N-1}ds\overset{(58)}{\leq}K\left(\frac{R}{2}\right)^{\alpha+N}\cdot$$

$$\left\{\frac{1}{(\alpha+N)\,\Gamma\left(\alpha+1\right)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\}=:\lambda_{1}. \tag{62}$$

Consequently we obtain

$$\int_{B(0,R)}\left|f\left(y\right)\right|dy=\int_{S^{N-1}}\left(\int_{0}^{R}\left|f\left(s\omega\right)\right|s^{N-1}ds\right)d\omega\leq$$

$$\lambda_{1}\int_{S^{N-1}}d\omega=\lambda_{1}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{63}$$

proving the claims. ■

We continue with

Theorem 18 *Let $f\in C\left(\overline{B\left(0,R\right)}\right)$ that is not necessarily radial, $1\leq\alpha<2$. Assume for any $\omega\in S^{N-1}$, that $f\left(\cdot\omega\right)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f\left(\cdot\omega\right)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f\left(0\right)=f\left(R\omega\right)=0$. We further assume*

$$\left\|\frac{\partial_{0+}^{\alpha}f\left(\cdot\omega\right)}{\partial r^{\alpha}}\right\|_{L_{1}\left(\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f\left(\cdot\omega\right)}{\partial r^{\alpha}}\right\|_{L_{1}\left(\left[\frac{R}{2},R\right]\right)}\leq M, \tag{64}$$

for every $\omega\in S^{N-1}$, where $M>0$.

Then

(i)

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{M\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \cdot \tag{65}$$

$$\left\{ \frac{1}{(\alpha+N-1)\,\Gamma(\alpha)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\},$$

and

(ii)

$$\frac{1}{Vol\,(B(0,R))} \int_{B(0,R)} |f(y)|\, dy \leq \tag{66}$$

$$\frac{MNR^{\alpha-1}}{2^{\alpha+N-1}} \left\{ \frac{1}{(\alpha+N-1)\,\Gamma(\alpha)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}.$$

Proof. In Remark 10, see (43), (44), (45), we proved that

$$\int_0^R |g(s)|\, s^{N-1} ds \leq \left(\frac{R}{2}\right)^{\alpha+N-1} \cdot$$

$$\left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\,\Gamma(\alpha)} + \left\|D_{R-}^{\alpha} g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{67}$$

In the above (67) we plug in $g(\cdot) = f(\cdot\omega)$, for $\omega \in S^{N-1}$ fixed, and we derive

$$\int_0^R |f(s\omega)|\, s^{N-1} ds \overset{(64)}{\leq} M \left(\frac{R}{2}\right)^{\alpha+N-1} \cdot$$

$$\left\{ \frac{1}{(\alpha+N-1)\,\Gamma(\alpha)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\} =: \lambda_2. \tag{68}$$

Hence

$$\int_{B(0,R)} |f(y)|\, dy = \int_{S^{N-1}} \left(\int_0^R |f(s\omega)|\, s^{N-1} ds \right) d\omega \leq$$

$$\lambda_2 \int_{S^{N-1}} d\omega = \lambda_2 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{69}$$

proving the claims. ■

We further have

Theorem 19 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\frac{1}{q} < \alpha < 2$. Let $f \in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial. Assume for any $\omega \in S^{N-1}$, that $f(\cdot\omega) \in C_{0+}^{\alpha}\left(\left[0, \frac{R}{2}\right]\right)$*

and $f(\cdot\omega) \in C^{\alpha}_{R-}\left(\left[\frac{R}{2}, R\right]\right)$, *such that* $f(0) = f(R\omega) = 0$. *When* $\frac{1}{q} < \alpha < 1$ *the last boundary condition is void. We further assume*

$$\left\| \frac{\partial^{\alpha}_{0+} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_q\left(\left[0, \frac{R}{2}\right]\right)}, \quad \left\| \frac{\partial^{\alpha}_{R-} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_q\left(\left[\frac{R}{2}, R\right]\right)} \leq \Phi, \tag{70}$$

for every $\omega \in S^{N-1}$, *where* $\Phi > 0$.

Then

(i)

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{\Phi \pi^{\frac{N}{2}} R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot \tag{71}$$

$$\left\{ \frac{1}{\left(\alpha+N-\frac{1}{q}\right)} + (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \right\},$$

and

(ii)

$$\frac{1}{Vol(B(0,R))} \int_{B(0,R)} |f(y)|\, dy \leq \frac{\Phi N R^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}}} \cdot \tag{72}$$

$$\left\{ \frac{1}{\left(\alpha+N-\frac{1}{q}\right)} + (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \right\}.$$

Proof. In Remark 12, see (50), (51), (52), (53), we proved that

$$\int_0^R |g(s)|\, s^{N-1} ds \leq$$

$$\left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}} \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \left\{ \frac{\left\| D^{\alpha}_{0+} g \right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right. \tag{73}$$

$$\left. (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \left\| D^{\alpha}_{R-} g \right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\}.$$

In the above (73) we plug in $g(\cdot) = f(\cdot\omega)$, for $\omega \in S^{N-1}$ fixed, and we find

$$\int_0^R |f(s\omega)|\, s^{N-1} ds \overset{(70)}{\leq} \Phi \left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}} \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \frac{1}{\left(\alpha+N-\frac{1}{q}\right)} + (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\} =: \lambda_3. \tag{74}$$

Thus

$$\int_{B(0,R)} |f(y)|\,dy = \int_{S^{N-1}}\left(\int_0^R |f(s\omega)|\, s^{N-1}ds\right) d\omega \leq$$

$$\lambda_3 \int_{S^{N-1}} d\omega = \lambda_3 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{75}$$

proving the claims. ■

We make

Remark 20 *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. *Consider first that* $f : \overline{A} \to \mathbb{R}$ *is radial; that is, there exists* g *such that* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$. *Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see ([14], p. 149-150 and [1], p. 421), furthermore for general* $F : \overline{A} \to \mathbb{R}$ *Lebesgue integrable function we have that*

$$\int_A F(x)\,dx = \int_{S^{N-1}}\left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1}dr\right) d\omega. \tag{76}$$

Let $d\omega$ *be the element of surface measure on* S^{N-1}, *then*

$$\omega_N := \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{77}$$

Here

$$Vol(A) = \frac{\omega_N\left(R_2^N - R_1^N\right)}{N} = \frac{2\pi^{\frac{N}{2}}\left(R_2^N - R_1^N\right)}{N\Gamma\left(\frac{N}{2}\right)}. \tag{78}$$

We assume that $g \in C([R_1, R_2])$, *and* $\alpha > 0$, $m = [\alpha]$, *such that* $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ *and* $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, *with* $g^{(k)}(R_1) = g^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$. *When* $0 < \alpha < 1$ *the last boundary conditions are void.*

By assumption here and Theorem 3 we have

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_{R_1}^{s} (s-t)^{\alpha-1}\left(D^{\alpha}_{R_1+}g\right)(t)\,dt, \tag{79}$$

all $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$,

also it holds, by assumption and Theorem 4, that

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_{s}^{R_2} (t-s)^{\alpha-1}\left(D^{\alpha}_{R_2-}g\right)(t)\,dt, \tag{80}$$

all $s\in\left[\frac{R_1+R_2}{2},R_2\right]$.

By (79) we get

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_{R_1}^{s}(s-t)^{\alpha-1}\left|\left(D_{R_1+}^{\alpha}g\right)(t)\right|dt \tag{81}$$

$$\leq\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\frac{(s-R_1)^{\alpha}}{\Gamma(\alpha+1)}, \tag{82}$$

for any $s\in\left[R_1,\frac{R_1+R_2}{2}\right]$.

Similarly we obtain by (80) that

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_{s}^{R_2}(t-s)^{\alpha-1}\left|\left(D_{R_2-}^{\alpha}g\right)(t)\right|dt \tag{83}$$

$$\leq\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\frac{(R_2-s)^{\alpha}}{\Gamma(\alpha+1)}, \tag{84}$$

for any $s\in\left[\frac{R_1+R_2}{2},R_2\right]$.

Next we observe that

$$\left|\int_A f(y)\,dy\right|\leq\int_A|f(y)|\,dy\overset{(76)}{=} \tag{85}$$

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2}|g(s)|\,s^{N-1}ds\right)d\omega=\left(\int_{R_1}^{R_2}|g(s)|\,s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}= \tag{86}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_{R_1}^{\frac{R_1+R_2}{2}}|g(s)|\,s^{N-1}ds+\int_{\frac{R_1+R_2}{2}}^{R_2}|g(s)|\,s^{N-1}ds\right\}\overset{(by\ (82)\ and\ (84))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\int_{R_1}^{\frac{R_1+R_2}{2}}(s-R_1)^{\alpha}s^{N-1}ds\right.$$

$$\left.+\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\int_{\frac{R_1+R_2}{2}}^{R_2}(R_2-s)^{\alpha}s^{N-1}ds\right\}= \tag{87}$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\right.$$

$$\left(\sum_{k=0}^{N-1}(-1)^{N+k-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right]\right\}. \tag{88}$$

We have proved that

$$\left|\int_A f(y)\,dy\right| \leq \int_A |f(y)|\,dy \leq \frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right.$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right]\right\}. \tag{89}$$

Consider now $f_* : \overline{A} \to \mathbb{R}$ *be radial such that* $f_*(x) = g_*(s)$, $s = |x|$, $s \in [R_1, R_2]$, $\forall\ x \in \overline{A}$, *where*

$$g_*(s) = \begin{cases} (s-R_1)^{\alpha}, & s \in \left[R_1, \frac{R_1+R_2}{2}\right], \\ (R_2-s)^{\alpha}, & s \in \left[\frac{R_1+R_2}{2}, R_2\right],\ \alpha > 0. \end{cases} \tag{90}$$

We have, as in [4], that

$$\left\|D^{\alpha}_{R_1+}g_*\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]} = \Gamma(\alpha+1), \quad \text{and} \quad \left\|D^{\alpha}_{R_2-}g_*\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]} = \Gamma(\alpha+1). \tag{91}$$

Hence

$$\text{R.H.S. (89) (applied on } g_*) = \frac{\Gamma(\alpha+1)\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\sum_{k=0}^{N-1}\left(1+(-1)^{N+k-1}\right)\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right\}. \tag{92}$$

Furthermore we find

$$\text{L.H.S. (89) (applied on } f_*) = \int_A f_*(y)\,dy =$$

$$\left(\int_{R_1}^{R_2} g_*(s)\,s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_{R_1}^{\frac{R_1+R_2}{2}}(s-R_1)^{\alpha}s^{N-1}ds + \int_{\frac{R_1+R_2}{2}}^{R_2}(R_2-s)^{\alpha}s^{N-1}ds\right\} = \tag{93}$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)+\right. \tag{94}$$

$$\left.\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\} =$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\sum_{k=0}^{N-1}\left((-1)^{N+k-1}+1\right)\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right\}. \tag{95}$$

So that we find

$$\text{R.H.S. (89) (applied on } g_*) = \text{L.H.S. (89) (applied on } f_*), \tag{96}$$

proving sharpness of (89).

We have proved the following

Theorem 21 *Let $f:\overline{A}\to\mathbb{R}$ be radial; that is, there exists a function g such that $f(x)=g(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, $\alpha>0$, $m=[\alpha]$. We assume that $g\in C([R_1,R_2])$, such that $g\in C^{\alpha}_{R_1+}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $g\in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, with $g^{(k)}(R_1)=g^{(k)}(R_2)=0$, $k=0,1,...,m-1$. When $0<\alpha<1$ the last boundary conditions are void. Then*

$$\left|\int_A f(y)\,dy\right|\leq\int_A|f(y)|\,dy\leq\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right. \tag{97}$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}.$$

Inequalities (97) are sharp, namely they are attained by the radial function $f_*:\overline{A}\to\mathbb{R}$ such that $f_*(x)=g_*(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, where

$$g_*(s)=\begin{cases}(s-R_1)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ (R_2-s)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right].\end{cases} \tag{98}$$

We continue with

Remark 22 *Here $\alpha\geq 1$. By (81) we get*

$$|g(s)|\leq\frac{(s-R_1)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{R_1+}g\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}, \tag{99}$$

for any $s\in\left[R_1,\frac{R_1+R_2}{2}\right]$.

And by (83) we derive

$$|g(s)|\leq\frac{(R_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{R_2-}g\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}, \tag{100}$$

for any $s\in\left[\frac{R_1+R_2}{2},R_2\right]$.

Hence

$$\int_A |f(y)|\, dy \overset{(86)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (99) and (100))}}{\leq} \tag{101}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s-R_1)^{\alpha-1} s^{N-1} ds \right) + \right. \tag{102}$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2-s)^{\alpha-1} s^{N-1} ds \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \tag{103}$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\}.$$

We have proved that

Theorem 23 *All terms and assumptions here as in Theorem 21, but with* $\alpha \geq 1$*. Then*

$$\int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\}. \tag{104}$$

We continue with

Remark 24 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Let* $\alpha > \frac{1}{q}$*. By (81) we get*

$$|g(s)| \leq \frac{(s-R_1)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \tag{105}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly by (83) we derive

$$\left|g\left(s\right)\right| \leq \frac{\left(R_2-s\right)^{\alpha-1+\frac{1}{p}}}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}} \left\|D^{\alpha}_{R_2-}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}, \tag{106}$$

for any $s\in\left[\frac{R_1+R_2}{2},R_2\right]$.

Hence

$$\int_A \left|f\left(y\right)\right|dy =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_{R_1}^{\frac{R_1+R_2}{2}}\left|g\left(s\right)\right|s^{N-1}ds+\int_{\frac{R_1+R_2}{2}}^{R_2}\left|g\left(s\right)\right|s^{N-1}ds\right\}\leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\int_{R_1}^{\frac{R_1+R_2}{2}}\left(s-R_1\right)^{\alpha-1+\frac{1}{p}}s^{N-1}ds\right)+\right.$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\int_{\frac{R_1+R_2}{2}}^{R_2}\left(R_2-s\right)^{\alpha-1+\frac{1}{p}}s^{N-1}ds\right)\right\}= \tag{107}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\frac{(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{2^{\alpha+N-\frac{1}{q}}}\right)\cdot\right.$$

$$\left(\sum_{k=0}^{N-1}\frac{\left(-1\right)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N-k+\alpha-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+ \tag{108}$$

$$\left\|D^{\alpha}_{R_2-}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\frac{(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{2^{\alpha+N-\frac{1}{q}}}\right)\cdot$$

$$\left.\left(\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right)\right\}=$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\sum_{k=0}^{N-1}\frac{\left(-1\right)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+\right.$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\right\}. \tag{109}$$

We have proved

Theorem 25 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. All terms and assumptions as in Theorem 21. Then*

$$\int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}. \tag{110}$$

Combining Theorems 21, 23, 25 we derive

Theorem 26 *Let any $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. And let $f : \overline{A} \to \mathbb{R}$ be radial; that is, there exists a function g such that $f(x) = g(s)$, $s = |x|$, $s \in [R_1, R_2]$, $\forall\ x \in \overline{A}$; $\alpha \geq 1$, $m = [\alpha]$. We assume that $g \in C([R_1, R_2])$, such that $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $g^{(k)}(R_1) = g^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$. Then*

$$\left| \int_A f(y)\, dy \right| \leq \int_A |f(y)|\, dy \leq \min \left\{ \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot \right.$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{\infty, \left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{\infty, \left[\frac{R_1+R_2}{2}, R_2\right]} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right\},$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \cdot \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) +$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\},$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \left\| D^{\alpha}_{R_2-}g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\} \Bigg\}. \quad (111)$$

The corresponding estimate on the average follows

Corollary 27 *Let all terms and assumptions as in Theorem 26. Then*

$$\left| \frac{1}{Vol(A)} \int_A f(y)\, dy \right| \le \frac{1}{Vol(A)} \int_A |f(y)|\, dy \le \left(\frac{N!}{2^{\alpha+N}\left(R_2^N - R_1^N\right)} \right) \cdot$$

$$\min \Bigg\{ \left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{\infty,\left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) + \left\| D^{\alpha}_{R_2-}g \right\|_{\infty,\left[\frac{R_1+R_2}{2}, R_2\right]} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right\},$$

$$2\left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \left\| D^{\alpha}_{R_2-}g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\},$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right) 2^{\frac{1}{q}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \left\| D^{\alpha}_{R_2-}g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\} \Bigg\}. \quad (112)$$

We need

Definition 28 *(see [1], p. 287) Let $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, and A is a spherical shell. Assume that there exists $\frac{\partial^{\alpha}_{R_1+} f(x)}{\partial r^{\alpha}} \in C\left(\overline{A}\right)$, given by*

$$\frac{\partial^{\alpha}_{R_1+} f(x)}{\partial r^{\alpha}} := \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r-t)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \quad (113)$$

where $x \in \overline{A}$; that is, $x = r\omega$, $r \in [R_1, R_2]$, and $\omega \in S^{N-1}$.

We call $\frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}}$ the left radial generalised fractional derivative of f of order α.

We also need to introduce

Definition 29 *Let* $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, *and* A *is a spherical shell. Assume that there exists* $\frac{\partial^\alpha_{R_2-} f(x)}{\partial r^\alpha} \in C\left(\overline{A}\right)$, *given by*

$$\frac{\partial^\alpha_{R_2-} f(x)}{\partial r^\alpha} := (-1)^{m-1} \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_r^{R_2} (t-r)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \tag{114}$$

where $x \in \overline{A}$; *that is,* $x = r\omega$, $r \in [R_1, R_2]$, *and* $\omega \in S^{N-1}$.

We call $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha}$ *the right radial generalised fractional derivative of* f *of order* α.

We present

Theorem 30 *Let the sperical shells* $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$; $A_1 := B\left(0, \frac{R_1+R_2}{2}\right) - \overline{B(0, R_1)}$, $A_2 := B(0, R_2) - \overline{B\left(0, \frac{R_1+R_2}{2}\right)}$. *Let* $f \in C\left(\overline{A}\right)$, *not necessarily radial,* $\alpha > 0$, $m = [\alpha]$. *Assume that* $\frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \in C\left(\overline{A_1}\right)$, $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \in C\left(\overline{A_2}\right)$. *For each* $\omega \in S^{N-1}$, *we assume further that* $f(\cdot\omega) \in C^\alpha_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ *and* $f(\cdot\omega) \in C^\alpha_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, *with* $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. *When* $0 < \alpha < 1$ *the last boundary conditions are void. Then*

(i)

$$\left| \int_A f(y)\, dy \right| \leq \int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot$$

$$\left\{ \left\| \frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}, \tag{115}$$

and

(ii)

$$\left| \frac{1}{Vol(A)} \int_A f(y)\, dy \right| \leq \frac{1}{Vol(A)} \int_A |f(y)|\, dy \leq \left(\frac{N!}{2^{\alpha+N} \left(R_2^N - R_1^N\right)} \right) \cdot \tag{116}$$

$$\left\{ \left\| \frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}.$$

Proof. By (86)-(88) we get

$$\int_{R_1}^{R_2} |g(s)| s^{N-1} ds \leq \left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\right) \left(\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}}\right) \cdot \tag{117}$$

$$\left\{ \left\| D_{R_1+}^{\alpha} g \right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| D_{R_2-}^{\alpha} g \right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]} \left[\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right] \right\}.$$

For fixed $\omega \in S^{N-1}$, $f(\cdot\omega)$ sets like a radial function on $\overline{A}$. Thus plugging $f(\cdot\omega)$ into (117), we get

$$\int_{R_1}^{R_2} |f(s\omega)| s^{N-1} ds \leq \left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\right) \left(\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}}\right) \cdot \tag{118}$$

$$\left\{ \left\| \frac{\partial_{R_1+}^{\alpha} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial_{R_2-}^{\alpha} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\} =: \gamma_1.$$

Therefore by (76) and (118) we derive

$$\int_A |f(y)| dy = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} |f(s\omega)| s^{N-1} ds \right) d\omega \leq$$

$$\gamma_1 \int_{S^{N-1}} d\omega = \gamma_1 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \left(\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}}\right) \cdot \tag{119}$$

$$\left\{ \left\| \frac{\partial_{R_1+}^{\alpha} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial_{R_2-}^{\alpha} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\},$$

proving the claims of the theorem. ■

We give also

Theorem 31 *Let $f \in C\left(\overline{A}\right)$, not necessarily radial, $\alpha \geq 1$, $m = [\alpha]$. For each $\omega \in S^{N-1}$, we assume that $f(\cdot\omega) \in C_{R_1+}^{\alpha}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega) \in$*

$C_{R_2-}^{\alpha}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, *with* $\frac{\partial^k f(R_1\omega)}{\partial r^k}=\frac{\partial^k f(R_2\omega)}{\partial r^k}=0$, $k=0,1,...,m-1$. *We further assume*

$$\left\|\frac{\partial_{R_1+}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)},\ \left\|\frac{\partial_{R_2-}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\leq\Psi_1, \tag{120}$$

for every $\omega\in S^{N-1}$, *where* $\Psi_1>0$.

Then

(i)

$$\int_A|f(y)|\,dy\leq\frac{\Psi_1\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-2}}\cdot \tag{121}$$

$$\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)+\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\right\},$$

and

(ii)

$$\frac{1}{Vol(A)}\int_A|f(y)|\,dy\leq\frac{\Psi_1 N!}{2^{\alpha+N-1}\left(R_2^N-R_1^N\right)}\cdot \tag{122}$$

$$\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)+\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\right\}.$$

Proof. Similar to Theorem 30, using (101)-(103). ■

We finish with

Theorem 32 *Let* $f\in C\left(\overline{A}\right)$, *not necessarily radial,* $\alpha>\frac{1}{q}$, *where* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $m=[\alpha]$. *For each* $\omega\in S^{N-1}$, *we assume that* $f(\cdot\omega)\in C_{R_1+}^{\alpha}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ *and* $f(\cdot\omega)\in C_{R_2-}^{\alpha}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, *with* $\frac{\partial^k f(R_1\omega)}{\partial r^k}=\frac{\partial^k f(R_2\omega)}{\partial r^k}=0$, $k=0,1,...,m-1$. *When* $\frac{1}{q}<\alpha<1$ *the last boundary conditions is void. We further assume*

$$\left\|\frac{\partial_{R_1+}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)},\ \left\|\frac{\partial_{R_2-}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\leq\Psi_2, \tag{123}$$

for every $\omega\in S^{N-1}$, *where* $\Psi_2>0$.

Then

(i)

$$\int_A |f(y)|\,dy \leq \frac{\Psi_2 \pi^{\frac{N}{2}} (N-1)! \Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right) \Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot \tag{124}$$

$$\left\{ \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\},$$

and

(ii)

$$\frac{1}{Vol(A)} \int_A |f(y)|\,dy \leq \frac{N! \Gamma\left(\alpha + \frac{1}{p}\right) \Psi_2}{2^{\alpha+N-\frac{1}{q}} \left(R_2^N - R_1^N\right) \Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \tag{125}$$

$$\left\{ \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}.$$

Proof. Similar to Theorem 30, using (107)-(109). ∎

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