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General Grüss and Ostrowski type inequalities involving s-convexity

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Abstract

Using the well known reoresentation formula for functions due to Fink [7], we establish a series of general Grüss and Ostrowski type inequalities involving s-convexity and s-concavity in the second sense, acting to all possible directions.

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1 Background

We are motivated by the following famous inequality due to G. Grüss of 1935.

Theorem 1 *([8]) Let f, g be integrable functions from $[a,b]$ into $\mathbb{R}$, that satisfy the conditions*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a,b],$$

where $m, M, n, N \in \mathbb{R}$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left(\frac{1}{b-a} \int_a^b f(x) \, dx \right) \left(\frac{1}{b-a} \int_a^b g(x) \, dx \right) \right| \tag{1}$$
$$\leq \frac{1}{4} (M-m)(N-n).$$

We are also motivated by [3], Chapter 25, pp. 305-317.

Another motivation is [2].

We are strongly motivated in our method by

Proposition 2 *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'\in L_1([a,b])$, then*

$$f(x)=\frac{1}{b-a}\int_a^b f(u)\,du+(a-b)\int_0^1 p(x,t)\,f'(ta+(1-t)b)\,dt, \tag{2}$$

for all $x\in[a,b]$, where

$$p(x,t)=\begin{cases} t, & t\in\left[0,\frac{b-x}{b-a}\right), \\ t-1, & t\in\left[\frac{b-x}{b-a},1\right]. \end{cases} \tag{3}$$

Proof. We use Montgomery identity ([5])

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b q(x,t)\,f'(t)\,dt, \tag{4}$$

for all $x\in[a,b]$, where

$$q(x,t)=\begin{cases} t-a, & t\in[a,x], \\ t-b, & t\in(x,b]. \end{cases} \tag{5}$$

We observe by change of variable that

$$\frac{1}{b-a}\int_a^b q(x,u)\,f'(u)\,du=\int_0^1 q(x,\lambda a+(1-\lambda)b)\,f'(\lambda a+(1-\lambda)b)\,d\lambda. \tag{6}$$

We notice that

$$q(x,\lambda a+(1-\lambda)b)=\begin{cases} (1-\lambda)(b-a), & \lambda a+(1-\lambda)b\in[a,x], \\ -\lambda(b-a), & \lambda a+(1-\lambda)b\in(x,b] \end{cases} \tag{7}$$

$$=\begin{cases} (1-\lambda)(b-a), & \lambda\in\left[\frac{b-x}{b-a},1\right], \\ -\lambda(b-a), & \lambda\in\left[0,\frac{b-x}{b-a}\right) \end{cases}$$

$$=(a-b)\begin{cases} \lambda, & \lambda\in\left[0,\frac{b-x}{b-a}\right), \\ \lambda-1, & \lambda\in\left[\frac{b-x}{b-a},1\right] \end{cases}=(a-b)\,p(x,\lambda). \tag{8}$$

Therefore it holds

$$\int_0^1 q(x,\lambda a+(1-\lambda)b)\,f'(\lambda a+(1-\lambda)b)\,d\lambda=$$

$$(a-b)\int_0^1 p(x,\lambda)\,f'(\lambda a+(1-\lambda)b)\,d\lambda, \tag{9}$$

that is

$$\frac{1}{b-a}\int_a^b q(x,u)\,f'(u)\,du=(a-b)\int_0^1 p(x,\lambda)\,f'(\lambda a+(1-\lambda)b)\,d\lambda, \tag{10}$$

proving the claim. ■

Note 3 *From (3) we notice that*

$$|p(x,t)| \le 1, \text{ all } x \in [a,b] \text{ and } t \in [0,1]. \tag{11}$$

Assume now that $|f'|$ is convex on $[a,b]$ with $f'(a), f'(b) \in \mathbb{R}$. Then $g(t) = |f'(ta + (1-t)b)| \ge 0$ is also a convex function in $t \in [0,1]$, as a composition of a convex and an affine function.

It is clear now by comparing the areas under g and the related trapezium that

$$\int_0^1 |f'(ta + (1-t)b)| \, dt \le \frac{|f'(a)| + |f'(b)|}{2}. \tag{12}$$

Hence when $|f'|$ is convex with $f'(a), f'(b) \in \mathbb{R}$, we get

$$\int_0^1 |p(x,t)| \, |f'(ta + (1-t)b)| \, dt \le \frac{|f'(a)| + |f'(b)|}{2}. \tag{13}$$

We need

Theorem 4 *([7], Fink) Let $a, b \in \mathbb{R}$, $a < b$, $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $x \in [a,b]$. Then*

$$f(x) = \frac{n}{b-a} \int_a^b f(t) \, dt \tag{14}$$

$$-\sum_{k=1}^{n-1} \left(\frac{n-k}{k!}\right) \left(\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}\right)$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t) \, dt,$$

where

$$q(x,t) = \begin{cases} t-a, & t \in [a,x], \\ t-b, & t \in (x,b]. \end{cases} \tag{15}$$

When $n = 1$ the sum $\sum_{k=1}^{n-1}$ is zero.

We make

Remark 5 *(on Theorem 4) Here we transform the remainder of (14) as follows*

$$\frac{1}{(n-1)!(b-a)} \int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t) \, dt = \tag{16}$$

$$\frac{1}{(n-1)!} \int_0^1 (x - \lambda a - (1-\lambda) b)^{n-1} q(x, \lambda a + (1-\lambda) b) f^{(n)}(\lambda a + (1-\lambda) b) \, d\lambda$$

$$\overset{(8)}{=} \frac{a-b}{(n-1)!} \int_0^1 ((x-b) + \lambda(b-a))^{n-1} p(x,\lambda) f^{(n)}(\lambda a + (1-\lambda) b) \, d\lambda. \tag{17}$$

We have established the following result which is our main tool here

Theorem 6 *Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is absolutely continuous on $[a, b]$, $x \in [a, b]$. Then*

$$f(x) = \frac{n}{b-a} \int_a^b f(t)\, dt - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right)$$

$$+ \frac{a-b}{(n-1)!} \int_0^1 ((x-b) + \lambda(b-a))^{n-1} p(x, \lambda) f^{(n)}(\lambda a + (1-\lambda) b)\, d\lambda, \tag{18}$$

where

$$p(x, \lambda) = \begin{cases} \lambda, & \lambda \in \left[0, \frac{b-x}{b-a}\right), \\ \lambda - 1, & \lambda \in \left[\frac{b-x}{b-a}, 1\right]. \end{cases} \tag{19}$$

When $n = 1$ the sum $\sum_{k=1}^{n-1}$ is zero.

Note 7 *When $n = 1$ formula (18) collapses to (2).*

We call the remainder of (18) as

$$Rem(18) = \frac{a-b}{(n-1)!} \int_0^1 ((x-b) + \lambda(b-a))^{n-1} p(x, \lambda) f^{(n)}(\lambda a + (1-\lambda) b)\, d\lambda. \tag{20}$$

We have that

$$|Rem(18)| \le \frac{(b-a)^n}{(n-1)!} \int_0^1 \left| f^{(n)}(\lambda a + (1-\lambda) b) \right| d\lambda. \tag{21}$$

Assume now that $\left|f^{(n)}\right|$ is convex on $[a, b]$ with $f^{(n)}(a), f^{(n)}(b) \in \mathbb{R}$, then

$$\int_0^1 \left| f^{(n)}(\lambda a + (1-\lambda) b) \right| d\lambda \le \frac{\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|}{2}. \tag{22}$$

So given that $\left|f^{(n)}\right|$ is convex on $[a, b]$ with $f^{(n)}(a), f^{(n)}(b) \in \mathbb{R}$, we derive

$$|Rem(18)| \le \frac{(b-a)^n}{(n-1)!} \frac{\left(\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right)}{2}. \tag{23}$$

We need

Definition 8 *([9]) A function $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1-\lambda) y) \le \lambda^s f(x) + (1-\lambda)^s f(y), \tag{24}$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

When $s = 1$, s-convexity in the second sense reduces to ordinary convexity.

If "$\ge$" holds in (24), we talk about s-concavity in the second sense.

We also need

Definition 9 *(see also [1])Let I be a subinterval of $\mathbb{R}_+$ and $f : I \to (0, \infty)$. We call f s-logarithmically convex (s-log-convex) in the second sense, iff $\log f(x)$ is s-convex in the second sense, iff*

$$f(\lambda x + (1-\lambda) y) \leq (f(x))^{\lambda^s} (f(y))^{(1-\lambda)^s}, \tag{25}$$

for all $x, y \in I$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

When $s = 1$, s-log-convexity in the second sense reduces to usual log-convexity.

If "$\geq$" holds in (25), we talk about s-log-concavity in the second sense.

We also need the s-convex Hadamard's inequality

Theorem 10 *([6]) Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1([a,b])$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a) + f(b)}{s+1}. \tag{26}$$

The constant $K = \frac{1}{s+1}$ is the best possible in the second inequality (26). The above inequalities are sharp.

We mention also the s-convex Ostrowski type inequality

Theorem 11 *([2]) Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \leq M$, $x \in [a,b]$, then it holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq \frac{M}{b-a} \left(\frac{(x-a)^2 + (b-x)^2}{s+1} \right), \tag{27}$$

for each $x \in [a,b]$.

We are also motivated by

Theorem 12 *([4],Čebyšev 1882) Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions. If $f', g' \in L_\infty([a,b])$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a} \int_a^b f(x)\,dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{28}$$

Next we derive general inequalities similar to (28) and (27).

2 Main Results

We present our first general main result that is about Grüss type inequalities and involves s-convexity and s-concavity in the second sense.

Theorem 13 *Let $a,b \in \mathbb{R}$, $a<b$, $f,g:[a,b] \to \mathbb{R}$, $n \in \mathbb{N}$; $f^{(n-1)}$, $g^{(n-1)}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$. Whenever we consider s-convexity or s-concavity of functions we take $a \geq 0$.*

Denote by

$$F_{n-1}^{f}(x) := \sum_{k=1}^{n-1} \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a} \right), \tag{29}$$

with $F_0^f(x) := 0$, and

$$\Delta_{(f,g)} := \int_a^b f(x) g(x) dx - \frac{n}{b-a} \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right)$$
$$- \frac{1}{2} \left[\int_a^b \left(g(x) F_{n-1}^f(x) + f(x) F_{n-1}^g(x) \right) dx \right]. \tag{30}$$

We distinguish the cases:

1) Here $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense and

$$\left|f^{(n)}(x)\right| \leq M, \quad \left|g^{(n)}(x)\right| \leq K, \quad x \in [a,b]. \tag{31}$$

Then

$$\left|\Delta_{(f,g)}\right| \leq \frac{2(b-a)^{n+1}}{(n-1)!(s+2)(s+3)} \left(K \|f\|_\infty + M \|g\|_\infty \right). \tag{32}$$

2) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, and $f^{(n)}$, $g^{(n)} \in L_{p_4}([a,b])$. Then

$$\left|\Delta_{(f,g)}\right| \leq$$
$$\frac{(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}} \left[\|f\|_{p_1} \left\|g^{(n)}\right\|_{p_4} + \|g\|_{p_1} \left\|f^{(n)}\right\|_{p_4} \right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}} (n-1)! \left((p_2(n-1)+1)(p_2(n-1)+2) \right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2) \right)^{\frac{1}{p_3}}}. \tag{33}$$

3) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, with $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $\left|g^{(n)}(x)\right| \leq K$, $x \in [a,b]$. Then

$$\left|\Delta_{(f,g)}\right| \leq$$
$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}} (b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} K + \|g\|_{p_1} M \right]}{(n-1)! \left((p_2(n-1)+1)(p_2(n-1)+2) \right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2) \right)^{\frac{1}{p_3}} (p_4 s+1)^{\frac{1}{p_4}}}. \tag{34}$$

4) Let $p_i > 1 : \sum_{i=1}^{4} \frac{1}{p_i} = 1$*, with* $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ *are s-convex in the second sense and* $\left|f^{(n)}(x)\right| \leq M$, $\left|g^{(n)}(x)\right| \leq K$, $x \in [a,b]$*. Then*

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1} K + \|g\|_{p_1} M\right]}{2^{\frac{1}{p_1}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(s+1)^{\frac{1}{p_4}}}. \tag{35}$$

5) Here $p_i > 1 : \sum_{i=1}^{4} \frac{1}{p_i} = 1$*. Let* $f^{(n)}, g^{(n)} \in L_{p_4}([a,b])$*, with* $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ *being s-concave in the second sense. Then*

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g^{(n)}\left(\frac{a+b}{2}\right)\right| + \|g\|_{p_1}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right]}{2^{\left(\frac{1}{p_1}+\frac{2-s}{p_4}\right)}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{36}$$

Proof. Since $f,g : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, with $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$, by Theorem 6 we obtain

$$f(x) = \frac{n}{b-a}\int_a^b f(t)\,dt + F_{n-1}^f(x) + R_n^f(x), \tag{37}$$

where

$$F_{n-1}^f(x) := \sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}\right), \tag{38}$$

and

$$R_n^f(x) := \frac{a-b}{(n-1)!}\int_0^1 \left((x-b)+\lambda(b-a)\right)^{n-1} p(x,\lambda) f^{(n)}(\lambda a + (1-\lambda)b)\,d\lambda$$

$$= \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t)\,dt. \tag{39}$$

Similarly we have

$$g(x) = \frac{n}{b-a}\int_a^b g(t)\,dt + F_{n-1}^g(x) + R_n^g(x). \tag{40}$$

Then

$$f(x)g(x) = \frac{n}{b-a}g(x)\int_a^b f(t)\,dt + g(x)F_{n-1}^f(x) + g(x)R_n^f(x), \tag{41}$$

$$f(x)g(x)=\frac{n}{b-a}f(x)\int_a^b g(t)\,dt+f(x)F_{n-1}^g(x)+f(x)R_n^g(x).$$

Then by integrating we obtain

$$\int_a^b f(x)g(x)\,dx=\frac{n}{b-a}\left(\int_a^b g(x)\,dx\right)\left(\int_a^b f(x)\,dx\right) \tag{42}$$

$$+\int_a^b g(x)F_{n-1}^f(x)\,dx+\int_a^b g(x)R_n^f(x)\,dx,$$

$$\int_a^b f(x)g(x)\,dx=\frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$+\int_a^b f(x)F_{n-1}^g(x)\,dx+\int_a^b f(x)R_n^g(x)\,dx.$$

That is

$$\int_a^b f(x)g(x)\,dx-\frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)=$$

$$\int_a^b g(x)F_{n-1}^f(x)\,dx+\int_a^b g(x)R_n^f(x)\,dx= \tag{43}$$

$$\int_a^b f(x)F_{n-1}^g(x)\,dx+\int_a^b f(x)R_n^g(x)\,dx.$$

Adding the last we derive

$$\Delta_{(f,g)}:=\int_a^b f(x)g(x)\,dx-\frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$-\frac{1}{2}\left[\int_a^b\left(g(x)F_{n-1}^f(x)+f(x)F_{n-1}^g(x)\right)dx\right]$$

$$=\frac{1}{2}\left[\int_a^b\left(f(x)R_n^g(x)+g(x)R_n^f(x)\right)dx\right]. \tag{44}$$

Next we estimate $\Delta_{(f,g)}$.

1) Estimate with respect to $\|\cdot\|_\infty$ and s-convexity in the second sense. We have that

$$\left|\Delta_{(f,g)}\right|\leq\frac{1}{2}\left[\|f\|_\infty\int_a^b\left|R_n^g(x)\right|dx+\|g\|_\infty\int_a^b\left|R_n^f(x)\right|dx\right]. \tag{45}$$

We notice under the assumption $\left|f^{(n)}\right|$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, that

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n}{(n-1)!} \int_0^1 |p(x,\lambda)| \left|f^{(n)}(\lambda a + (1-\lambda) b)\right| d\lambda \tag{46}$$

$$\leq \frac{(b-a)^n}{(n-1)!} \int_0^1 |p(x,\lambda)| \left(\lambda^s \left|f^{(n)}(a)\right| + (1-\lambda)^s \left|f^{(n)}(b)\right|\right) d\lambda$$

$$\leq \frac{M(b-a)^n}{(n-1)!} \int_0^1 |p(x,\lambda)| (\lambda^s + (1-\lambda)^s) d\lambda \tag{47}$$

$$= \frac{M(b-a)^n}{(n-1)!} \left[\int_0^{\frac{b-x}{b-a}} \lambda (\lambda^s + (1-\lambda)^s) d\lambda + \int_{\frac{b-x}{b-a}}^1 (1-\lambda)(\lambda^s + (1-\lambda)^s) d\lambda\right]$$

$$= \frac{M(b-a)^n}{(n-1)!} \left[\int_0^{\frac{b-x}{b-a}} \lambda^{s+1} d\lambda + \int_0^{\frac{b-x}{b-a}} \lambda (1-\lambda)^s d\lambda + \right. \tag{48}$$

$$\left. \int_{\frac{b-x}{b-a}}^1 \left(\lambda^s - \lambda^{s+1}\right) d\lambda + \int_{\frac{b-x}{b-a}}^1 (1-\lambda)^{s+1} d\lambda\right]$$

$$= \frac{M(b-a)^n}{(n-1)!} \left[\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right) + \right.$$

$$\left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)}\right) +$$

$$\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)}\right) +$$

$$\left.\left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\right)\right] =$$

$$\frac{M(b-a)^n}{(n-1)!} \left[\frac{2}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} + \frac{2}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\right.$$

$$\left. - \frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1} - \frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1} + \frac{2}{(s+1)(s+2)}\right] = \tag{49}$$

$$\frac{M(b-a)^n}{(n-1)!} \left[\frac{2}{s+2}\left(\frac{(b-x)^{s+2} + (x-a)^{s+2}}{(b-a)^{s+2}}\right)\right.$$

$$\left. - \frac{1}{s+1}\left(\frac{(b-x)^{s+1} + (x-a)^{s+1}}{(b-a)^{s+1}}\right) + \frac{2}{(s+1)(s+2)}\right] =: \frac{M(b-a)^n}{(n-1)!} \varphi(x,s). \tag{50}$$

We have proved that

$$\left|R_n^f(x)\right| \leq \frac{M(b-a)^n}{(n-1)!}\varphi(x,s), \quad x \in [a,b]. \tag{51}$$

Given that $\left|g^{(n)}\right|$ is s-convex in the second sense and $\left|g^{(n)}(x)\right| \leq K$, $x \in [a,b]$, we similarly get

$$\left|R_n^g(x)\right| \leq \frac{K(b-a)^n}{(n-1)!}\varphi(x,s), \quad x \in [a,b]. \tag{52}$$

We notice that

$$\int_a^b \varphi(x,s)\,dx = \frac{4(b-a)}{(s+2)(s+3)}. \tag{53}$$

Hence it holds

$$\int_a^b \left|R_n^f(x)\right| dx \leq \frac{4M(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}, \tag{54}$$

and similarly

$$\int_a^b \left|R_n^g(x)\right| dx \leq \frac{4K(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}. \tag{55}$$

Therefore we proved that

$$\left|\Delta_{(f,g)}\right| \leq \frac{1}{2}\left[\|f\|_\infty \frac{4K(b-a)^{n+1}}{(n-1)!(s+2)(s+3)} + \|g\|_\infty \frac{4M(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}\right]. \tag{56}$$

Hence we derive

$$\left|\Delta_{(f,g)}\right| \leq \frac{2(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}\left(K\|f\|_\infty + M\|g\|_\infty\right). \tag{57}$$

2) Estimate with respect to $\|\cdot\|_{p_i}$ norms, $i = 1,2,3,4$; $p_i > 1$, $\sum_{i=1}^4 \frac{1}{p_i} = 1$.

We use Hölder's inequality for four functions, and we use the old remainder form, see (14).

So here

$$R_n^f(x) = \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t)\,dt, \tag{58}$$

where $q(x,t)$ as in (15).

We have that

$$\left|\int_a^b g(x) R_n^f(x)\,dx\right| \leq \int_a^b |g(x)|\left|R_n^f(x)\right| dx \leq \tag{59}$$

$$\frac{1}{(n-1)!\,(b-a)}\int_a^b\int_a^b |g(x)|\,|x-t|^{n-1}\,|q(x,t)|\left|f^{(n)}(t)\right|dtdx \leq$$

$$\frac{1}{(n-1)!\,(b-a)}\left(\int_a^b\int_a^b |g(x)|^{p_1}\,dtdx\right)^{\frac{1}{p_1}}\left(\int_a^b\int_a^b |x-t|^{(n-1)p_2}\,dtdx\right)^{\frac{1}{p_2}}\cdot$$

$$\left(\int_a^b\int_a^b |q(x,t)|^{p_3}\,dtdx\right)^{\frac{1}{p_3}}\left(\int_a^b\int_a^b \left|f^{(n)}(t)\right|^{p_4}dtdx\right)^{\frac{1}{p_4}} = \tag{60}$$

$$\frac{1}{(n-1)!\,(b-a)^{\frac{1}{p_2}+\frac{1}{p_3}}}\,\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}\left(\int_a^b\left(\int_a^b |x-t|^{(n-1)p_2}\,dt\right)dx\right)^{\frac{1}{p_2}}\cdot$$

$$\left(\int_a^b\left(\int_a^b |q(x,t)|^{p_3}\,dt\right)dx\right)^{\frac{1}{p_3}} =$$

$$\frac{\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}}{(n-1)!\,(b-a)^{\frac{1}{p_2}+\frac{1}{p_3}}}\cdot\frac{2^{\frac{1}{p_2}}\,(b-a)^{n-1+\frac{2}{p_2}}}{\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}}\cdot \tag{61}$$

$$\frac{2^{\frac{1}{p_3}}\,(b-a)^{1+\frac{2}{p_3}}}{\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}} =$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}\,(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\,\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{62}$$

We have established that

$$\left|\int_a^b g(x)\,R_n^f(x)\,dx\right| \leq$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}\,(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\,\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{63}$$

Similarly we obtain

$$\left|\int_a^b f(x)\,R_n^g(x)\,dx\right| \leq$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}\,(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\,\|f\|_{p_1}\left\|g^{(n)}\right\|_{p_4}}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{64}$$

Consequently by (44), (63), (64), we get

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\left[\|f\|_{p_1}\left\|g^{(n)}\right\|_{p_4}+\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{65}$$

3) We notice also that if $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense, then

$$\left\|g^{(n)}\right\|_{p_4}^{p_4}=\int_a^b\left|g^{(n)}(t)\right|^{p_4}dt=$$

$$(b-a)\int_0^1\left|g^{(n)}(\lambda a+(1-\lambda)b)\right|^{p_4}d\lambda\leq \tag{66}$$

$$(b-a)\int_0^1\left(\lambda^s\left|g^{(n)}(a)\right|+(1-\lambda)^s\left|g^{(n)}(b)\right|\right)^{p_4}d\lambda\leq$$

$$K^{p_4}(b-a)\int_0^1\left(\lambda^s+(1-\lambda)^s\right)^{p_4}d\lambda\leq$$

$$2^{p_4-1}K^{p_4}(b-a)\left[\int_0^1\left(\lambda^{p_4 s}+(1-\lambda)^{p_4 s}\right)d\lambda\right]=\frac{(2K)^{p_4}(b-a)}{(p_4 s+1)}. \tag{67}$$

Hence it holds

$$\left\|g^{(n)}\right\|_{p_4}\leq 2K\left(\frac{b-a}{p_4 s+1}\right)^{\frac{1}{p_4}}. \tag{68}$$

Similarly we find

$$\left\|f^{(n)}\right\|_{p_4}\leq 2M\left(\frac{b-a}{p_4 s+1}\right)^{\frac{1}{p_4}}. \tag{69}$$

So if $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense and $\left|f^{(n)}(x)\right|\leq M$, $\left|g^{(n)}(x)\right|\leq K$, $x\in[a,b]$, then by (65), (68), (69), we get

$$\left|\Delta_{(f,g)}\right|\leq$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}K+\|g\|_{p_1}M\right]}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(p_4 s+1)^{\frac{1}{p_4}}}. \tag{70}$$

4) Next assuming that $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ are s-convex in the second sense, then

$$\left\|g^{(n)}\right\|_{p_4}^{p_4}=(b-a)\int_0^1\left|g^{(n)}(\lambda a+(1-\lambda)b)\right|^{p_4}d\lambda\leq$$

$$(b-a)\int_0^1\left(\lambda^s\left|g^{(n)}(a)\right|^{p_4}+(1-\lambda)^s\left|g^{(n)}(b)\right|^{p_4}\right)d\lambda\leq \tag{71}$$

$$K^{p_4}(b-a)\int_0^1\left(\lambda^s+(1-\lambda)^s\right)d\lambda=\frac{2K^{p_4}(b-a)}{s+1}. \tag{72}$$

That is

$$\left\| g^{(n)} \right\|_{p_4} \leq \frac{2^{\frac{1}{p_4}} K (b-a)^{\frac{1}{p_4}}}{(s+1)^{\frac{1}{p_4}}}. \tag{73}$$

Similarly we get

$$\left\| f^{(n)} \right\|_{p_4} \leq \frac{2^{\frac{1}{p_4}} M (b-a)^{\frac{1}{p_4}}}{(s+1)^{\frac{1}{p_4}}}. \tag{74}$$

So if $\left| f^{(n)} \right|^{p_4}$, $\left| g^{(n)} \right|^{p_4}$ are s-convex in the second sense and $\left| f^{(n)}(x) \right| \leq M$, $\left| g^{(n)}(x) \right| \leq K$, $x \in [a,b]$, then by (65), (73), (74), we derive

$$\left| \Delta_{(f,g)} \right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} K + \|g\|_{p_1} M \right]}{2^{\frac{1}{p_1}} (n-1)! \left((p_2 (n-1)+1)(p_2 (n-1)+2) \right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2) \right)^{\frac{1}{p_3}} (s+1)^{\frac{1}{p_4}}}. \tag{75}$$

5) Assume finally that $\left| g^{(n)} \right|^{p_4}$, $\left| f^{(n)} \right|^{p_4}$ are s-concave in the second sense, then by (26) we get

$$\int_a^b \left| g^{(n)}(t) \right|^{p_4} dt \leq 2^{s-1} \left| g^{(n)} \left(\frac{a+b}{2} \right) \right|^{p_4} (b-a) \tag{76}$$

and

$$\int_a^b \left| f^{(n)}(t) \right|^{p_4} dt \leq 2^{s-1} \left| f^{(n)} \left(\frac{a+b}{2} \right) \right|^{p_4} (b-a). \tag{77}$$

Therefore

$$\left\| g^{(n)} \right\|_{p_4} \leq 2^{\frac{s-1}{p_4}} \left| g^{(n)} \left(\frac{a+b}{2} \right) \right| (b-a)^{\frac{1}{p_4}}, \tag{78}$$

and

$$\left\| f^{(n)} \right\|_{p_4} \leq 2^{\frac{s-1}{p_4}} \left| f^{(n)} \left(\frac{a+b}{2} \right) \right| (b-a)^{\frac{1}{p_4}}. \tag{79}$$

Then by using (65), (78), (79), we get

$$\left| \Delta_{(f,g)} \right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} \left| g^{(n)} \left(\frac{a+b}{2} \right) \right| + \|g\|_{p_1} \left| f^{(n)} \left(\frac{a+b}{2} \right) \right| \right]}{2^{\frac{1}{p_1}+\frac{2-s}{p_4}} (n-1)! \left((p_2 (n-1)+1)(p_2 (n-1)+2) \right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2) \right)^{\frac{1}{p_3}}}. \tag{80}$$

The proof of the theorem is complete. ■

We apply Theorem 13 for $n = 1$.

Proposition 14 *Let $a, b \in \mathbb{R}$, $a < b$, $f, g : [a,b] \to \mathbb{R}$; f, g are absolutely continuous on $[a,b]$, $x \in [a,b]$. Whenever we consider s-convexity or s-concavity of functions we take $a \geq 0$.*

Denote by

$$\Delta^*_{(f,g)} := \int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right). \tag{81}$$

We distinguish the cases:

1) Here $|f'|$, $|g'|$ are s-convex in the second sense and

$$|f'(x)| \leq M, \quad |g'(x)| \leq K, \quad x \in [a,b]. \tag{82}$$

Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{2(b-a)^2}{(s+2)(s+3)}\left(K\|f\|_\infty + M\|g\|_\infty\right). \tag{83}$$

2) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, and f', $g' \in L_{p_4}([a,b])$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^{1+\frac{1}{p_2}+\frac{1}{p_3}}\left[\|f\|_{p_1}\|g'\|_{p_4} + \|g\|_{p_1}\|f'\|_{p_4}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_4}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{84}$$

3) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, with $|f'|$, $|g'|$ are s-convex in the second sense and $|f'(x)| \leq M$, $|g'(x)| \leq K$, $x \in [a,b]$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{2^{\frac{1}{p_3}}(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}K + \|g\|_{p_1}M\right]}{\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(p_4 s+1)^{\frac{1}{p_4}}}. \tag{85}$$

4) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, with $|f'|^{p_4}$, $|g'|^{p_4}$ are s-convex in the second sense and $|f'(x)| \leq M$, $|g'(x)| \leq K$, $x \in [a,b]$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}K + \|g\|_{p_1}M\right]}{2^{\frac{1}{p_1}+\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(s+1)^{\frac{1}{p_4}}}. \tag{86}$$

5) Here $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$. Let $f', g' \in L_{p_4}([a,b])$, with $|f'|^{p_4}$, $|g'|^{p_4}$ being s-concave in the second sense. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g'\left(\frac{a+b}{2}\right)\right| + \|g\|_{p_1}\left|f'\left(\frac{a+b}{2}\right)\right|\right]}{2^{\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{2-s}{p_4}\right)}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{87}$$

We continue with

Theorem 15 *Let $0 \leq a < b$, $f,g : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$; $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$. We assume that $\left|f^{(n)}\right|, \left|g^{(n)}\right|$ are s-logarithmically convex in the second sense, and $\left|f^{(n)}(a)\right|$, $\left|f^{(n)}(b)\right|$, $\left|g^{(n)}(a)\right|$, $\left|g^{(n)}(b)\right| \in (0,1]$. Call $A := \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$, $B := \frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}$, $s \in (0,1]$, and*

$$\psi_s(z) := \begin{cases} \frac{z^s-1}{s\ln z}, & \text{if } z \in (0,\infty) - \{1\}, \\ 1, & \text{if } z = 1. \end{cases} \tag{88}$$

Denote by

$$F_{n-1}^{f}(x) := \sum_{k=1}^{n-1} \left(\frac{n-k}{k!}\right) \left(\frac{f^{(k-1)}(b)(x-b)^{k} - f^{(k-1)}(a)(x-a)^{k}}{b-a}\right), \quad (89)$$

with $F_0^f(x) := 0$*, and*

$$\Delta_{(f,g)} := \int_a^b f(x) g(x) dx - \frac{n}{b-a} \left(\int_a^b f(x) dx\right) \left(\int_a^b g(x) dx\right)$$
$$-\frac{1}{2} \left[\int_a^b \left(g(x) F_{n-1}^{f}(x) + f(x) F_{n-1}^{g}(x)\right) dx\right]. \quad (90)$$

We distinguish the cases:

1) Estimate with respect to $\|\cdot\|_\infty$*. It holds*

$$\left|\Delta_{(f,g)}\right| \le \frac{(b-a)^{n+1}}{2(n-1)!} \left[\|f\|_\infty \left|g^{(n)}(b)\right|^{s} \psi_s(B) + \|g\|_\infty \left|f^{(n)}(b)\right|^{s} \psi_s(A)\right]. \quad (91)$$

2) Let $p_i > 1 : \sum_{i=1}^{4} \frac{1}{p_i} = 1$*. Then*

$$\left|\Delta_{(f,g)}\right| \le$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} \left|g^{(n)}(b)\right|^{s} \left(\psi_s(B^{p_4})\right)^{\frac{1}{p_4}} + \|g\|_{p_1} \left|f^{(n)}(b)\right|^{s} \left(\psi_s(A^{p_4})\right)^{\frac{1}{p_4}}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}} (n-1)! \left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \quad (92)$$

Proof. 1) See also [1]. Here we assume that $0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \le 1$, set $A := \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$, $\left|f^{(n)}\right|$ is s-logarithmically convex in the second sense, $s \in (0,1]$, $\lambda \in [0,1]$.

We observe that

$$\left|R_n^f(x)\right| \overset{\text{(by (46), (11))}}{\le} \frac{(b-a)^n}{(n-1)!} \int_0^1 \left|f^{(n)}(\lambda a + (1-\lambda) b)\right| d\lambda \quad (93)$$

$$\le \frac{(b-a)^n}{(n-1)!} \int_0^1 \left|f^{(n)}(a)\right|^{\lambda^s} \left|f^{(n)}(b)\right|^{(1-\lambda)^s} d\lambda$$

$$\le \frac{(b-a)^n}{(n-1)!} \int_0^1 \left|f^{(n)}(a)\right|^{\lambda s} \left|f^{(n)}(b)\right|^{(1-\lambda)s} d\lambda$$

$$= \frac{(b-a)^n \left|f^{(n)}(b)\right|^{s}}{(n-1)!} \int_0^1 \left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\lambda s} d\lambda \quad (94)$$

$$= \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \int_0^1 A^{\lambda s} d\lambda =: (*) .$$

If $A = 1$, then

$$(*) = \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!}. \tag{95}$$

If $A \neq 1$ we get

$$(*) = \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \int_0^1 e^{s(\ln A)\lambda} d\lambda = \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \frac{1}{s \ln A} \left(A^{s\lambda}|_0^1\right)$$

$$= \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \left(\frac{A^s - 1}{s \ln A}\right). \tag{96}$$

Therefore we derive that

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \psi_s(A). \tag{97}$$

We further assume $0 < \left|g^{(n)}(a)\right|, \left|g^{(n)}(b)\right| \leq 1$, set $B := \frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}$, $\left|g^{(n)}\right|$ is s-log-convex in the second sense. Then, similarly, we obtain that

$$\left|R_n^g(x)\right| \leq \frac{(b-a)^n \left|g^{(n)}(b)\right|^s}{(n-1)!} \psi_s(B). \tag{98}$$

Notice that

$$\int_a^b \left|R_n^f(x)\right| dx \leq \frac{(b-a)^{n+1} \left|f^{(n)}(b)\right|^s}{(n-1)!} \psi_s(A) \tag{99}$$

and

$$\int_a^b \left|R_n^g(x)\right| dx \leq \frac{(b-a)^{n+1} \left|g^{(n)}(b)\right|^s}{(n-1)!} \psi_s(B). \tag{100}$$

It is clear now

$$\left|\Delta_{(f,g)}\right| \leq \frac{1}{2} \left[\|f\|_\infty \left|g^{(n)}(b)\right|^s \psi_s(B) + \|g\|_\infty \left|f^{(n)}(b)\right|^s \psi_s(A)\right] \frac{(b-a)^{n+1}}{(n-1)!}. \tag{101}$$

2) Here we assume that $0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \left|g^{(n)}(a)\right|, \left|g^{(n)}(b)\right| \leq 1$, set $C := \left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{p_4}$ and $D := \left(\frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}\right)^{p_4}$; $\left|f^{(n)}\right|, \left|g^{(n)}\right|$ are s-log-convex in the second sense, $s \in (0,1]$, $\lambda \in [0,1]$.

We have as in (66) that

$$\left\|g^{(n)}\right\|_{p_4}^{p_4} = (b-a) \int_0^1 \left|g^{(n)}(\lambda a + (1-\lambda) b)\right|^{p_4} d\lambda \tag{102}$$

$$\leq (b-a) \int_0^1 \left(\left|g^{(n)}(a)\right|^{\lambda^s}\right)^{p_4} \left(\left|g^{(n)}(b)\right|^{(1-\lambda)^s}\right)^{p_4} d\lambda$$

$$= (b-a)\int_0^1 \left(\left|g^{(n)}(a)\right|^{p_4}\right)^{\lambda^s}\left(\left|g^{(n)}(b)\right|^{p_4}\right)^{(1-\lambda)^s} d\lambda$$

$$\leq (b-a)\int_0^1 \left(\left|g^{(n)}(a)\right|^{p_4}\right)^{\lambda s}\left(\left|g^{(n)}(b)\right|^{p_4}\right)^{(1-\lambda)s} d\lambda$$

$$= (b-a)\left|g^{(n)}(b)\right|^{p_4 s}\int_0^1 \left(\left(\frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}\right)^{p_4}\right)^{s\lambda} d\lambda$$

$$= (b-a)\left|g^{(n)}(b)\right|^{p_4 s}\int_0^1 D^{s\lambda} d\lambda. \tag{103}$$

Hence

$$\left\|g^{(n)}\right\|_{p_4}^{p_4} \leq (b-a)\left|g^{(n)}(b)\right|^{p_4 s}\psi_s(D), \tag{104}$$

and

$$\left\|g^{(n)}\right\|_{p_4} \leq (b-a)^{\frac{1}{p_4}}\left|g^{(n)}(b)\right|^s(\psi_s(D))^{\frac{1}{p_4}}. \tag{105}$$

Similarly we obtain

$$\left\|f^{(n)}\right\|_{p_4} \leq (b-a)^{\frac{1}{p_4}}\left|f^{(n)}(b)\right|^s(\psi_s(C))^{\frac{1}{p_4}}. \tag{106}$$

Finally using (33) we find

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g^{(n)}(b)\right|^s(\psi_s(D))^{\frac{1}{p_4}} + \|g\|_{p_1}\left|f^{(n)}(b)\right|^s(\psi_s(C))^{\frac{1}{p_4}}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{107}$$

The proof of the theorem now is complete. ■

We apply Theorem 15 for $n = 1$.

Proposition 16 *Let $0 \leq a < b$, $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$. Assume that $|f'|, |g'|$ are s-log-convex in the second sense, and $|f'(a)|, |f'(b)|, |g'(a)|, |g'(b)| \in (0,1]$. Call $A^* := \frac{|f'(a)|}{|f'(b)|}$, $B^* := \frac{|g'(a)|}{|g'(b)|}$, $s \in (0,1]$, and*

$$\psi_s(z) := \begin{cases} \frac{z^s-1}{s\ln z}, & \text{if } z \in (0,\infty)-\{1\}, \\ 1, & \text{if } z = 1. \end{cases} \tag{108}$$

Denote by

$$\Delta^*_{(f,g)} := \int_a^b f(x)g(x)dx - \frac{1}{b-a}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right). \tag{109}$$

We distinguish the cases:

1) It holds

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^2}{2}\left[\|f\|_\infty \left|g'(b)\right|^s \psi_s(B^*) + \|g\|_\infty \left|f'(b)\right|^s \psi_s(A^*)\right]. \quad (110)$$

2) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g'(b)\right|^s \left(\psi_s\left(B^{*p_4}\right)\right)^{\frac{1}{p_4}} + \|g\|_{p_1}\left|f'(b)\right|^s \left(\psi_s\left(A^{*p_4}\right)\right)^{\frac{1}{p_4}}\right]}{2^{1-\frac{1}{p_3}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \quad (111)$$

Next we present s-convexity general Ostrowski type inequalities.

Theorem 17 *Let $a, b \in \mathbb{R}$, $a < b$, $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$; $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $x \in [a,b]$. When we consider s-convexity or s-concavity of functions we take $a \geq 0$.*

Denote by

$$F^f_{n-1}(x) := \sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}\right), \quad (112)$$

with $F^f_0(x) := 0$, and

$$R^f_n(x) := f(x) - \frac{n}{b-a}\int_a^b f(t)\,dt - F^f_{n-1}(x). \quad (113)$$

We distinguish the cases:

1) Here $\left|f^{(n)}\right|$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$. Then

$$\left|R^f_n(x)\right| \leq \frac{M(b-a)^n}{(n-1)!}\left[\frac{2}{(s+2)}\left(\frac{(b-x)^{s+2}+(x-a)^{s+2}}{(b-a)^{s+2}}\right)\right. \quad (114)$$

$$\left. -\frac{1}{(s+1)}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right) + \frac{2}{(s+1)(s+2)}\right].$$

2) Let $p_i > 1 : \sum_{i=1}^3 \frac{1}{p_i} = 1$, and $f^{(n)} \in L_{p_3}([a,b])$. Then

$$\left|R^f_n(x)\right| \leq \frac{\left\|f^{(n)}\right\|_{p_3}}{(n-1)!(b-a)}\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\cdot$$

$$\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \quad (115)$$

3) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. If $\left|f^{(n)}\right|$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, then

$$\left|R_n^f(x)\right| \leq \frac{2M(b-a)^{\frac{1}{p_3}-1}}{(p_3 s+1)^{\frac{1}{p_3}}(n-1)!} \cdot \tag{116}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

4) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. If $\left|f^{(n)}\right|^{p_3}$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, then

$$\left|R_n^f(x)\right| \leq \frac{2^{\frac{1}{p_3}}M(b-a)^{\frac{1}{p_3}-1}}{(s+1)^{\frac{1}{p_3}}(n-1)!} \cdot \tag{117}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

5) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$, $f^{(n)} \in L_{p_3}([a,b])$, with $\left|f^{(n)}\right|^{p_3}$ being s-concave in the second sense. Then

$$\left|R_n^f(x)\right| \leq \frac{2^{\frac{s-1}{p_3}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|(b-a)^{\frac{1}{p_3}-1}}{(n-1)!} \cdot \tag{118}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

Proof. 1) As in (50), (51), we get

$$\left|R_n^f(x)\right| \leq \frac{M(b-a)^n}{(n-1)!}\varphi(x,s), \quad x \in [a,b]. \tag{119}$$

2) Let $p_i > 1, \sum_{i=1}^{3} \frac{1}{p_i} = 1$. Then

$$\left|R_n^f(x)\right| \leq \frac{1}{(n-1)!(b-a)}\int_a^b |x-t|^{n-1}|q(x,t)|\left|f^{(n)}(t)\right|dt$$

$$\leq \frac{1}{(n-1)!(b-a)}\left(\int_a^b |x-t|^{p_1(n-1)}dt\right)^{\frac{1}{p_1}}\left(\int_a^b |q(x,t)|^{p_2}dt\right)^{\frac{1}{p_2}}\left\|f^{(n)}\right\|_{p_3} \tag{120}$$

$$= \frac{\left\|f^{(n)}\right\|_{p_3}}{(n-1)!(b-a)}\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}.$$

$$\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{121}$$

3) If $\left|f^{(n)}\right|$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, we get

$$\left\|f^{(n)}\right\|_{p_3} \leq 2M\left(\frac{b-a}{p_3 s+1}\right)^{\frac{1}{p_3}}, \tag{122}$$

see also (69).

4) If $\left|f^{(n)}\right|^{p_3}$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, we get

$$\left\|f^{(n)}\right\|_{p_3} \leq \frac{2^{\frac{1}{p_3}} M (b-a)^{\frac{1}{p_3}}}{(s+1)^{\frac{1}{p_3}}}, \tag{123}$$

see also (74).

5) If $\left|f^{(n)}\right|^{p_3}$ is s-concave in the second sense, then by (26) we find

$$\left\|f^{(n)}\right\|_{p_3} \leq 2^{\frac{s-1}{p_3}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|(b-a)^{\frac{1}{p_3}}, \tag{124}$$

see also (79). ■

We apply Theorem 17 for $n=1$.

Proposition 18 *Let $a,b \in \mathbb{R}$, $a<b$, $f:[a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, $x \in [a,b]$. In case of s-convexity or s-concavity we take $a \geq 0$.*

Denote by

$$R_1^f(x) = f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{125}$$

We distinguish the cases:

1) Here $|f'|$ is s-convex in the second sense and $|f'(x)| \leq M$, $x \in [a,b]$. Then

$$\left|R_1^f(x)\right| \leq M(b-a)\left[\frac{2}{(s+2)}\left(\frac{(b-x)^{s+2}+(x-a)^{s+2}}{(b-a)^{s+2}}\right)\right. \tag{126}$$

$$\left.-\frac{1}{(s+1)}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\right].$$

2) Let $p_i>1 : \sum_{i=1}^{3}\frac{1}{p_i}=1$, and $f' \in L_{p_3}([a,b])$. Then

$$\left|R_1^f(x)\right| \leq \|f'\|_{p_3}(b-a)^{\frac{1}{p_1}-1}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{127}$$

3) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. If $|f'|$ is s-convex in the second sense and $|f'(x)| \leq M$, $x \in [a,b]$, then

$$\left|R_1^f(x)\right| \leq \frac{2M}{(p_3 s+1)^{\frac{1}{p_3}} (b-a)^{\frac{1}{p_2}}} \left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{128}$$

4) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. If $|f'|^{p_3}$ is s-convex in the second sense and $|f'(x)| \leq M$, $x \in [a,b]$, then

$$\left|R_1^f(x)\right| \leq \frac{2^{\frac{1}{p_3}} M}{(s+1)^{\frac{1}{p_3}} (b-a)^{\frac{1}{p_2}}} \left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{129}$$

5) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$, $f' \in L_{p_3}([a,b])$, with $|f'|^{p_3}$ being s-concave in the second sense. Then

$$\left|R_1^f(x)\right| \leq \frac{2^{\frac{s-1}{p_3}} \left|f'\left(\frac{a+b}{2}\right)\right|}{(b-a)^{\frac{1}{p_2}}} \left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{130}$$

We continue with

Theorem 19 *Let $0 \leq a < b$, $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$; $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $x \in [a,b]$. We assume that $\left|f^{(n)}\right|$ is* s-log*-convex in the second sense, and $\left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \in (0,1]$. Call $A := \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$, $s \in (0,1]$, and*

$$\psi_s(z) := \begin{cases} \frac{z^s-1}{s\ln z}, & \text{if } z \in (0,\infty)-\{1\}, \\ 1, & \text{if } z=1. \end{cases} \tag{131}$$

Denote by

$$F_{n-1}^f(x) := \sum_{k=1}^{n-1} \left(\frac{n-k}{k!}\right) \left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}\right), \tag{132}$$

with $F_0^f(x) := 0$, and

$$R_n^f(x) := f(x) - \frac{n}{b-a}\int_a^b f(t)\,dt - F_{n-1}^f(x). \tag{133}$$

We distinguish the cases:

1) It holds generally

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \psi_s(A). \tag{134}$$

2) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. Then

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^{\frac{1}{p_3}-1}}{(n-1)!} \left|f^{(n)}(b)\right|^s \left(\psi_s\left(A^{p_3}\right)\right)^{\frac{1}{p_3}}. \tag{135}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

Proof. 1) As in (97) we get

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!}\psi_s(A). \tag{136}$$

2) As in (106) we obtain

$$\left\|f^{(n)}\right\|_{p_3} \leq (b-a)^{\frac{1}{p_3}}\left|f^{(n)}(b)\right|^s \left(\psi_s\left(A^{p_3}\right)\right)^{\frac{1}{p_3}}. \tag{137}$$

Then we use (115). ■

At last we apply Theorem 19 for $n=1$.

Proposition 20 *Let $0 \leq a < b$, $f:[a,b] \to \mathbb{R}$ absolutely continuous on $[a,b]$, $x \in [a,b]$. We assume that $|f'|$ is s-*log*-convex in the second sense, and $|f'(a)|, |f'(b)| \in (0,1]$. Call $A^* := \frac{|f'(a)|}{|f'(b)|}$, $s \in (0,1]$, and $\psi_s(z)$ as in (131).*

Denote by

$$R_1^f(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{138}$$

We distinguish the cases:

1) It holds

$$\left|R_1^f(x)\right| \leq (b-a)\left|f'(b)\right|^s \psi_s\left(A^*\right). \tag{139}$$

2) Let $p_i > 1 : \sum_{i=1}^3 \frac{1}{p_i} = 1$. Then

$$\left|R_1^f(x)\right| \leq (b-a)^{-\frac{1}{p_2}}\left|f'(b)\right|^s \left(\psi_s\left(A^{*p_3}\right)\right)^{\frac{1}{p_3}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{140}$$

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