

# Univariate Fractional Polya type integral inequalities

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## Abstract

Here we establish a series of various fractional Polya type integral inequalities with the help of generalised right and left fractional derivatives. We give application to complex valued functions defined on the unit circle.

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## 1 Introduction

We mention the following famous Polya's integral inequality, see [12], [13, p. 62], [14] and [15, p. 83].

**Theorem 1** *Let  $f(x)$  be differentiable and not identically a constant on  $[a, b]$  with  $f(a) = f(b) = 0$ . Then there exists at least one point  $\xi \in [a, b]$  such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x) dx. \quad (1)$$

In [16], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

**Theorem 2** *Let  $f(x)$  be differentiable and not identically constant on  $[a, b]$  with  $f(a) = f(b) = 0$  and  $M = \sup_{x \in [a, b]} |f'(x)|$ . Then*

$$\left| \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{4} M, \quad (2)$$

where  $\frac{(b-a)^2}{4}$  in (2) is the best constant.

The above motivate the current paper.

In this article we present univariate fractional Polya type integral inequalities in various cases, similar to (2).

For the last we need the following fractional calculus background.

Let  $\alpha > 0$ ,  $m = [\alpha]$ ,  $\beta = \alpha - m$ ,  $0 < \beta < 1$ ,  $f \in C([a, b])$ ,  $[a, b] \subset \mathbb{R}$ ,  $x \in [a, b]$ . The gamma function  $\Gamma$  is given by  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ . We define the left Riemann-Liouville integral

$$(J_{\alpha}^{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (3)$$

$a \leq x \leq b$ . We define the subspace  $C_{a+}^{\alpha}([a, b])$  of  $C^m([a, b])$ :

$$C_{a+}^{\alpha}([a, b]) = \left\{ f \in C^m([a, b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a, b]) \right\}. \quad (4)$$

For  $f \in C_{a+}^{\alpha}([a, b])$ , we define the left generalized  $\alpha$ -fractional derivative of  $f$  over  $[a, b]$  as

$$D_{a+}^{\alpha} f := \left( J_{1-\beta}^{a+} f^{(m)} \right)', \quad (5)$$

see [1], p. 24. Canavati first in [5], introduced the above over  $[0, 1]$ .

Notice that  $D_{a+}^{\alpha} f \in C([a, b])$ .

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [5] the same over  $[0, 1]$  that appeared first.

**Theorem 3** *Let  $f \in C_{a+}^{\alpha}([a, b])$ .*

*(i) If  $\alpha \geq 1$ , then*

$$f(x) = f(a) + f'(a)(x-a) + f''(a) \frac{(x-a)^2}{2} + \dots + f^{(m-1)}(a) \frac{(x-a)^{m-1}}{(m-1)!} \quad (6)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (D_{a+}^{\alpha} f)(t) dt, \quad \text{all } x \in [a, b].$$

*(ii) If  $0 < \alpha < 1$ , we have*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (D_{a+}^{\alpha} f)(t) dt, \quad \text{all } x \in [a, b]. \quad (7)$$

Furthermore we need:

Let again  $\alpha > 0$ ,  $m = [\alpha]$ ,  $\beta = \alpha - m$ ,  $f \in C([a, b])$ , call the right Riemann-Liouville fractional integral operator by

$$(J_{\alpha}^{b-} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad (8)$$

$x \in [a, b]$ , see also [2], [8], [9], [10], [17]. Define the subspace of functions

$$C_{b-}^{\alpha}([a, b]) := \left\{ f \in C^m([a, b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a, b]) \right\}. \quad (9)$$

Define the right generalized  $\alpha$ -fractional derivative of  $f$  over  $[a, b]$  as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left( J_{b-}^{1-\beta} f^{(m)} \right)', \quad (10)$$

see [2]. We set  $D_{b-}^0 f = f$ . Notice that  $D_{b-}^{\alpha} f \in C([a, b])$ .

From [2], we need the following right Taylor fractional formula.

**Theorem 4** *Let  $f \in C_{b-}^{\alpha}([a, b])$ ,  $\alpha > 0$ ,  $m := [\alpha]$ . Then*

(i) *If  $\alpha \geq 1$ , we get*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + (J_{b-}^{\alpha} D_{b-}^{\alpha} f)(x), \quad \text{all } x \in [a, b]. \quad (11)$$

(ii) *If  $0 < \alpha < 1$ , we get*

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} (D_{b-}^{\alpha} f)(t) dt, \quad \text{all } x \in [a, b]. \quad (12)$$

We need from [3]

**Definition 5** *Let  $f \in C([a, b])$ ,  $x \in [a, b]$ ,  $\alpha > 0$ ,  $m := [a]$ . Assume that  $f \in C_{b-}^{\alpha}([\frac{a+b}{2}, b])$  and  $f \in C_{a+}^{\alpha}([a, \frac{a+b}{2}])$ . We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \quad (13)$$

In [4] we proved the following fractional Polya type integral inequality without any boundary conditions.

**Theorem 6** *Let  $0 < \alpha < 1$ ,  $f \in C([a, b])$ . Assume  $f \in C_{a+}^{\alpha}([a, \frac{a+b}{2}])$  and  $f \in C_{b-}^{\alpha}([\frac{a+b}{2}, b])$ . Set*

$$M_1(f) = \max \left\{ \|D_{a+}^{\alpha} f\|_{\infty, [a, \frac{a+b}{2}]}, \|D_{b-}^{\alpha} f\|_{\infty, [\frac{a+b}{2}, b]} \right\}. \quad (14)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \leq \frac{\left( \|D_{a+}^{\alpha} f\|_{\infty, [a, \frac{a+b}{2}]} + \|D_{b-}^{\alpha} f\|_{\infty, [\frac{a+b}{2}, b]} \right) \left( \frac{b-a}{2} \right)^{\alpha+1}}{\Gamma(\alpha+2)} \leq M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2) 2^{\alpha}}. \quad (15)$$

Inequalities (15), (16) are sharp, namely they are attained by

$$f_*(x) = \begin{cases} (x-a)^{\alpha}, & x \in [a, \frac{a+b}{2}] \\ (b-x)^{\alpha}, & x \in [\frac{a+b}{2}, b] \end{cases}, \quad 0 < \alpha < 1. \quad (17)$$

Clearly here non zero constant functions  $f$  are excluded.

The last result also motivates this work.

**Remark 7** (see [4]) When  $\alpha \geq 1$ , thus  $m = [\alpha] \geq 1$ , and by assuming that  $f^{(k)}(a) = f^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, m-1$ , we can prove the same statements (15), (16), (17) as in Theorem 6. If we set there  $\alpha = 1$  we derive exactly Theorem 2. So we have generalized Theorem 2. Again here  $f^{(m)}$  cannot be a constant different than zero, equivalently,  $f$  cannot be a non-trivial polynomial of degree  $m$ .

We continue here with other interesting univariate fractional Polya type inequalities.

## 2 Main Results

We present our first main result.

**Theorem 8** Let  $\alpha \geq 1$ ,  $m = [\alpha]$ ,  $f \in C([a, b])$ . Assume  $f \in C_{a+}^\alpha([a, \frac{a+b}{2}])$  and  $f \in C_{b-}^\alpha([\frac{a+b}{2}, b])$ , such that  $f^{(k)}(a) = f^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, m-1$ . Set

$$M_2(f) = \max \left\{ \|D_{a+}^\alpha f\|_{L_1([a, \frac{a+b}{2}])}, \|D_{b-}^\alpha f\|_{L_1([\frac{a+b}{2}, b])} \right\}. \quad (18)$$

Then

$$\left| \int_a^b f(x) dx \right| \leq \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \|D^\alpha f\|_{L_1([a, b])} \leq \frac{M_2(f)}{\Gamma(\alpha+1) 2^{\alpha-1}} (b-a)^\alpha. \quad (20)$$

Here  $f$  cannot be a non-trivial polynomial of degree  $m$ .

**Proof.** By assumption and Theorem 3 we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (D_{a+}^\alpha f)(t) dt, \quad \text{all } x \in \left[ a, \frac{a+b}{2} \right], \quad (21)$$

also it holds, by assumption and Theorem 4, that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} (D_{b-}^\alpha f)(t) dt, \quad \text{all } x \in \left[ \frac{a+b}{2}, b \right]. \quad (22)$$

By (21) we get

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |(D_{a+}^\alpha f)(t)| dt \\ &\leq \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} |(D_{a+}^\alpha f)(t)| dt, \quad \text{all } x \in \left[ a, \frac{a+b}{2} \right]. \end{aligned} \quad (23)$$

By (22) we derive

$$\begin{aligned} |f(x)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} |(D_{b-}^\alpha f)(t)| dt \\ &\leq \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b |(D_{b-}^\alpha f)(t)| dt, \quad \text{all } x \in \left[ \frac{a+b}{2}, b \right]. \end{aligned} \quad (24)$$

Consequently we have

$$\begin{aligned} \int_a^{\frac{a+b}{2}} |f(x)| dx &\leq \frac{1}{\Gamma(\alpha)} \left( \int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1} dx \right) \|D_{a+}^\alpha f\|_{L_1([a, \frac{a+b}{2}])} \\ &= \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \|D_{a+}^\alpha f\|_{L_1([a, \frac{a+b}{2}])}, \end{aligned} \quad (25)$$

and

$$\begin{aligned} \int_{\frac{a+b}{2}}^b |f(x)| dx &\leq \frac{1}{\Gamma(\alpha)} \left( \int_{\frac{a+b}{2}}^b (b-x)^{\alpha-1} dx \right) \|D_{b-}^\alpha f\|_{L_1([\frac{a+b}{2}, b])} \\ &= \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \|D_{b-}^\alpha f\|_{L_1([\frac{a+b}{2}, b])}. \end{aligned} \quad (26)$$

Therefore we obtain by adding (25) and (26) that

$$\begin{aligned} \int_a^b |f(x)| dx &\leq \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \left[ \|D_{a+}^\alpha f\|_{L_1([a, \frac{a+b}{2}])} + \|D_{b-}^\alpha f\|_{L_1([\frac{a+b}{2}, b])} \right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \|D^\alpha f\|_{L_1([a, b])} \leq \\ &\quad \max \left\{ \|D_{a+}^\alpha f\|_{L_1([a, \frac{a+b}{2}])}, \|D_{b-}^\alpha f\|_{L_1([\frac{a+b}{2}, b])} \right\} \frac{(b-a)^\alpha}{\Gamma(\alpha+1) 2^{\alpha-1}}, \end{aligned} \quad (28)$$

proving the claim. ■

We continue with

**Theorem 9** Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ ,  $\alpha > \frac{1}{q}$ ,  $m = [\alpha]$ ,  $f \in C([a, b])$ . Assume  $f \in C_{a+}^\alpha([a, \frac{a+b}{2}])$  and  $f \in C_{b-}^\alpha([\frac{a+b}{2}, b])$ , such that  $f^{(k)}(a) = f^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, m-1$ . When  $\frac{1}{q} < \alpha < 1$ , the last boundary conditions are void.

Set

$$M_3(f) = \max \left\{ \|D_{a+}^\alpha f\|_{L_q([a, \frac{a+b}{2}])}, \|D_{b-}^\alpha f\|_{L_q([\frac{a+b}{2}, b])} \right\}. \quad (29)$$

Then

$$\int_a^b |f(x)| dx \leq \frac{1}{\Gamma(\alpha) (p(\alpha-1) + 1)^{\frac{1}{p}} \left( \alpha + \frac{1}{p} \right)} \left( \frac{b-a}{2} \right)^{\alpha + \frac{1}{p}}. \quad (30)$$

$$\begin{aligned} & \left[ \|D_{a+}^{\alpha} f\|_{L_q([a, \frac{a+b}{2}])} + \|D_{b-}^{\alpha} f\|_{L_q([\frac{a+b}{2}, b])} \right] \leq \\ & \frac{M_3(f)}{\Gamma(\alpha) (p(\alpha-1) + 1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right) 2^{\alpha - \frac{1}{q}}} (b-a)^{\alpha + \frac{1}{p}}. \end{aligned} \quad (31)$$

Again here  $f$  cannot be a non-trivial polynomial of degree  $m$ .

**Proof.** By Theorem 3 we have

$$\begin{aligned} |f(x)| & \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |(D_{a+}^{\alpha} f)(t)| dt \\ & \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^x (x-t)^{p(\alpha-1)} dt \right)^{\frac{1}{p}} \left( \int_a^x |(D_{a+}^{\alpha} f)(t)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{(p(\alpha-1)+1)}{p}}}{(p(\alpha-1) + 1)^{\frac{1}{p}}} \|D_{a+}^{\alpha} f\|_{L_q([a, \frac{a+b}{2}])}, \quad \text{for all } x \in \left[ a, \frac{a+b}{2} \right]. \end{aligned} \quad (32)$$

That is

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1) + 1)^{\frac{1}{p}}} \|D_{a+}^{\alpha} f\|_{L_q([a, \frac{a+b}{2}])}, \quad \text{for all } x \in \left[ a, \frac{a+b}{2} \right] \quad (33)$$

Similarly from Theorem 4 we get

$$\begin{aligned} |f(x)| & \leq \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} |(D_{b-}^{\alpha} f)(t)| dt \\ & \leq \frac{1}{\Gamma(\alpha)} \left( \int_x^b (t-x)^{p(\alpha-1)} dt \right)^{\frac{1}{p}} \left( \int_x^b |(D_{b-}^{\alpha} f)(t)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1) + 1)^{\frac{1}{p}}} \|D_{b-}^{\alpha} f\|_{L_q([\frac{a+b}{2}, b])}, \quad \text{for all } x \in \left[ \frac{a+b}{2}, b \right]. \end{aligned} \quad (34)$$

That is

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1) + 1)^{\frac{1}{p}}} \|D_{b-}^{\alpha} f\|_{L_q([\frac{a+b}{2}, b])}, \quad \text{for all } x \in \left[ \frac{a+b}{2}, b \right]. \quad (36)$$

Consequently we obtain by (33) that

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} |f(x)| dx \leq \\ & \frac{1}{\Gamma(\alpha) (p(\alpha-1) + 1)^{\frac{1}{p}}} \left( \int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1+\frac{1}{p}} dx \right) \|D_{a+}^{\alpha} f\|_{L_q([a, \frac{a+b}{2}])} = \end{aligned}$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\|D_{a+}^{\alpha}f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}. \quad (37)$$

Similarly it holds by (36) that

$$\begin{aligned} & \int_{\frac{a+b}{2}}^b |f(x)| dx \leq \\ & \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_{\frac{a+b}{2}}^b (b-x)^{\alpha-1+\frac{1}{p}} dx\right)\|D_{b-}^{\alpha}f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} = \\ & \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\|D_{b-}^{\alpha}f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}. \end{aligned} \quad (38)$$

Adding (37) and (38) we have

$$\begin{aligned} \int_a^b |f(x)| dx & \leq \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}} \\ & \cdot \left[\|D_{a+}^{\alpha}f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)} + \|D_{b-}^{\alpha}f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right] \leq \\ & \frac{\max\left\{\|D_{a+}^{\alpha}f\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}, \|D_{b-}^{\alpha}f\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right\}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}, \end{aligned} \quad (40)$$

proving the claim. ■

Combining Theorem 6, Remark 7, Theorem 8 and Theorem 9, we obtain

**Theorem 10** *Let any  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ ,  $\alpha \geq 1$ ,  $m = [\alpha]$ ,  $f \in C([a, b])$ . Assume  $f \in C_{a+}^{\alpha}([a, \frac{a+b}{2}])$  and  $f \in C_{b-}^{\alpha}([\frac{a+b}{2}, b])$ , such that  $f^{(k)}(a) = f^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, m-1$ .*

Then

$$\begin{aligned} & \int_a^b |f(x)| dx \leq \\ & \min \left\{ \frac{\left(\|D_{a+}^{\alpha}f\|_{\infty, \left[a, \frac{a+b}{2}\right]} + \|D_{b-}^{\alpha}f\|_{\infty, \left[\frac{a+b}{2}, b\right]}\right)}{\Gamma(\alpha+2)}\left(\frac{b-a}{2}\right)^{\alpha+1}, \right. \\ & \quad \frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{2}\right)^{\alpha}\|D^{\alpha}f\|_{L_1([a, b])}, \\ & \quad \left. \frac{\left[\|D_{a+}^{\alpha}f\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)} + \|D_{b-}^{\alpha}f\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)}\right]}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\right\} \leq \end{aligned} \quad (41)$$

$$\min \left\{ M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^\alpha}, \frac{M_2(f)}{\Gamma(\alpha+1)2^{\alpha-1}} (b-a)^\alpha, \right. \\ \left. \frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right) 2^{\alpha-\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{p}} \right\}, \quad (42)$$

where  $M_1(f)$  as in (14),  $M_2(f)$  as in (18) and  $M_3(f)$  as in (29).  
Here  $f$  cannot be a non-trivial polynomial of degree  $m$ .

**Corollary 11** Here all as in Theorem 10. Then

$$\left| \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{b-a} \int_a^b |f(x)| dx \leq \\ \min \left\{ \frac{\left( \|D_{a+}^\alpha f\|_{\infty, [a, \frac{a+b}{2}]} + \|D_{b-}^\alpha f\|_{\infty, [\frac{a+b}{2}, b]} \right)}{\Gamma(\alpha+2)2^{\alpha+1}} (b-a)^\alpha, \quad (43) \right. \\ \frac{1}{2^\alpha \Gamma(\alpha+1)} (b-a)^{\alpha-1} \|D^\alpha f\|_{L_1([a,b])}, \\ \left. \frac{\left[ \|D_{a+}^\alpha f\|_{L_q([a, \frac{a+b}{2}])} + \|D_{b-}^\alpha f\|_{L_q([\frac{a+b}{2}, b])} \right]}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right) 2^{\alpha+\frac{1}{p}}} (b-a)^{\alpha+\frac{1}{p}-1} \right\} \leq \\ \min \left\{ M_1(f) \frac{(b-a)^\alpha}{\Gamma(\alpha+2)2^\alpha}, \frac{M_2(f)}{\Gamma(\alpha+1)2^{\alpha-1}} (b-a)^{\alpha-1}, \right. \\ \left. \frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right) 2^{\alpha-\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{p}-1} \right\}. \quad (44)$$

In 1938, A. Ostrowski [11] proved the following important inequality:

**Theorem 12** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.,  $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty$ . Then

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty, \quad (45)$$

for any  $x \in [a, b]$ . The constant  $\frac{1}{4}$  is the best possible.



In (45) for  $x = \frac{a+b}{2}$  we get

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \left(\frac{b-a}{4}\right) \|f'\|_\infty. \quad (46)$$

We have proved the following

**Theorem 13** *Let  $f \in C^1([a, b])$ , with  $f\left(\frac{a+b}{2}\right) = 0$ . Then*

$$\left| \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{4} \|f'\|_\infty, \quad (47)$$

where the constant  $\frac{1}{4}$  is the best possible.

So we reproved (2) with only one initial condition.

### 3 Application

Inequalities for complex valued functions defined on the unit circle were studied extensively by S. Dragomir, see [6], [7].

We give here our version for these functions involved in a Polya type inequality, by applying a result of this article.

Let  $t \in [a, b] \subseteq [0, 2\pi)$ , the unit circle arc  $A = \{z \in \mathbb{C} : z = e^{it}, t \in [a, b]\}$ , and  $f : A \rightarrow \mathbb{C}$  be a continuous function. Clearly here there exist functions  $u, v : A \rightarrow \mathbb{R}$  continuous, the real and the complex part of  $f$ , respectively, such that

$$f(e^{it}) = u(e^{it}) + iv(e^{it}). \quad (48)$$

So that  $f$  is continuous, iff  $u, v$  are continuous.

Call  $g(t) = f(e^{it})$ ,  $l_1(t) = u(e^{it})$ ,  $l_2(t) = v(e^{it})$ ,  $t \in [a, b]$ ; so that  $g : [a, b] \rightarrow \mathbb{C}$  and  $l_1, l_2 : [a, b] \rightarrow \mathbb{R}$  are continuous functions in  $t$ .

If  $g$  has a derivative with respect to  $t$ , then  $l_1, l_2$  have also derivatives with respect to  $t$ . In that case

$$f_t(e^{it}) = u_t(e^{it}) + iv_t(e^{it}), \quad (49)$$

(i.e.  $g'(t) = l_1'(t) + il_2'(t)$ ), which means

$$f_t(\cos t + i \sin t) = u_t(\cos t + i \sin t) + iv_t(\cos t + i \sin t). \quad (50)$$

Let us call  $x = \cos t$ ,  $y = \sin t$ . Then

$$\begin{aligned} u_t(e^{it}) &= u_t(\cos t + i \sin t) = u_t(x + iy) = u_t(x, y) = \\ &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial u(e^{it})}{\partial x} (-\sin t) + \frac{\partial u(e^{it})}{\partial y} \cos t. \end{aligned} \quad (51)$$

Similarly we find that

$$v_t(e^{it}) = \frac{\partial v(e^{it})}{\partial x}(-\sin t) + \frac{\partial v(e^{it})}{\partial y} \cos t. \quad (52)$$

Since  $g$  is continuous on  $[a, b]$ , then  $\int_a^b f(e^{it}) dt$  exists. Furthermore it holds

$$\int_a^b f(e^{it}) dt = \int_a^b u(e^{it}) dt + i \int_a^b v(e^{it}) dt. \quad (53)$$

We have here that

$$\left| \int_a^b f(e^{it}) dt \right| \leq \int_a^b |f(e^{it})| dt \leq \quad (54)$$

$$\int_a^b |u(e^{it})| dt + \int_a^b |v(e^{it})| dt = \int_a^b |l_1(t)| dt + \int_a^b |l_2(t)| dt. \quad (55)$$

We give the following application of Theorem 10.

**Theorem 14** *Let  $f \in C(A, \mathbb{C})$ ,  $[a, b] \subseteq [0, 2\pi]$ ; any  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ ,  $\alpha \geq 1$ ,  $m = [\alpha]$ . Assume  $l_1, l_2 \in C_{a+}^\alpha([a, \frac{a+b}{2}])$  and  $l_1, l_2 \in C_{b-}^\alpha([\frac{a+b}{2}, b])$ , such that  $l_1^{(k)}(a) = l_2^{(k)}(a) = l_1^{(k)}(b) = l_2^{(k)}(b) = 0$ ,  $k = 0, 1, \dots, m-1$ . Then*

$$\begin{aligned} & \left| \int_a^b f(e^{it}) dt \right| \leq \int_a^b |f(e^{it})| dt \leq \\ & \min \left\{ \frac{\left( \|D_{a+}^\alpha l_1\|_{\infty, [a, \frac{a+b}{2}]} + \|D_{b-}^\alpha l_1\|_{\infty, [\frac{a+b}{2}, b]} \right) \left( \frac{b-a}{2} \right)^{\alpha+1}}{\Gamma(\alpha+2)}, \quad (56) \right. \\ & \quad \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \|D^\alpha l_1\|_{L_1([a, b])}, \\ & \quad \left. \frac{\left[ \|D_{a+}^\alpha l_1\|_{L_q([a, \frac{a+b}{2}])} + \|D_{b-}^\alpha l_1\|_{L_q([\frac{a+b}{2}, b])} \right] \left( \frac{b-a}{2} \right)^{\alpha+\frac{1}{p}}}{\Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} \left( \alpha + \frac{1}{p} \right)} \right\} + \\ & \min \left\{ \frac{\left( \|D_{a+}^\alpha l_2\|_{\infty, [a, \frac{a+b}{2}]} + \|D_{b-}^\alpha l_2\|_{\infty, [\frac{a+b}{2}, b]} \right) \left( \frac{b-a}{2} \right)^{\alpha+1}}{\Gamma(\alpha+2)}, \right. \\ & \quad \frac{1}{\Gamma(\alpha+1)} \left( \frac{b-a}{2} \right)^\alpha \|D^\alpha l_2\|_{L_1([a, b])}, \\ & \quad \left. \frac{\left[ \|D_{a+}^\alpha l_2\|_{L_q([a, \frac{a+b}{2}])} + \|D_{b-}^\alpha l_2\|_{L_q([\frac{a+b}{2}, b])} \right] \left( \frac{b-a}{2} \right)^{\alpha+\frac{1}{p}}}{\Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} \left( \alpha + \frac{1}{p} \right)} \right\} \leq \end{aligned}$$

$$\begin{aligned}
& \min \left\{ M_1(l_1) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^\alpha}, \frac{M_2(l_1)}{\Gamma(\alpha+1)2^{\alpha-1}} (b-a)^\alpha, \right. \\
& \left. \frac{M_3(l_1)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right) 2^{\alpha-\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{p}} \right\} + \\
& \min \left\{ M_1(l_2) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^\alpha}, \frac{M_2(l_2)}{\Gamma(\alpha+1)2^{\alpha-1}} (b-a)^\alpha, \right. \\
& \left. \frac{M_3(l_2)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right) 2^{\alpha-\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{p}} \right\}, \tag{57}
\end{aligned}$$

where  $M_1(l_i)$  as in (14),  $M_2(l_i)$  as in (18) and  $M_3(l_i)$  as in (29),  $i = 1, 2$ .

Here  $l_1, l_2$  cannot be non-trivial polynomials of degree  $m$ .

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