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The Reduction Method in Fractional Calculus and Fractional Ostrowski type inequalities

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Abstract

Here we study generalised fractional integrals and fractional derivatives. We present the reduction method of Fractional Calculus and we reduce them to basic fractional integrals and fractional derivatives. We give a series of generalised Ostrowski type fractional inequalities involving s–convexity. We apply all of the above to Hadamard and Erdélyi-Kober fractional integrals and fractional derivatives. We produce also important generalised fractional Taylor formulae.

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1 The Reduction Method in Fractional Calculus

We use a lot here the following generalised fractional integrals.

Definition 1 *(see also [8, p. 99]) The left and right fractional integrals, respectively, of a function f with respect to given function g are defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a,b])$ (absolutely continuous functions) and is strictly increasing, $f \in L_\infty([a,b])$. We set

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) \, dt, \quad x \geq a, \tag{1}$$

clearly $\left(I_{a+;g}^{\alpha} f\right)(a) = 0$, *and*

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)f(t)\,dt,\quad x\leq b, \tag{2}$$

clearly $\left(I_{b-;g}^{\alpha}f\right)(b)=0.$

When g is the identity function id, we get that $I_{a+;id}^{\alpha}=I_{a+}^{\alpha}$ and $I_{b-;id}^{\alpha}=I_{b-}^{\alpha}$ the ordinary left and right Riemann-Liouville fractional integrals, where

$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt,\quad x\geq a, \tag{3}$$

$\left(I_{a+}^{\alpha}f\right)(a)=0,$ *and*

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt,\quad x\leq b, \tag{4}$$

$\left(I_{b-}^{\alpha}f\right)(b)=0.$

We need

Lemma 2 *Let $g\in AC([a,b])$ which is strictly increasing and $f\in L_{\infty}([a,b])$. Then*

$$\|f\|_{\infty,[a,b]}\geq\left\|f\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}, \tag{5}$$

i.e. $\left(f\circ g^{-1}\right)\in L_{\infty}([g(a),g(b)])$.

If additionally $g^{-1}\in AC([g(a),g(b)])$ then

$$\|f\|_{\infty,[a,b]}=\left\|f\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}. \tag{6}$$

Proof. Here m stands for the Lebesgue measure. By definition we have

$$\|f\|_{\infty,[a,b]}=ess\sup|f(t)|$$

$$=\inf\left\{M:m\left\{t:\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}=m\{t:|f(t)|>M\}=0\right\}. \tag{7}$$

Furthermore we have

$$\left\|f\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}=\inf\left\{L:m\left\{g(t):\left|\left(f\circ g^{-1}\right)(g(t))\right|>L\right\}=0\right\}. \tag{8}$$

Because g is absolutely continuous and strictly increasing function on $[a,b]$, by [9, p. 108] exercise 14, we get that

$$m\left\{g(t):\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}=m\left(g\left(\left\{t:\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}\right)\right)=0, \tag{9}$$

given that $m\left\{t:\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}=0.$

Therefore each M of (7) fulfills $M\in\left\{L:m\left\{g(t):\left|\left(f\circ g^{-1}\right)(g(t))\right|>L\right\}=0\right\}.$

The last implies (5). Similarly arguing reverse we derive (6). ■

We use (5) in the next

Remark 3 *We observe that*

$$\left(I_{a+;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(g(x)-g(t))^{\alpha-1}\left(f\circ g^{-1}\right)(g(t))\,g'(t)\,dt=$$

(by change of variable for Lebesgue integrals)

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha-1}\left(f\circ g^{-1}\right)(z)\,dz=\left(I_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad x\geq a, \tag{11}$$

equivalently $g(x)\geq g(a)$.

That is in the terms and assumtions of Definition 1 we get

$$\left(I_{a+;g}^{\alpha}f\right)(x)=\left(I_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad \text{for } x\geq a. \tag{12}$$

Similarly we observe that

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}\left(f\circ g^{-1}\right)(g(t))\,g'(t)\,dt$$

$$=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(z-g(x))^{\alpha-1}\left(f\circ g^{-1}\right)(z)\,dz=\left(I_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)), \tag{13}$$

for $x\leq b$.

That is

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\left(I_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad \text{for } x\leq b. \tag{14}$$

So by (12) and (14) we have reduced the general fractional integrals to the ordinary left and right Riemann-Liouville fractional integrals.

We need

Definition 4 *([7]) Let* $0<a<b<\infty$, $\alpha>0$. *The left and right Hadamard fractional integrals of order* α *are given by*

$$\left(J_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left(\ln\frac{x}{y}\right)^{\alpha-1}\frac{f(y)}{y}dy,\quad x\geq a, \tag{15}$$

and

$$\left(J_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\left(\ln\frac{y}{x}\right)^{\alpha-1}\frac{f(y)}{y}dy,\quad x\leq b, \tag{16}$$

respectively.

Here we take $f\in L_{\infty}([a,b])$.

Comparing to Definition 1 we have $g(x) = \ln x$ on $[a,b]$, $0 < a < b < \infty$. Comparing to (12) and (14) we get

$$\left(J_{a+}^{\alpha} f\right)(x) = \left(I_{(\ln a)+}^{\alpha} (f \circ \exp)\right)(\ln x), \quad \text{for } x \geq a, \tag{17}$$

and

$$\left(J_{b-}^{\alpha} f\right)(x) = \left(I_{(\ln b)-}^{\alpha} (f \circ \exp)\right)(\ln x), \quad \text{for } x \leq b. \tag{18}$$

We also consider

Definition 5 *Let $0 < a < b < \infty$; $\alpha, \sigma > 0$ and $\eta > -1$. Let $f \in L_{\infty}([a,b])$. We mention here the left and right Erdélyi-Kober type fractional integrals, respectively: as in [10] we define*

$$\left(I_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \left(x^{\sigma} - t^{\sigma}\right)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt, \quad x \geq a, \tag{19}$$

and similarly we also define

$$\left(I_{b-;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_x^b \left(t^{\sigma} - x^{\sigma}\right)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt, \quad x \leq b. \tag{20}$$

Remark 6 *(following Definition 5) The above give rise to the following generalised weighted left and right fractional integrals.*

We set

$$\left(K_{a+;\sigma,\eta}^{\alpha} f\right)(x) = x^{\sigma(\alpha+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \tag{21}$$
$$\frac{\sigma}{\Gamma(\alpha)} \int_a^x \left(x^{\sigma} - t^{\sigma}\right)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt =$$
$$\frac{1}{\Gamma(\alpha)} \int_a^x \left(x^{\sigma} - t^{\sigma}\right)^{\alpha-1} \left(t^{\sigma\eta} f(t)\right) \sigma t^{\sigma-1} dt =$$
$$\frac{1}{\Gamma(\alpha)} \int_a^x \left(x^{\sigma} - t^{\sigma}\right)^{\alpha-1} \left(t^{\sigma\eta} f(t)\right) dt^{\sigma} =$$

(setting $z = t^{\sigma}$)

$$\frac{1}{\Gamma(\alpha)} \int_{a^{\sigma}}^{x^{\sigma}} \left(x^{\sigma} - z\right)^{\alpha-1} \left(z^{\eta} f\left(z^{\frac{1}{\sigma}}\right)\right) dz = \left(I_{a^{\sigma}+}^{\alpha} \left(z^{\eta} f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}), \quad x \geq a, \tag{22}$$

that is

$$\left(K_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \left(I_{a^{\sigma}+}^{\alpha} \left(z^{\eta} f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}), \quad x \geq a. \tag{23}$$

Similarly we put

$$\left(K_{b-;\sigma,\eta}^{\alpha} f\right)(x) = x^{\sigma(\alpha+\eta)} \left(I_{b-;\sigma,\eta}^{\alpha} f\right)(x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{x^{\sigma}}^{b^{\sigma}}(z-x^{\sigma})^{\alpha-1}\left(z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)\right)dz=\left(I_{b^{\sigma}-}^{\alpha}\left(z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}),\quad x\leq b, \tag{24}$$

that is

$$\left(K_{b-;\sigma,\eta}^{\alpha}f\right)(x)=\left(I_{b^{\sigma}-}^{\alpha}\left(z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}),\qquad x\leq b. \tag{25}$$

Comparing to Definition 1 here, we have that $g(x)=x^{\sigma}\in C^{1}([a,b])$, *thus* $x^{\sigma}\in AC([a,b])$ *and it is strictly increasing. Clearly* $g^{-1}(z)=z^{\frac{1}{\sigma}}$, $z\in[a^{\sigma},b^{\sigma}]$. *We set* $F(t)=t^{\sigma\eta}f(t)$, $t\in[a,b]$. *Clearly we have* $F\in L_{\infty}([a,b])$. *Notice that* $F\circ g^{-1}=F\circ(id)^{\frac{1}{\sigma}}$, *and* $F(t)=\left(F\circ g^{-1}\right)(g(t))=\left(F\circ g^{-1}\right)(z)=z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)$. *Thus a formal description of (23) and (25) follows.*

We have

$$\left(K_{a+;\sigma,\eta}^{\alpha}f\right)(x)=\left(I_{a^{\sigma}+}^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma}),\qquad x\geq a, \tag{26}$$

and

$$\left(K_{b-;\sigma,\eta}^{\alpha}f\right)(x)=\left(I_{b^{\sigma}-}^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma}),\qquad x\leq b, \tag{27}$$

where $F(x)=x^{\sigma\eta}f(x)$, $x\in[a,b]$.

We introduce

Definition 7 *Let* $\alpha>0$, $m=[\alpha]$, $\beta=\alpha-m$, $0<\beta<1$, $f\in C([a,b])$, $[a,b]\subset\mathbb{R}$; $g\in AC([a,b])$, g *is strictly increasing. We define the subspace* $C_{a+;g}^{\alpha}([a,b])$ *of* $C^{m}([a,b])$:

$$C_{a+;g}^{\alpha}([a,b])=\left\{f\in C^{m}([a,b]):\left(I_{a+;g}^{1-\beta}f^{(m)}\right)\in C^{1}([a,b])\right\}. \tag{28}$$

Denote $C_{a+}^{\alpha}=C_{a+;id}^{\alpha}$.

For $f\in C_{a+;g}^{\alpha}([a,b])$, *we define the left* g*-generalised* α*-fractional derivative of* f *over* $[a,b]$ *as*

$$D_{a+;g}^{\alpha}(f)=\left(I_{a+;g}^{1-\beta}f^{(m)}\right)'. \tag{29}$$

When $g=id$, *we denote*

$$D_{a+}^{\alpha}f=\left(I_{a+}^{1-\beta}f^{(m)}\right)', \tag{30}$$

called the left generalized α*-fractional derivative of* f *over* $[a,b]$, *see [4], [2], p. 24.*

We also introduce

Definition 8 *Let* $\alpha>0$, $m=[\alpha]$, $\beta=\alpha-m$, $f\in C([a,b])$, $[a,b]\subset\mathbb{R}$; $g\in AC([a,b])$, g *is strictly increasing. We define the subspace* $C_{b-;g}^{\alpha}([a,b])$ *of* $C^{m}([a,b])$:

$$C_{b-;g}^{\alpha}([a,b])=\left\{f\in C^{m}([a,b]):\left(I_{b-;g}^{1-\beta}f^{(m)}\right)\in C^{1}([a,b])\right\}. \tag{31}$$

Denote $C^{\alpha}_{b-} = C^{\alpha}_{b-;id}$.

For $f \in C^{\alpha}_{b-;g}([a,b])$, *we define the right* g*-generalised* α*-fractional derivative of* f *over* $[a,b]$ *as*

$$D^{\alpha}_{b-;g}(f) = (-1)^{m-1}\left(I^{1-\beta}_{b-;g} f^{(m)}\right)'. \tag{32}$$

When $g = id$, *we denote*

$$D^{\alpha}_{b-} f = (-1)^{m-1}\left(I^{1-\beta}_{b-} f^{(m)}\right)', \tag{33}$$

called the right generalized α*-fractional derivative of* f *over* $[a,b]$, *see [3].*

Regarding fractional derivatives in this article from now on we consider only $0 < \alpha < 1$, i.e. $m = 0$ and $\beta = \alpha$.

So in this case we get

$$\left(D^{\alpha}_{a+;g} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (g(x) - g(t))^{-\alpha} g'(t) f(t)\, dt, \tag{34}$$

$$\left(D^{\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\, dt, \tag{35}$$

and

$$\left(D^{\alpha}_{b-;g} f\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (g(t) - g(x))^{-\alpha} g'(t) f(t)\, dt, \tag{36}$$

$$\left(D^{\alpha}_{b-} f\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)\, dt, \tag{37}$$

for any $x \in [a,b]$.

We mention the following fractional Taylor formulae.

Theorem 9 *1) (see [2], pp. 8-10, [4]) Let* $f \in C^{\alpha}_{a+}([a,b])$, $0 < \alpha < 1$. *Then*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D^{\alpha}_{a+} f\right)(t)\, dt = \left(I^{\alpha}_{a+}\left(D^{\alpha}_{a+} f\right)\right)(x), \quad x \in [a,b]. \tag{38}$$

2) (see [3]) Let $f \in C^{\alpha}_{b-}([a,b])$, $0 < \alpha < 1$. *Then*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D^{\alpha}_{b-} f\right)(t)\, dt = \left(I^{\alpha}_{b-}\left(D^{\alpha}_{b-} f\right)\right)(x), \quad x \in [a,b]. \tag{39}$$

We make

Remark 10 *Here* $0<\alpha<1$ *and* $g\in C^{1}([a,b])$, g *is strictly increasing. Furthermore we assume that* $\left(D_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x))$ *exists. By (12) we have*

$$\left(I_{a+;g}^{1-\alpha}f\right)(x)=\left(I_{g(a)+}^{1-\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad x\in[a,b]. \tag{40}$$

Hence there exists

$$\begin{aligned}\left(D_{a+;g}^{\alpha}(f)\right)(x)&=\left(I_{a+;g}^{1-\alpha}f\right)'(x)\overset{(40)}{=}\left(I_{g(a)+}^{1-\alpha}\left(f\circ g^{-1}\right)\right)'(g(x))\,g'(x)\\&=\left(D_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x))\,g'(x),\quad x\in[a,b].\end{aligned} \tag{41}$$

We have established that there exists

$$\left(D_{a+;g}^{\alpha}(f)\right)(x)=\left(D_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x))\,g'(x),\quad x\in[a,b],\ f\in C([a,b]). \tag{42}$$

Next we assume that there exists $\left(D_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x))$. *By (14) we get*

$$\left(I_{b-;g}^{1-\alpha}f\right)(x)=\left(I_{g(b)-}^{1-\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad x\in[a,b]. \tag{43}$$

Hence there exists

$$\begin{aligned}\left(D_{b-;g}^{\alpha}(f)\right)(x)&=-\left(I_{b-;g}^{1-\alpha}f\right)'(x)\overset{(43)}{=}-\left(I_{g(b)-}^{1-\alpha}\left(f\circ g^{-1}\right)\right)'(g(x))\,g'(x)\\&=\left(D_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x))\,g'(x),\quad x\in[a,b].\end{aligned} \tag{44}$$

We have proved that there exists

$$\left(D_{b-;g}^{\alpha}(f)\right)(x)=\left(D_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x))\,g'(x),\quad x\in[a,b],\ f\in C([a,b]). \tag{45}$$

Next we apply (42) and (45).

We make

Remark 11 *(all as in Definition 4) We introduce the following Hadamard type fractional derivatives, see (46), (47). Here* $f\in C([a,b])$. *Let* $0<\alpha<1$, *and that* $\left(D_{(\ln a)+}^{\alpha}(f\circ\exp)\right)(\ln x)$ *exists for* $x\in[a,b]$, $a>0$.

Then by (42), we get

$$\left(D_{a+;\ln}^{\alpha}(f)\right)(x)=\frac{\left(D_{(\ln a)+}^{\alpha}(f\circ\exp)\right)(\ln x)}{x},\quad x\in[a,b]. \tag{46}$$

Assume next that $\left(D_{(\ln b)-}^{\alpha}(f\circ\exp)\right)(\ln x)$ *exists for* $x\in[a,b]$.

Then by (45), we find

$$\left(D_{b-;\ln}^{\alpha}(f)\right)(x)=\frac{\left(D_{(\ln b)-}^{\alpha}(f\circ\exp)\right)(\ln x)}{x},\quad x\in[a,b]. \tag{47}$$

We make

Remark 12 *(refer to Definition 5, Remark 6) Let $0 < \alpha < 1$. By (26) we get*

$$\left(K^{1-\alpha}_{a+;\sigma,\eta} f\right)(x) = \left(I^{1-\alpha}_{a^\sigma +}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma), \quad x \in [a,b]. \tag{48}$$

And by (27)

$$\left(K^{1-\alpha}_{b-;\sigma,\eta} f\right)(x) = \left(I^{1-\alpha}_{b^\sigma -}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma), \quad x \in [a,b]. \tag{49}$$

Above $F(x) = x^{\sigma\eta} f(x)$, $x \in [a,b]$.

Assume that

$$\frac{d\left(I^{1-\alpha}_{a^\sigma +}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)}{dx^\sigma} \tag{50}$$

and

$$\frac{d\left(I^{1-\alpha}_{b^\sigma -}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)}{dx^\sigma} \tag{51}$$

exist and are continuous in $x^\sigma \in [a^\sigma, b^\sigma]$, $f \in C([a,b])$.

Then

$$\frac{d\left(K^{1-\alpha}_{a+;\sigma,\eta} f\right)(x)}{dx} = \frac{d\left(I^{1-\alpha}_{a^\sigma +}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)}{dx^\sigma}\sigma x^{\sigma-1}, \tag{52}$$

and

$$\frac{d\left(K^{1-\alpha}_{b-;\sigma,\eta} f\right)(x)}{dx} = \frac{d\left(I^{1-\alpha}_{b^\sigma -}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)}{dx^\sigma}\sigma x^{\sigma-1}, \tag{53}$$

exist and are continuous in $x \in [a,b]$.

So we introduce the modified Erdélyi-Kober type left and right fractional derivatives of $f \in C([a,b])$, as follows:

$$\left(D^{\alpha}_{a+:\sigma,\eta} f\right)(x) = \frac{d\left(K^{1-\alpha}_{a+;\sigma,\eta} f\right)(x)}{dx}, \tag{54}$$

and

$$\left(D^{\alpha}_{b-:\sigma,\eta} f\right)(x) = -\frac{d\left(K^{1-\alpha}_{b-;\sigma,\eta} f\right)(x)}{dx}, \tag{55}$$

$x \in [a,b]$, $0 < \alpha < 1$.

That is, it holds

$$\left(D^{\alpha}_{a+:\sigma,\eta} f\right)(x) = \left(D^{\alpha}_{a^\sigma +}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)\,\sigma x^{\sigma-1}, \tag{56}$$

and

$$\left(D^{\alpha}_{b-:\sigma,\eta} f\right)(x) = \left(D^{\alpha}_{b^\sigma -}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)\,\sigma x^{\sigma-1}, \tag{57}$$

$x \in [a,b]$, $0 < \alpha < 1$, $a > 0$.

We make

Remark 13 *(continuation of Remark 11) Hence $f \in C([a,b])$. By (46) we get*

$$\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)(\ln x) = x\left(D^{\alpha}_{a+;\ln}(f)\right)(x) = e^{\ln x}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{\ln x}\right), \tag{58}$$

$x \in [a,b]$.

Hence by (38) we obtain

$$f(x) = f\left(e^{\ln x}\right) = \left(I^{\alpha}_{(\ln a)+}\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)\right)(\ln x) = \tag{59}$$

$$\left(I^{\alpha}_{(\ln a)+}\left(e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{\ln a}^{\ln x}(\ln x - t)^{\alpha-1} e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)dt, \tag{60}$$

$x \in [a,b]$.

By (47) we have

$$\left(D^{\alpha}_{(\ln b)-}(f \circ \exp)\right)(\ln x) = x\left(D^{\alpha}_{b-;\ln}(f)\right)(x) = e^{\ln x}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{\ln x}\right), \tag{61}$$

$x \in [a,b]$.

Hence by (39) we obtain

$$f(x) = f\left(e^{\ln x}\right) = \left(I^{\alpha}_{(\ln b)-}\left(D^{\alpha}_{(\ln b)-}(f \circ \exp)\right)\right)(\ln x) =$$

$$\left(I^{\alpha}_{(\ln b)-}\left(e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{\ln x}^{\ln b}(t - \ln x)^{\alpha-1} e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)dt, \tag{62}$$

$x \in [a,b]$.

We have proved the following Taylor Hadamard type fractional formulae.

Theorem 14 *Let $0 < \alpha < 1$, and all as in Definition 4, $f \in C([a,b])$, $a > 0$.*

1) Assume that $\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)(\ln x)$ exists and it is continuous, $x \in [a,b]$. Then

$$f(x) = \left(I^{\alpha}_{(\ln a)+}\left(e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{\ln a}^{\ln x}(\ln x - t)^{\alpha-1} e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)dt, \tag{63}$$

$x \in [a,b]$.

2) Assume that $\left(D^{\alpha}_{(\ln b)-}(f\circ\exp)\right)(\ln x)$ *exists and it is continuous,* $x\in[a,b]$. *Then*

$$f(x)=\left(I^{\alpha}_{(\ln b)-}\left(e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x)$$

$$=\frac{1}{\Gamma(\alpha)}\int_{\ln x}^{\ln b}(t-\ln x)^{\alpha-1}e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)dt, \tag{64}$$

$x\in[a,b]$.

We make

Remark 15 *(continuation of Remark 12) By (56) and (57) we get*

$$\left(D^{\alpha}_{a^{\sigma}+}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})=\frac{x^{1-\sigma}}{\sigma}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)(x)$$

$$=\frac{(x^{\sigma})^{\left(\frac{1}{\sigma}-1\right)}}{\sigma}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)\left((x^{\sigma})^{\frac{1}{\sigma}}\right), \tag{65}$$

and

$$\left(D^{\alpha}_{b^{\sigma}-}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})=\frac{x^{1-\sigma}}{\sigma}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)(x)$$

$$=\frac{(x^{\sigma})^{\left(\frac{1}{\sigma}-1\right)}}{\sigma}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)\left((x^{\sigma})^{\frac{1}{\sigma}}\right), \tag{66}$$

$x\in[a,b]$, $0<\alpha<1$, $f\in C([a,b])$. *Above assume* $\left(D^{\alpha}_{a^{\sigma}+}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})$, $\left(D^{\alpha}_{b^{\sigma}-}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})$ *exist and are continuous in* $x^{\sigma}\in[a^{\sigma},b^{\sigma}]$.

Hence, by (38) it holds

$$x^{\sigma\eta}f(x)=\left(F\circ(id)^{\frac{1}{\sigma}}\right)(x^{\sigma})=\left(I^{\alpha}_{a^{\sigma}+}\left(D^{\alpha}_{a^{\sigma}+}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma})$$

$$=\frac{1}{\sigma}\left(I^{\alpha}_{a^{\sigma}+}\left(t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}) \tag{67}$$

$$=\frac{1}{\sigma\Gamma(\alpha)}\int_{a^{\sigma}}^{x^{\sigma}}(x^{\sigma}-t)^{\alpha-1}t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)dt,\quad x\in[a,b]. \tag{68}$$

Similarly, by (39) we derive

$$x^{\sigma\eta}f(x)=\left(F\circ(id)^{\frac{1}{\sigma}}\right)(x^{\sigma})=\left(I^{\alpha}_{b^{\sigma}-}\left(D^{\alpha}_{b^{\sigma}-}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}) \tag{69}$$

$$=\frac{1}{\sigma}\left(I^{\alpha}_{b^{\sigma}-}\left(t^{\left(\frac{1}{\alpha}-1\right)}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma})$$

$$=\frac{1}{\sigma\Gamma(\alpha)}\int_{x^{\sigma}}^{b^{\sigma}}(t-x^{\sigma})^{\alpha-1}t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)dt,\quad x\in[a,b]. \tag{70}$$

We give the following Taylor Erdélyi-Kober type fractional formulae.

Theorem 16 *Let $0 < \alpha < 1$, all as in Definition 5, (21), (24), (54), (55), $f \in C([a,b])$, $a > 0$; $F(x) = x^{\sigma\eta} f(x)$, $x \in [a,b]$.*

1) Assume that $\left(D^{\alpha}_{a^{\sigma}+}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})$ exists and it is continuous in $x^{\sigma} \in [a^{\sigma}, b^{\sigma}]$. Then

$$f(x) = \frac{x^{-\sigma\eta}}{\sigma}\left(I^{\alpha}_{a^{\sigma}+}\left(t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{a+;\sigma,\eta} f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma})$$

$$= \frac{x^{-\sigma\eta}}{\sigma\Gamma(\alpha)}\int_{a^{\sigma}}^{x^{\sigma}} (x^{\sigma} - t)^{\alpha-1} t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{a+;\sigma,\eta} f\right)\left(t^{\frac{1}{\sigma}}\right) dt, \quad x \in [a,b]. \tag{71}$$

2) Assume that $\left(D^{\alpha}_{b^{\sigma}-}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})$ exists and it is continuous in $x^{\sigma} \in [a^{\sigma}, b^{\sigma}]$. Then

$$f(x) = \frac{x^{-\sigma\eta}}{\sigma}\left(I^{\alpha}_{b^{\sigma}-}\left(t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{b-;\sigma,\eta} f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma})$$

$$= \frac{x^{-\sigma\eta}}{\sigma\Gamma(\alpha)}\int_{x^{\sigma}}^{b^{\sigma}} (t - x^{\sigma})^{\alpha-1} t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{b-;\sigma,\eta} f\right)\left(t^{\frac{1}{\sigma}}\right) dt, \quad x \in [a,b]. \tag{72}$$

2 Fractional Ostrowski type Inequalities

We need the following

Lemma 17 *([11]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L_1([a,b])$, then for all $x \in [a,b]$ and $\alpha > 0$ we have:*

$$\left(\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[I^{\alpha}_{x-} f(a) + I^{\alpha}_{x+} f(b)\right] =$$

$$\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx + (1-t)a)\, dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha} f'(tx + (1-t)b)\, dt. \tag{73}$$

By (73), (12), (14), we obtain

Lemma 18 *Let $f \in C([a,b])$, $g \in C^1([a,b])$, g strictly increasing on $[a,b]$, $f \circ g^{-1}$ differentiable on $(g(a), g(b))$ with $\left(f \circ g^{-1}\right)' \in L_1([g(a), g(b)])$, $x \in [a,b]$, $a < b$, $a, b \in \mathbb{R}$, $\alpha > 0$. Then*

$$\left(\frac{(g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha}}{g(b) - g(a)}\right) f(x) -$$

$$\frac{\Gamma(\alpha+1)}{(g(b) - g(a))}\left[\left(I^{\alpha}_{x-;g} f\right)(a) + \left(I^{\alpha}_{x+;g} f\right)(b)\right] =$$

$$\frac{(g(x)-g(a))^{\alpha+1}}{(g(b)-g(a))}\int_0^1 t^{\alpha}\left(f\circ g^{-1}\right)'(tg(x)+(1-t)g(a))\,dt \tag{74}$$
$$-\frac{(g(b)-g(x))^{\alpha+1}}{(g(b)-g(a))}\int_0^1 t^{\alpha}\left(f\circ g^{-1}\right)'(tg(x)+(1-t)g(b))\,dt.$$

We apply (74) for $g(x)=\ln x$, $x\in[a,b]$.

Lemma 19 *Let $0<a<b<\infty$, $\alpha>0$. Let $f\in C([a,b])$, $(f\circ\exp)$ is differentiable on $(\ln a,\ln b)$ with $(f\circ\exp)'\in L_1([\ln a,\ln b])$, $x\in[a,b]$. Then*

$$\left(\frac{\left(\ln\frac{x}{a}\right)^{\alpha}+\left(\ln\frac{b}{x}\right)^{\alpha}}{\ln\frac{b}{a}}\right)f(x)-\frac{\Gamma(\alpha+1)}{\ln\left(\frac{b}{a}\right)}\left[\left(J_{x-}^{\alpha}f\right)(a)+\left(J_{x+}^{\alpha}f\right)(b)\right]=$$

$$\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}}{\ln\frac{b}{a}}\int_0^1 t^{\alpha}(f\circ\exp)'(t\ln x+(1-t)\ln a)\,dt$$
$$-\frac{\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\ln\frac{b}{a}}\int_0^1 t^{\alpha}(f\circ\exp)'(t\ln x+(1-t)\ln b)\,dt, \tag{75}$$

where $J_{x\pm}^{\alpha}f$ are the left and right Hadamard fractional integrals of order α anchored at $x\in[a,b]$, see (15), (16).

We apply (74) for $g(x)=x^{\sigma}$, $\sigma>0$, $x\in[a,b]$.

Lemma 20 *Let $0<a<b<\infty$, $\alpha>0$, $f\in C([a,b])$. Assume $\left(F\circ(id)^{\frac{1}{\sigma}}\right)$ is differentiable on (a^{σ},b^{σ}) with $\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\in L_1([a^{\sigma},b^{\sigma}])$, $x\in[a,b]$. Here $F(x)=x^{\sigma\eta}f(x)$, $x\in[a,b]$, $\eta>-1$. Then*

$$\left(\frac{(x^{\sigma}-a^{\sigma})^{\alpha}+(b^{\sigma}-x^{\sigma})^{\alpha}}{b^{\sigma}-a^{\sigma}}\right)x^{\sigma\eta}f(x)-$$

$$\frac{\Gamma(\alpha+1)}{(b^{\sigma}-a^{\sigma})}\left[\left(K_{x-;\sigma,\eta}^{\alpha}f\right)(a)+\left(K_{x+;\sigma,\eta}^{\alpha}f\right)(b)\right]=$$

$$\frac{(x^{\sigma}-a^{\sigma})^{\alpha+1}}{(b^{\sigma}-a^{\sigma})}\int_0^1 t^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)'(tx^{\sigma}+(1-t)a^{\sigma})\,dt$$
$$-\frac{(b^{\sigma}-x^{\sigma})^{\alpha+1}}{(b^{\sigma}-a^{\sigma})}\int_0^1 t^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)'(tx^{\sigma}+(1-t)b^{\sigma})\,dt, \tag{76}$$

where $\left(K_{x\pm;\sigma,\eta}^{\alpha}f\right)$ as in (26), (27).

We need

Definition 21 *([6]) A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1 - \lambda) y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y), \tag{77}$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

This class of s-convex functions is denoted by K_s^2.

When $s = 1$, s-convexity reduces to ordinary convexity.

If "$\geq$" holds in (77), we talk about s-concavity in the second sense.

In our proofs it is used a lot and it is built in the following

Theorem 22 *([5]) Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1]$, and let $a, b \in [0, \infty)$, $a < b$. If $f' \in L_1([a, b])$, then it holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}, \tag{78}$$

where the constant $\frac{1}{s+1}$ is the best possible in the second inequality.

We are also motivated by the following Ostrowski type inequality in

Theorem 23 *([1]) Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1([a, b])$, where $a, b \in I$, $a < b$. If $|f'|$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $|f'(x)| \leq M$, for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{s+1} \right], \tag{79}$$

for each $x \in [a, b]$.

We need

Theorem 24 *([11]) Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$, be a differentiable mapping on (a, b) with $a < b$, such that $f' \in L_1([a, b])$. If $|f'|$ is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $|f'(x)| \leq M$, $x \in [a, b]$, $\alpha > 0$, then*

$$\Delta_x(f) := \left| \left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[I_{x-}^\alpha f(a) + I_{x+}^\alpha f(b) \right] \right|$$

$$\leq \frac{M}{b-a} \left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right) \left[\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+s+1} \right]. \tag{80}$$

We give the following general fractional Ostrowski type inequality. The proof comes by Lemma 18 and Theorem 24.

Theorem 25 *Let $f \in C([a,b])$, $g \in C^1([a,b])$, g strictly increasing on $[a,b]$, $f \circ g^{-1}$ differentiable on $(g(a), g(b))$ with $\left(f \circ g^{-1}\right)' \in L_1([g(a), g(b)])$, $x \in [a,b]$, $a < b$, $a, b \in \mathbb{R}$, $\alpha > 0$. Assume $\left|\left(f \circ g^{-1}\right)'\right|$ is s-convex in the second sense on $[g(a), g(b)] \subset [0, \infty)$ for some fixed $s \in (0,1]$ and $\left|\left(f \circ g^{-1}\right)'(g(x))\right| \leq M$, $x \in [a,b]$. Then*

$$\Delta_{g(x)}(f) := \left|\left(\frac{(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}}{g(b)-g(a)}\right) f(x) - \right.$$
$$\left. \frac{\Gamma(\alpha+1)}{(g(b)-g(a))}\left[\left(I_{x-;g}^{\alpha} f\right)(a)+\left(I_{x+;g}^{\alpha} f\right)(b)\right]\right|$$
$$\leq \frac{M}{(g(b)-g(a))}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right) \cdot$$
$$\left[\frac{(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}}{\alpha+s+1}\right]. \tag{81}$$

We need

Theorem 26 *([11]) All as in Theorem 24, but here $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p, q > 1 : \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Delta_x(f) \leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]. \tag{82}$$

We apply Theorem 26 and Lemma 18. We give the following fractional Ostrowski type inequality.

Theorem 27 *All as in Theorem 25, however here $\left|\left(f \circ g^{-1}\right)'\right|^q$ is s-convex in the second sense on $[g(a), g(b)] \subset [0, \infty)$ for some fixed $s \in (0,1]$, $p, q > 1$: $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Delta_{g(x)}(f) \leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}}{g(b)-g(a)}\right]. \tag{83}$$

We need

Theorem 28 *([11]) All as in Theorem 24, but here $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, with $q \geq 1$. Then*

$$\Delta_x(f) \leq M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}} \cdot$$
$$\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]. \tag{84}$$

We give with the use of (84) the following

Theorem 29 *Here all as in Theorem 27, however $q \geq 1$, p is not related. Then*

$$\Delta_{g(x)}(f) \leq M \left(\frac{1}{1+\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+s+1} \right)^{\frac{1}{q}} \cdot$$

$$\left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right)^{\frac{1}{q}} \left[\frac{(g(x)-g(a))^{\alpha+1} + (g(b)-g(x))^{\alpha+1}}{g(b)-g(a)} \right]. \quad (85)$$

We need

Theorem 30 *([11]) All as in Theorem 24, but here $|f'|^q$ is s-concave in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Delta_x(f) \leq \frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}(b-a)} \cdot$$

$$\left[(x-a)^{\alpha+1} \left| f'\left(\frac{x+a}{2} \right) \right| + (b-x)^{\alpha+1} \left| f'\left(\frac{b+x}{2} \right) \right| \right]. \quad (86)$$

Using (86) we get

Theorem 31 *All as in Theorem 25, but here $\left| \left(f \circ g^{-1} \right)' \right|^q$ is s-concave in the second sense on $[g(a), g(b)] \subset [0,\infty)$ for some fixed $s \in (0,1]$, $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Delta_{g(x)}(f) \leq \frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}(g(b)-g(a))} \cdot$$

$$\left[(g(x)-g(a))^{\alpha+1} \left| \left(f \circ g^{-1} \right)' \left(\frac{g(x)+g(a)}{2} \right) \right| + \right. \quad (87)$$

$$\left. (g(b)-g(x))^{\alpha+1} \left| \left(f \circ g^{-1} \right)' \left(\frac{g(b)+g(x)}{2} \right) \right| \right].$$

We make

Remark 32 *Let $0 < a < b < \infty$, $\alpha > 0$. We have that*

$$\Delta_{\ln x}(f) = \left| \left(\frac{\left(\ln \frac{x}{a} \right)^{\alpha} + \left(\ln \frac{b}{x} \right)^{\alpha}}{\ln \frac{b}{a}} \right) f(x) \right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{\ln \frac{b}{a}} \left[\left(J_{x-}^{\alpha} f \right)(a) + \left(J_{x+}^{\alpha} f \right)(b) \right] \right|, \quad (88)$$

where $J_{x\pm}^{\alpha}f$ are the Hadamard fractional integrals, see (15), (16), and

$$\Delta_{x^{\sigma}}(f)=\left|\left(\frac{(x^{\sigma}-a^{\sigma})^{\alpha}+(b^{\sigma}-x^{\sigma})^{\alpha}}{b^{\sigma}-a^{\sigma}}\right)x^{\sigma\eta}f(x)\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{(b^{\sigma}-a^{\sigma})}\left[\left(K_{x-;\sigma,\eta}^{\alpha}f\right)(a)+\left(K_{x+;\sigma,\eta}^{\alpha}f\right)(b)\right]\right|, \tag{89}$$

where $K_{x\pm;\sigma,\eta}^{\alpha}(f)$ as in (26), (27), the modified Erdélyi-Kober type fractional integrals, see also (19), (20), (21), and (24), where $\sigma>0$, $\eta>-1$.

Using Theorem 25 we get

Theorem 33 *Let $0<a<b<\infty$, $\alpha>0$. Let $f\in C([a,b])$, $(f\circ\exp)$ is differentiable on $(\ln a,\ln b)$ with $(f\circ\exp)'\in L_1([\ln a,\ln b])$, $x\in[a,b]$. Assume $\left|(f\circ\exp)'\right|$ is s-convex in the second sense on $[\ln a,\ln b]\subset[0,\infty)$ for some fixed $s\in(0,1]$ and $\left|(f\circ\exp)'(\ln x)\right|\leq M$, $x\in[a,b]$. Then*

$$\Delta_{\ln x}(f)\leq\frac{M}{\ln\frac{b}{a}}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\cdot$$

$$\left[\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}+\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\alpha+s+1}\right]. \tag{90}$$

Using Theorem 27 we derive

Theorem 34 *All as in Theorem 33, but here $\left|(f\circ\exp)'\right|^{q}$ is s-convex in the second sense on $[\ln a,\ln b]\subset[0,\infty)$ for some fixed $s\in(0,1]$, $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Delta_{\ln x}(f)\leq\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}+\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\ln\frac{b}{a}}\right]. \tag{91}$$

Using Theorem 29 we derive

Theorem 35 *All as in Theorem 34, however $q\geq1$, p is not related. Then*

$$\Delta_{\ln x}(f)\leq M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\cdot$$

$$\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}+\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\ln\frac{b}{a}}\right]. \tag{92}$$

Based on Theorem 31 we produce

Theorem 36 *All as in Theorem 33, however here* $\left|(f\circ\exp)'\right|^q$ *is s-concave in the second sense on* $[\ln a, \ln b] \subset [0,\infty)$ *for some fixed* $s \in (0,1]$*,* $p,q>1$ *:* $\frac{1}{p}+\frac{1}{q}=1$*. Then*

$$\Delta_{\ln x}(f) \leq \frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}\left(\ln\frac{b}{a}\right)}\cdot$$

$$\left[\left(\ln\frac{x}{a}\right)^{\alpha+1}\left|(f\circ\exp)'\left(\frac{\ln(xa)}{2}\right)\right| + \left(\ln\frac{b}{x}\right)^{\alpha+1}\left|(f\circ\exp)'\left(\frac{\ln(bx)}{2}\right)\right|\right]. \quad (93)$$

Based on Theorem 25 we give

Theorem 37 *Let* $0<a<b<\infty$*,* $f\in C([a,b])$*,* $\alpha,\sigma>0$*,* $\eta>-1$*. Assume* $\left(F\circ(id)^{\frac{1}{\sigma}}\right)$ *is differentiable on* (a^σ,b^σ) *with* $\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\in L_1([a^\sigma,b^\sigma])$*,* $x\in[a,b]$*. Here* $F(x)=x^{\sigma\eta}f(x)$*,* $x\in[a,b]$*. Assume* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\right|$ *is s-convex in the second sense on* $[a^\sigma,b^\sigma]$ *for some fixed* $s\in(0,1]$ *and* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'(x^\sigma)\right|\leq M$*,* $x\in[a,b]$*. Then*

$$\Delta_{x^\sigma}(f) \leq \frac{M}{(b^\sigma-a^\sigma)}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\cdot$$

$$\left[\frac{(x^\sigma-a^\sigma)^{\alpha+1}+(b^\sigma-x^\sigma)^{\alpha+1}}{\alpha+s+1}\right]. \quad (94)$$

By Theorem 27 we get

Theorem 38 *All as in Theorem 37, however here* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\right|^q$ *is s-convex in the second sense on* $[a^\sigma,b^\sigma]$ *for some fixed* $s\in(0,1]$*,* $p,q>1$ *:* $\frac{1}{p}+\frac{1}{q}=1$*. Then*

$$\Delta_{x^\sigma}(f) \leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x^\sigma-a^\sigma)^{\alpha+1}+(b^\sigma-x^\sigma)^{\alpha+1}}{b^\sigma-a^\sigma}\right]. \quad (95)$$

Using Theorem 29 we get

Theorem 39 *Here all as in Theorem 38, however* $q\geq 1$*,* p *is not related. Then*

$$\Delta_{x^\sigma}(f) \leq M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\cdot$$

$$\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{(x^\sigma-a^\sigma)^{\alpha+1}+(b^\sigma-x^\sigma)^{\alpha+1}}{b^\sigma-a^\sigma}\right]. \quad (96)$$

Using Theorem 31 we obtain

Theorem 40 *All as in Theorem 37, however here* $\left|\left(F \circ (id)^{\frac{1}{\sigma}}\right)'\right|^q$ *is s-concave in the second sense on* $[a^\sigma, b^\sigma]$ *for some fixed* $s \in (0,1]$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\Delta_{x^\sigma}(f) \le \frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}(b^\sigma - a^\sigma)} \cdot$$

$$\left[(x^\sigma - a^\sigma)^{\alpha+1}\left|\left(F \circ (id)^{\frac{1}{\sigma}}\right)'\left(\frac{x^\sigma + a^\sigma}{2}\right)\right| + \right.$$

$$\left.(b^\sigma - x^\sigma)^{\alpha+1}\left|\left(F \circ (id)^{\frac{1}{\sigma}}\right)'\left(\frac{b^\sigma + x^\sigma}{2}\right)\right|\right]. \tag{97}$$

3 Addendum

We make

Remark 41 *Let* $0 < \alpha < 1$, $f \in C([a,b])$, $g \in C^1([a,b])$, *g strictly increasing;* $\left(D^\alpha_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x))$, $\left(D^\alpha_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x))$ *exist and are continuous on* $[g(a), g(b)]$. *Also assume* $g'(x) \neq 0$, *almost all* $x \in [a,b]$.

Then by (42) we get

$$\left(D^\alpha_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x)) = \left(g'(x)\right)^{-1}\left(D^\alpha_{a+;g}(f)\right)(x), \tag{98}$$

almost all $x \in [a,b]$.

Also by (45) we get

$$\left(D^\alpha_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x)) = \left(g'(x)\right)^{-1}\left(D^\alpha_{b-;g}(f)\right)(x), \tag{99}$$

almost all $x \in [a,b]$.

Then by (38) and (39) we obtain

$$f(x) = \left(f \circ g^{-1}\right)(g(x)) = I^\alpha_{g(a)+}\left(D^\alpha_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x)) \overset{(98)}{=}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x) - t)^{\alpha-1}\left(g'(t)\right)^{-1}\left(D^\alpha_{a+;g}(f)\right)(t)\,dt, \tag{100}$$

and

$$f(x) = \left(f \circ g^{-1}\right)(g(x)) = I^\alpha_{g(b)-}\left(D^\alpha_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x)) \overset{(99)}{=}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(t - g(x))^{\alpha-1}\left(g'(t)\right)^{-1}\left(D^\alpha_{b-;g}(f)\right)(t)\,dt, \tag{101}$$

for any $x \in [a,b]$.

We have proved the following generalized fractional Taylor formulae.

Theorem 42 *Let* $0 < \alpha < 1$, $f \in C([a,b])$, $g \in C^1([a,b])$, *g strictly increasing; each of* $\left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x))$, $\left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x))$ *exists and it is continuous on* $[g(a), g(b)]$. *Assume that* $g'(x) \neq 0$, *for almost all* $x \in [a,b]$. *Then*

1)

$$f(x) = I^{\alpha}_{g(a)+}\left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x)) =$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - t)^{\alpha-1} \left(g'(t)\right)^{-1} \left(D^{\alpha}_{a+;g}(f)\right)(t)\, dt, \tag{102}$$

and

2)

$$f(x) = I^{\alpha}_{g(b)-}\left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x)) =$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (t - g(x))^{\alpha-1} \left(g'(t)\right)^{-1} \left(D^{\alpha}_{b-;g}(f)\right)(t)\, dt, \tag{103}$$

for any $x \in [a,b]$.

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