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Fractional Polya type integral inequality

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we establish a fractional Polya type integral inequality with the help of generalised right and left fractional derivatives. The amazing fact here is that we do not need any boundary conditions as the classical Polya integral inequality requires.

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1 Introduction

We mention the following famous Polya's integral inequality, see [7], [8, p, 62], [9] and [10, p. 83].

Theorem 1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a)=f(b)=0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$\left|f'(\xi)\right| > \frac{4}{(b-a)^2}\int_a^b f(x)\,dx. \tag{1}$$

In [11], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

Theorem 2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a)=f(b)=0$ and $M=\sup_{x\in[a,b]}\left|f'(x)\right|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}M, \tag{2}$$

where $\frac{(b-a)^2}{4}$ in (2) is the best constant.

In this short note we present a fractional Polya type integral inequality, similar to (2), without the boundary conditions $f(a) = f(b) = 0$.

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{4}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{5}$$

see [1], p. 24. Canavati first in [3] introduced the above over $[0,1]$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [3] the same over $[0,1]$ that appeared first.

Theorem 3 *Let $f \in C_{a+}^\alpha([a,b])$.*

(i) If $\alpha \geq 1$, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \text{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{7}$$

We will use (7).

Notice that

$$\int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt = \int_a^x \left(D_{a+}^\alpha f\right)(t)\, d\left(\frac{(x-t)^\alpha}{-\alpha}\right)$$

$$= \left(D_{a+}^\alpha f\right)(\xi_x) \frac{(x-a)^\alpha}{\alpha}, \text{ where } \xi_x \in [a,x], \tag{8}$$

by first integral mean value theorem. Hence, when $0 < \alpha < 1$, we get

$$f(x) = \left(D_{a+}^\alpha f\right)(\xi_x) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \quad \text{all } x \in [a,b]. \tag{9}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{10}$$

$x \in [a,b]$, see also [2], [4], [5], [6], [12]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{11}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{12}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \text{all } x \in [a,b]. \tag{13}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{14}$$

We will use (14).

Notice that

$$\int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt = \int_x^b \left(D_{b-}^{\alpha} f \right)(t)\, d\left(\frac{(t-x)^{\alpha}}{\alpha} \right)$$

$$= \left(D_{b-}^{\alpha} f \right)(\eta_x) \frac{(b-x)^{\alpha}}{\alpha}, \text{ where } \eta_x \in [x,b], \tag{15}$$

by first integral mean value theorem. Hence, when $0 < \alpha < 1$, we obtain

$$f(x) = \left(D_{b-}^{\alpha} f \right)(\eta_x) \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{all } x \in [a,b]. \tag{16}$$

2 Main Result

We present the following fractional Polya type integral inequality without any boundary conditions.

Theorem 5 *Let* $0 < \alpha < 1$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. *Set*

$$M(f) = \max\left\{\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}. \tag{17}$$

Then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq M(f)\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}. \tag{18}$$

Inequality (18) is sharp, namely it is attained by

$$f_*(x) = \left\{\begin{array}{ll} (x-a)^{\alpha}, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^{\alpha}, & x \in \left[\frac{a+b}{2}, b\right] \end{array}\right\}, \quad 0 < \alpha < 1. \tag{19}$$

Clearly here non zero constant functions f *are excluded.*

Proof. By (9) we get

$$|f(x)| \leq \left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{for any } x \in \left[a, \frac{a+b}{2}\right]. \tag{20}$$

By (16) we derive

$$|f(x)| \leq \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{for any } x \in \left[\frac{a+b}{2}, b\right]. \tag{21}$$

Hence we get

$$\int_a^b |f(x)|\,dx = \int_a^{\frac{a+b}{2}} |f(x)|\,dx + \int_{\frac{a+b}{2}}^b |f(x)|\,dx$$

(by (20), (21))

$$\begin{aligned} &\leq \frac{\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} (x-a)^{\alpha}\,dx + \frac{\left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^b (b-x)^{\alpha}\,dx \\ &= \frac{\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{(\Gamma(\alpha+1))(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha+1} + \frac{\left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{(\Gamma(\alpha+1))(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha+1} \end{aligned} \tag{22}$$

$$= \frac{\left(\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2}\right)^{\alpha+1}. \tag{23}$$

So we have proved that

$$\int_a^b |f(x)|\,dx \le \max\left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\} \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}, \quad (24)$$

proving (18).

Notice that

$$f_*\left(\left(\frac{a+b}{2}\right)_-\right) = f_*\left(\left(\frac{a+b}{2}\right)_+\right) = \left(\frac{b-a}{2}\right)^{\alpha},$$

so that $f_* \in C([a,b])$.

Here $m = 0$. We see that

$$\left(J_{1-\beta}^{\alpha+}(\cdot - a)^{\alpha}\right)(x) = \left(J_{1-\alpha}^{a+}(\cdot - a)^{\alpha}\right)(x) =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} (t-a)^{\alpha}\,dt =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{(1-\alpha)-1} (t-a)^{(\alpha+1)-1}\,dt =$$

(by [13], p. 256)

$$\frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\,\Gamma(\alpha+1)}{\Gamma(2)} (x-a) = \Gamma(\alpha+1)(x-a).$$

Hence

$$D_{a+}^{\alpha}(x-a)^{\alpha} = \Gamma(\alpha+1), \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right]. \quad (25)$$

Therefore

$$\left\| D_{a+}^{\alpha}(x-a)^{\alpha} \right\|_{\infty,\left[a,\frac{a+b}{2}\right]} = \Gamma(a+1). \quad (26)$$

Furthermore we have

$$\left(J_{b-}^{1-\alpha}(b-\cdot)^{\alpha}\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} (b-t)^{\alpha}\,dt =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_x^b (b-t)^{(\alpha+1)-1} (t-x)^{(1-\alpha)-1}\,dt =$$

(by [13], p. 256)

$$\frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(\alpha+1)\,\Gamma(1-\alpha)}{\Gamma(2)} (b-x) = \Gamma(\alpha+1)(b-x).$$

Therefore

$$D_{b-}^{\alpha}(b-x)^{\alpha} = \Gamma(\alpha+1), \quad \text{for all } x \in \left[\frac{a+b}{2}, b\right], \quad (27)$$

and

$$\left\| D_{b-}^{\alpha} (b-x)^{\alpha} \right\|_{\infty,\left[\frac{a+b}{2},b\right]} = \Gamma (a+1) . \tag{28}$$

Consequently we find that

$$M (f_*) = \Gamma (\alpha+1) . \tag{29}$$

Applying f_* into (18) we obtain:

$$\text{R.H.S.(18) for } f_* \ = \Gamma (\alpha+1) \frac{(b-a)^{\alpha+1}}{\Gamma (\alpha+2) 2^{\alpha}} = \frac{(b-a)^{\alpha+1}}{(\alpha+1) 2^{\alpha}}, \tag{30}$$

while we get the same result from

$$\text{L.H.S. (18) for } f_* \ = \left| \int_a^b f_* (x) \, dx \right| =$$

$$\int_a^{\frac{a+b}{2}} (x-a)^{\alpha} \, dx + \int_{\frac{a+b}{2}}^b (b-x)^{\alpha} \, dx = \frac{(b-a)^{\alpha+1}}{(\alpha+1) 2^{\alpha}}, \tag{31}$$

proving sharpness of (18). ■

We make

Remark 6 *When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)} (a) = f^{(k)} (b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements as in Theorem 5. If we set there $\alpha = 1$ we derive exactly Theorem 2. So we generalize Theorem 2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.*

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