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Canavati fractional Ostrowski type inequalities

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Abstract

Here we present Ostrowski type inequalities involving left and right Canavati type generalised fractional derivatives. Combining these we obtain fractional Ostrowski type inequalities of mixed form. Then we establish Ostrowski type inequalities for ordinary and fractional derivatives involving complex valued functions defined on the unit circle.

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1 Introduction

In 1938, A. Ostrowski [12] proved the following important inequality:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \|f'\|_\infty , \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

In this article we present various general Ostrowski type inequalities involving fractional derivatives of Canavati type.

At the end we give applications to complex valued functions defined on the unit circle.

2 Background

Let $\nu > 0$, $n := [\nu]$ (integral part of ν), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. The gamma function Γ is given by $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. Here $[a,b] \subseteq \mathbb{R}$, $x, x_0 \in [a,b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a,b])$ and define the left Riemann-Liouville integral

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} f(t)\, dt, \tag{2}$$

$x_0 \leq x \leq b$. We define the subspace $C_{x_0}^\nu([a.b])$ of $C^n([a,b])$:

$$C_{x_0}^\nu([a,b]) := \left\{ f \in C^n([a,b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0,b]) \right\}. \tag{3}$$

For $f \in C_{x_0}^\nu([a,b])$, we define the left generalized ν-fractional derivative of f over $[x_0,b]$ as

$$D_{x_0}^\nu f := \left(J_{1-\alpha}^{x_0} f^{(n)} \right)', \tag{4}$$

see [4], p. 24, and Canavati derivative in [6].

Notice that $D_{x_0}^\nu f \in C([x_0,b])$.

We need the following generalization of Taylor's formula at the fractional level, see [4], pp. 8-10, and [6].

Theorem 2 *Let $f \in C_{x_0}^\nu([a,b])$, $x_0 \in [a,b]$ fixed.*

(i) If $\nu \geq 1$ then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + ... + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \left(J_\nu^{x_0} D_{x_0}^\nu f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0. \tag{5}$$

(ii) If $0 < \nu < 1$ we get

$$f(x) = \left(J_\nu^{x_0} D_{x_0}^\nu f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0 \tag{6}$$

We will use (5).

Furthermore we need:

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J)\, dJ, \tag{7}$$

$x \in [a,b]$, see also [5], [9], [10], [11], [13]. Define the subspace of functions

$$C_{b-}^\alpha([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{8}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f := (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{9}$$

see [5]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [5], we need the following Taylor fractional formula.

Theorem 3 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

1) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b_{-})}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \forall \, x \in [a,b]. \tag{10}$$

2) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x), \quad \forall \, x \in [a,b]. \tag{11}$$

We will use (10).

In [4], pp. 589-594, and [3], we proved the first fractional Ostrowski inequality.

Theorem 4 *Let $a \leq x_0 < b$, x_0 is fixed. Let $f \in C_{x_0}^{\nu}([a,b])$, $\nu \geq 1$, $n := [\nu]$. Assume $f^{(i)}(x_0) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0) \right| \leq \frac{\left\| D_{x_0}^{\nu} f \right\|_{\infty, [x_0, b]}}{\Gamma(\nu+2)} \cdot (b-x_0)^{\nu}. \tag{12}$$

Inequality (12) is sharp, namely it is attained by

$$f(x) := (x-x_0)^{\nu}, \ \ \nu \geq 1, \ \ x \in [a,b]. \tag{13}$$

When $1 \leq \nu < 2$ the assumption $f^{(i)}(x_0) = 0$, $i = 1, ..., n-1$ is void.

3 Main Results

We give

Theorem 5 *Same assumptions as in Theorem 4. Then*

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0) \right| \leq \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_1([x_0, b])} (b-x_0)^{\nu-1}}{\Gamma(\nu+1)}. \tag{14}$$

Proof. By (5) we get

$$f(y) - f(x_0) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{y} (y-w)^{\nu-1} \left(D_{x_0}^{\nu} f\right)(w)\, dw, \quad \forall\, y \geq x_0. \tag{15}$$

Hence

$$|f(y) - f(x_0)| \leq \frac{1}{\Gamma(\nu)} \int_{x_0}^{y} (y-w)^{\nu-1} \left|\left(D_{x_0}^{\nu} f\right)(w)\right| dw$$

$$\leq \frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)} \int_{x_0}^{y} \left|\left(D_{x_0}^{\nu} f\right)(w)\right| dw \leq \frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)} \int_{x_0}^{b} \left|\left(D_{x_0}^{\nu} f\right)(w)\right| dw$$

$$= \frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)} \left\|D_{x_0}^{\nu} f\right\|_{L_1([x_0,b])}.$$

I.e.

$$|f(y) - f(x_0)| \leq \frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)} \left\|D_{x_0}^{\nu} f\right\|_{L_1([x_0,b])}, \quad \forall\, y \in [x_0, b]. \tag{16}$$

Therefore we get

$$\left|\frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0)\right| = \frac{1}{b-x_0} \left|\int_{x_0}^{b} (f(y) - f(x_0))\, dy\right| \leq$$

$$\frac{1}{b-x_0} \int_{x_0}^{b} |f(y) - f(x_0)|\, dy \overset{(16)}{\leq}$$

$$\frac{1}{b-x_0} \left(\int_{x_0}^{b} (y-x_0)^{\nu-1} dy\right) \frac{\left\|D_{x_0}^{\nu} f\right\|_{L_1([x_0,b])}}{\Gamma(\nu)} = \frac{(b-x_0)^{\nu-1}}{\nu} \frac{\left\|D_{x_0}^{\nu} f\right\|_{L_1([x_0,b])}}{\Gamma(\nu)} =$$

$$\frac{(b-x_0)^{\nu-1}}{\Gamma(\nu+1)} \left\|D_{x_0}^{\nu} f\right\|_{L_1([x_0,b])},$$

proving the claim. ■

We continue with

Theorem 6 *Same assumptions as in Theorem 4. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left|\frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0)\right| \leq \frac{\left\|D_{x_0}^{\nu} f\right\|_{L_q([x_0,b])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-x_0)^{\nu-1+\frac{1}{p}}. \tag{17}$$

Proof. We notice that

$$|f(y) - f(x_0)| \leq \frac{1}{\Gamma(\nu)} \int_{x_0}^{y} (y-w)^{\nu-1} \left|\left(D_{x_0}^{\nu} f\right)(w)\right| dw$$

$$\leq \frac{1}{\Gamma(\nu)} \left(\int_{x_0}^{y} (y-w)^{p(\nu-1)} dw \right)^{\frac{1}{p}} \left(\int_{x_0}^{y} \left| \left(D_{x_0}^{\nu} f \right)(w) \right|^q dw \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\nu)} \left(\frac{(y-x_0)^{p(\nu-1)+1}}{p(\nu-1)+1} \right)^{\frac{1}{p}} \left(\int_{x_0}^{b} \left| \left(D_{x_0}^{\nu} f \right)(w) \right|^q dw \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\nu)} \frac{(y-x_0)^{(\nu-1)+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| \left(D_{x_0}^{\nu} f \right) \right\|_{L_q([x_0,b])}.$$

That is

$$|f(y) - f(x_0)| \leq \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])}}{\Gamma(\nu)} \frac{(y-x_0)^{(\nu-1)+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}, \quad \forall\, y \in [x_0, b]. \tag{18}$$

Consequently we obtain

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0) \right| \leq \frac{1}{b-x_0} \int_{x_0}^{b} |f(y) - f(x_0)|\, dy \overset{(18)}{\leq}$$

$$\left(\frac{1}{b-x_0} \right) \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \int_{x_0}^{b} (y-x_0)^{(\nu-1)+\frac{1}{p}} dy =$$

$$\frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-x_0)^{\nu+\frac{1}{p}-1}}{\left(\nu + \frac{1}{p} \right)}, \tag{19}$$

proving the claim. ■

Combining Theorems 4-6 we derive

Proposition 7 *Let all as in Theorem 6. Then*

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0) \right| \leq \min \left\{ \frac{\left\| D_{x_0}^{\nu} f \right\|_{\infty,[x_0,b]}}{\Gamma(\nu+2)} (b-x_0)^{\nu}, \right. \tag{20}$$

$$\left. \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_1([x_0,b])} (b-x_0)^{\nu-1}}{\Gamma(\nu+1)}, \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])} (b-x_0)^{\nu-1+\frac{1}{p}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p} \right)} \right\}.$$

We continue with right Canavati fractional Ostrowski inequalities.

Theorem 8 *Let $\alpha \geq 1$, $m = [\alpha]$, $f \in C_{b-}^{\alpha}([a,b])$. Assume $f^{(k)}(b_-) = 0$, $k = 1, ..., m-1$; which is void when $1 \leq \alpha < 2$. Then*

$$\left| \frac{1}{b-a} \int_{a}^{b} f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}. \tag{21}$$

Proof. Let $x \in [a,b]$. By (10) we get

$$f(x) - f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha-1} D_{b-}^{\alpha} f(J) \, dJ. \tag{22}$$

Then, as before, we get

$$|f(x) - f(b)| \leq \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}, \quad \forall \, x \in [a,b]. \tag{23}$$

Hence it holds

$$\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)| \, dx \overset{(23)}{\leq}$$

$$\frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)(b-a)} \int_a^b (b-x)^{\alpha} \, dx = \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(\Gamma(\alpha+1))(\alpha+1)} (b-a)^{\alpha}$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}, \tag{24}$$

proving the claim. ■

We continue with

Theorem 9 *Same assumptions as in Theorem 8. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])} (b-a)^{\alpha-1}}{\Gamma(\alpha+1)}. \tag{25}$$

Proof. We have again

$$|f(x) - f(b)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ \tag{26}$$

$$\leq \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}, \quad \forall \, x \in [a,b].$$

Hence

$$\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)| \, dx \overset{(26)}{\leq}$$

$$\frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(\Gamma(\alpha))(b-a)} \int_a^b (b-x)^{\alpha-1} \, dx = \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-a)^{\alpha-1}, \tag{27}$$

proving the claim. ■

We also have

Theorem 10 *Same assumptions as in Theorem 8. Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$ *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(b)\right| \leq \frac{\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}(b-a)^{\alpha-1+\frac{1}{p}}. \tag{28}$$

Proof. As before we obtain

$$|f(x)-f(b)| \leq \frac{\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}}(b-x)^{\alpha-1+\frac{1}{p}}, \quad \forall\, x\in[a,b]. \tag{29}$$

Hence

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(b)\right| \leq \frac{1}{b-a}\int_a^b |f(x)-f(b)|\,dx \overset{(29)}{\leq}$$

$$\frac{\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}(b-a)}\left(\int_a^b (b-x)^{\alpha-1+\frac{1}{p}}\,dx\right)=$$

$$\frac{\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b])}}{\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\Gamma(\alpha)}\frac{(b-a)^{\alpha-1+\frac{1}{p}}}{\left(\alpha+\frac{1}{p}\right)}, \tag{30}$$

proving the claim. ■

Combining Theorems 8-10 we derive

Proposition 11 *Here all as in Theorem 10. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(b)\right| \leq \min\left\{\frac{\left\|D_{b-}^{\alpha}f\right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)}(b-a)^{\alpha}, \right. \tag{31}$$

$$\left.\frac{\left\|D_{b-}^{\alpha}f\right\|_{L_1([a,b])}(b-a)^{\alpha-1}}{\Gamma(\alpha+1)}, \frac{\left\|D_{b-}^{\alpha}f\right\|_{L_q([a,b])}(b-a)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right\}.$$

We also give optimality of (21).

Proposition 12 *Inequality (21) is sharp, namely it is attained by*

$$f_*(J)=(b-J)^{\alpha}, \quad \alpha\geq 1, \quad J\in[a,b], \tag{32}$$

$m:=[\alpha]$.

Proof. We have that

$$f_*^{(m)}(J) = (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)(b-J)^{\alpha-m}. \tag{33}$$

We also notice

$$\left(J_{b-}^{1-(\alpha-m)} f_*^{(m)}\right)(x) \overset{(7)}{=} \frac{1}{\Gamma(1-\alpha+m)} \int_x^b (J-x)^{-\alpha+m} f_*^{(m)}(J)\, dJ$$

$$= \frac{1}{\Gamma(1-\alpha+m)} \int_x^b (J-x)^{m-\alpha} (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)$$

$$\cdot (\alpha-m+1)(b-J)^{\alpha-m}\, dJ$$

$$= \frac{(-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)}{\Gamma(1-\alpha+m)} \cdot$$

$$\int_x^b (b-J)^{(\alpha-m+1)-1} (J-x)^{(1+m-\alpha)-1}\, dJ$$

$$= \frac{(-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)}{\Gamma(1-\alpha+m)} \cdot$$

$$\frac{\Gamma(\alpha-m+1)\Gamma(1+m-\alpha)}{\Gamma(2)}(b-x)$$

$$= (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)\Gamma(\alpha-m+1)(b-x)$$

$$= (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)\Gamma(\alpha-m+2)(b-x)$$

$$= \ldots = (-1)^m \Gamma(\alpha+1)(b-x). \tag{34}$$

That is

$$\left(J_{b-}^{1-\alpha+m} f_*^{(m)}\right)(x) = (-1)^m \Gamma(\alpha+1)(b-x). \tag{35}$$

Therefore it holds

$$\left(D_{b-}^{\alpha} f_*\right)(x) \overset{(9)}{=} (-1)^{m-1} (-1)^m \Gamma(\alpha+1)(-1) = \Gamma(\alpha+1). \tag{36}$$

So that

$$\left\|D_{b-}^{\alpha} f_*\right\|_{\infty,[a,b]} = \Gamma(\alpha+1). \tag{37}$$

We also notice that $f_*^{(k)}(b_-) = 0$, $k = 1, \ldots, m-1$, and $f_*(b) = 0$.

We observe further that

$$L.H.S.(21) = \frac{(b-a)^{\alpha}}{\alpha+1}, \tag{38}$$

and

$$R.H.S.(21) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}(b-a)^{\alpha} = \frac{(b-a)^{\alpha}}{\alpha+1}, \tag{39}$$

proving the claim. ■

Next we present mixed Canavati fractional Ostrowski type inequalities.

Theorem 13 *Let $\alpha \geq 1$, $m = [\alpha]$, $x \in [a,b]$ fixed, $f \in C([a,b])$ with $f \in C_{x-}^{\alpha}([a,x])$ and $f \in C_{x}^{\alpha}([x,b])$. Assume that $f^{(k)}(x) = 0$, $k = 1,...,m-1$, which is void when $1 \leq \alpha < 2$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq \tag{40}$$

$$\frac{1}{(b-a)\,\Gamma(\alpha+2)} \left\{ \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha+1} + \left\| D_{x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha+1} \right\} \leq$$

$$\left(\frac{(b-x)^{\alpha+1} + (x-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)} \right) \max \left\{ \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}, \left\| D_{x}^{\alpha} f \right\|_{\infty,[x,b]} \right\}. \tag{41}$$

Proof. Let $x \in [a,b]$. By (10) we get

$$f(y) - f(x) = \frac{1}{\Gamma(\alpha)} \int_y^x (J-y)^{\alpha-1} D_{x-}^{\alpha} f(J)\,dJ, \quad \forall\, y \in [a,x]. \tag{42}$$

Hence

$$|f(y) - f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_y^x (J-y)^{\alpha-1} \left| D_{x-}^{\alpha} f(J) \right| dJ \leq \frac{(x-y)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}, \tag{43}$$

$\forall\, y \in [a,x]$.

Similarly, by (5) we get

$$f(y) - f(x) = \frac{1}{\Gamma(\alpha)} \int_x^y (y-w)^{\alpha-1} \left(D_x^{\alpha} f \right)(w)\,dw, \quad \forall\, y \in [x,b]. \tag{44}$$

Hence

$$|f(y) - f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^y (y-w)^{\alpha-1} \left| D_x^{\alpha} f(w) \right| dw \leq \left\| D_x^{\alpha} f \right\|_{\infty,[x,b]} \frac{(y-x)^{\alpha}}{\Gamma(\alpha+1)}, \tag{45}$$

$\forall\, y \in [x,b]$.

We observe that

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq \frac{1}{b-a} \int_a^b |f(y) - f(x)|\,dy = \tag{46}$$

$$\frac{1}{b-a} \left\{ \int_a^x |f(y) - f(x)|\,dy + \int_x^b |f(y) - f(x)|\,dy \right\} \overset{\text{by } ((43),\,(45))}{\leq}$$

$$\frac{1}{b-a} \left\{ \frac{\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \int_a^x (x-y)^{\alpha}\,dy + \frac{\left\| D_x^{\alpha} f \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \int_x^b (y-x)^{\alpha}\,dy \right\} = \tag{47}$$

$$\frac{1}{(b-a)\,\Gamma(\alpha+2)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha+1}+\left\|D_{x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha+1}\right\}, \tag{48}$$

proving the claim. ■

We continue with the optimality of Theorem 13.

Proposition 14 *Inequalities (40), (41) are sharp, namely are attained by*

$$\overline{f}(J)=\begin{cases}(x-J)^{\alpha}, & J\in[a,x],\\ (J-x)^{\alpha}, & J\in[x,b],\end{cases} \tag{49}$$

where $\alpha\geq 1$, $x\in[a,b]$ is fixed.

See that $\overline{f}^{(k)}(x_{-})=\overline{f}^{(k)}(x_{+})=0$, $k=0,1,...,m-1$.

We have that

$$\left\|D_{x-}^{\alpha}\overline{f}\right\|_{\infty,[a,x]}=\left\|D_{x}^{\alpha}\overline{f}\right\|_{\infty,[x,b]}=\Gamma(\alpha+1). \tag{50}$$

Furthermore we notice

$$L.H.S.\,(40)=\frac{1}{(\alpha+1)(b-a)}\left\{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right\}, \tag{51}$$

and

$$\begin{aligned} R.H.S.\,(41)&=\frac{\left((b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right)}{(b-a)\,\Gamma(\alpha+2)}\Gamma(\alpha+1)\\ &=\frac{\left((b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right)}{(\alpha+1)(b-a)}, \end{aligned} \tag{52}$$

proving the claim.

We continue with

Theorem 15 *All as in Theorem 13. Then ($x\in[a,b]$)*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(y)\,dy-f(x)\right|\leq \tag{53}$$

$$\frac{1}{(b-a)\,\Gamma(\alpha+1)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_{1}([a,x])}(x-a)^{\alpha}+\left\|D_{x}^{\alpha}f\right\|_{L_{1}([x,b])}(b-x)^{\alpha}\right\}\leq$$

$$\left(\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_{1}([a,x])},\left\|D_{x}^{\alpha}f\right\|_{L_{1}([x,b])}\right\}. \tag{54}$$

Proof. Let $x\in[a,b]$. From (42) we get ($y\in[a,x]$)

$$|f(y)-f(x)|\leq\frac{1}{\Gamma(\alpha)}\int_{y}^{x}(J-y)^{\alpha-1}\left|D_{x-}^{\alpha}f(J)\right|dJ\leq \tag{55}$$

$$\frac{1}{\Gamma(\alpha)}(x-y)^{\alpha-1}\int_y^x \left|D_{x-}^{\alpha}f(J)\right| dJ \leq \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}\int_a^x \left|D_{x-}^{\alpha}f(J)\right| dJ. \tag{56}$$

That is

$$|f(y)-f(x)| \leq (x-y)^{\alpha-1}\frac{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}}{\Gamma(\alpha)}, \quad \forall\, y \in [a,x]. \tag{57}$$

Similarly from (44) we get

$$|f(y)-f(x)| \leq \frac{1}{\Gamma(\alpha)}\int_x^y (y-w)^{\alpha-1}\left|\left(D_x^{\alpha}f\right)(w)\right| dw \tag{58}$$

$$\leq \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}\left\|D_x^{\alpha}f\right\|_{L_1([x,b])}, \quad \forall\, y \in [x,b]. \tag{59}$$

From (46) we obtain

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \tag{60}$$

$$\frac{1}{b-a}\left\{\frac{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy + \frac{\left\|D_x^{\alpha}f\right\|_{L_1([x,b])}}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1}dy\right\} \tag{61}$$

$$= \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}(x-a)^{\alpha} + \left\|D_x^{\alpha}f\right\|_{L_1([x,b])}(b-x)^{\alpha}\right\} \leq$$

$$\left(\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{(b-a)\Gamma(\alpha+1)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}, \left\|D_x^{\alpha}f\right\|_{L_1([x,b])}\right\}, \tag{62}$$

proving the claim. ∎

We also give

Theorem 16 *All as in Theorem 13. Let $p,q>1: \frac{1}{p}+\frac{1}{q}=1$. Then $(x\in[a,b])$*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \tag{63}$$

$$\left\{(x-a)^{\alpha+\frac{1}{p}}\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])} + (b-x)^{\alpha+\frac{1}{p}}\left\|D_x^{\alpha}f\right\|_{L_q([x,b])}\right\} \leq$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{p}}+(x-a)^{\alpha+\frac{1}{p}}}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right).$$

$$\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])}, \left\|D_x^{\alpha}f\right\|_{L_q([x,b])}\right\}. \tag{64}$$

Proof. By (55) we get

$$|f(y) - f(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\int_y^x (J-y)^{p(\alpha-1)} \, dJ \right)^{\frac{1}{p}} \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{(x-y)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])}, \quad \forall\, y \in [a,x]. \tag{65}$$

Similarly from (58) we derive

$$|f(y) - f(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\int_x^y (y-w)^{p(\alpha-1)} \, dw \right)^{\frac{1}{p}} \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{(y-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])}, \quad \forall\, y \in [x,b]. \tag{66}$$

By (46) we derive

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\left\{ \left(\int_a^x (x-y)^{\alpha-1+\frac{1}{p}} \, dy \right) \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])} + \left(\int_x^b (y-x)^{\alpha-1+\frac{1}{p}} \, dy \right) \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])} \right\} \tag{67}$$

$$= \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \cdot$$

$$\left\{ (x-a)^{\alpha+\frac{1}{p}} \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])} + (b-x)^{\alpha+\frac{1}{p}} \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])} \right\} \leq \tag{68}$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{p}} + (x-a)^{\alpha+\frac{1}{p}}}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \right) \max \left\{ \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])}, \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])} \right\}, \tag{69}$$

proving the claim. ■

Corollary 17 *All as in Theorem 13. Then*

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \tag{70}$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{2}} + (x-a)^{\alpha+\frac{1}{2}}}{(b-a)\Gamma(\alpha)\sqrt{2\alpha-1}\left(\alpha+\frac{1}{2}\right)} \right) \max \left\{ \left\| D_{x-}^{\alpha} f \right\|_{L_2([a,x])}, \left\| D_{x}^{\alpha} f \right\|_{L_2([x,b])} \right\}.$$

Proof. By Theorem 16. ■

Combining Theorems 13, 15, 16 we derive

Theorem 18 *Here all as in Theorem 13. Let any* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq$$

$$\min\left\{\frac{1}{(b-a)\Gamma(\alpha+2)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha+1}+\left\|D_{x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha+1}\right\},\right.$$

$$\frac{1}{(b-a)\Gamma(\alpha+1)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}(x-a)^{\alpha}+\left\|D_{x}^{\alpha}f\right\|_{L_1([x,b])}(b-x)^{\alpha}\right\}, \tag{71}$$

$$\frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\cdot$$

$$\left.\left\{(x-a)^{\alpha+\frac{1}{p}}\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])}+(b-x)^{\alpha+\frac{1}{p}}\left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}\right\}\right\} \leq$$

$$\min\left\{\left(\frac{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}}{(b-a)\Gamma(\alpha+2)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]},\left\|D_{x}^{\alpha}f\right\|_{\infty,[x,b]}\right\},\right.$$

$$\left(\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{(b-a)\Gamma(\alpha+1)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])},\left\|D_{x}^{\alpha}f\right\|_{L_1([x,b])}\right\}, \tag{72}$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{p}}+(x-a)^{\alpha+\frac{1}{p}}}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right)\cdot$$

$$\left.\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])},\left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}\right\}\right\}.$$

4 Applications

Inequalities for complex valued functions defined on the unit circle were studied extensively by S. Dragomir, see [7], [8].

We give here our version for these functions involved in Ostrowski type inequalities, by applying results of this article.

Let $t\in[a,b]\subseteq[0,2\pi)$, the unit circle arc $A=\left\{z\in\mathbb{C}:z=e^{it},\ t\in[a,b]\right\}$, and $f:A\to\mathbb{C}$ be a continuous function. Clearly here there exist functions $u,v:A\to\mathbb{R}$ continuous, the real and the complex part of f, respectively, such that

$$f\left(e^{it}\right)=u\left(e^{it}\right)+iv\left(e^{it}\right). \tag{73}$$

So that f is continuous, iff u,v are continuous.

Call $g(t)=f\left(e^{it}\right)$, $l_1(t)=u\left(e^{it}\right)$, $l_2(t)=v\left(e^{it}\right)$, $t\in[a,b]$; so that $g:[a,b]\to\mathbb{C}$ and $l_1,l_2:[a,b]\to\mathbb{R}$ are continuous functions in t.

If g has a derivative with respect to t, then l_1, l_2 have also derivatives with respect to t. In that case

$$f_t\left(e^{it}\right) = u_t\left(e^{it}\right) + iv_t\left(e^{it}\right), \tag{74}$$

(i.e. $g'(t) = l_1'(t) + il_2'(t)$), which means

$$f_t(\cos t + i\sin t) = u_t(\cos t + i\sin t) + iv_t(\cos t + i\sin t). \tag{75}$$

Let us call $x = \cos t$, $y = \sin t$. Then

$$u_t\left(e^{it}\right) = u_t(\cos t + i\sin t) = u_t(x+iy) = u_t(x,y) =$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u\left(e^{it}\right)}{\partial x}(-\sin t) + \frac{\partial u\left(e^{it}\right)}{\partial y}\cos t. \tag{76}$$

Similarly we find that

$$v_t\left(e^{it}\right) = \frac{\partial v\left(e^{it}\right)}{\partial x}(-\sin t) + \frac{\partial v\left(e^{it}\right)}{\partial y}\cos t. \tag{77}$$

So that

$$\left\|u_t\left(e^{it}\right)\right\|_{\infty,[a,b]} \leq \left\|\frac{\partial u\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial u\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}, \tag{78}$$

and

$$\left\|v_t\left(e^{it}\right)\right\|_{\infty,[a,b]} \leq \left\|\frac{\partial v\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial v\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}, \tag{79}$$

Consequently it holds

$$\left\|f_t\left(e^{it}\right)\right\|_{\infty,[a,b]} \leq$$

$$\left\|\frac{\partial u\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial v\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial u\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]} + \left\|\frac{\partial v\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}. \tag{80}$$

Since g is continuous on $[a,b]$, then $\int_a^b f\left(e^{it}\right)dt$ exists. Furthermore it holds

$$\int_a^b f\left(e^{it}\right)dt = \int_a^b u\left(e^{it}\right)dt + i\int_a^b v\left(e^{it}\right)dt. \tag{81}$$

Let now $t_0 \in [a,b]$. We observe that

$$\left|\frac{1}{b-a}\int_a^b f\left(e^{it}\right)dt - f\left(e^{it_0}\right)\right| =$$

$$\left|\frac{1}{b-a}\int_a^b u\left(e^{it}\right)dt + i\frac{1}{b-a}\int_a^b v\left(e^{it}\right)dt - u\left(e^{it_0}\right) - iv\left(e^{it_0}\right)\right| \leq \tag{82}$$

$$\left|\frac{1}{b-a}\int_a^b u\left(e^{it}\right)dt - u\left(e^{it_0}\right)\right| + \left|\frac{1}{b-a}\int_a^b v\left(e^{it}\right)dt - v\left(e^{it_0}\right)\right| \overset{\text{(by (1))}}{\leq}$$

$$\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left[\left\|u_t\left(e^{it}\right)\right\|_{\infty,[a,b]}+\left\|v_t\left(e^{it}\right)\right\|_{\infty,[a,b]}\right]. \quad (83)$$

We have proved the following version of Ostrowski inequality for complex functions.

Theorem 19 *Let $f \in C(A,\mathbb{C})$ with its real and complex part $u\left(e^{it}\right), v\left(e^{it}\right) \in C^1([a,b])$ as functions of t, where $t_0 \in [a,b] \subseteq [0,2\pi)$. Then*

$$\left|\frac{1}{b-a}\int_a^b f\left(e^{it}\right)dt - f\left(e^{it_0}\right)\right| \leq$$

$$\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left[\left\|u_t\left(e^{it}\right)\right\|_{\infty,[a,b]}+\left\|v_t\left(e^{it}\right)\right\|_{\infty,[a,b]}\right] \leq \quad (84)$$

$$\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\cdot \quad (85)$$

$$\left[\left\|\frac{\partial u\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]}+\left\|\frac{\partial v\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]}+\left\|\frac{\partial u\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}+\left\|\frac{\partial v\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}\right].$$

Inequality (84) is sharp.

An explanation follows next.

For $z \in \mathbb{C}-\{0\}$ we call principal value of $\log(z)$ the complex valued function

$$Log(z) := \ln|z| + iArg(z), \quad (86)$$

where $0 \leq Arg(z) < 2\pi$.

For $t \in [0,2\pi)$ we have that

$$Log\left(e^{it}\right) = it. \quad (87)$$

Let here $a = 0 < b < 2\pi$, and $t_0 = 0$. Here $l_1(t) = 0$ and $l_2(t) = t$, with $l_2'(t) = 1$.

Notice in general that

$$\left(\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a) = \frac{(b-t_0)^2+(t_0-a)^2}{2(b-a)}, \quad (88)$$

for any $t_0 \in [a,b]$, see [9], p. 498 and [1].

Hence we have

$$L.H.S.\,(84) = \left| \frac{1}{b} \int_0^b Log\left(e^{it}\right) dt \right| = \left| \frac{1}{b} \int_0^b it dt \right| = \frac{1}{b} \int_0^b t dt = \frac{b}{2}. \quad (89)$$

Furthermore it holds

$$R.H.S.\,(84) = \frac{b}{2} \cdot 1 = \frac{b}{2}. \quad (90)$$

By (89) and (90) we conclude that inequality (84) is attained by Log at $t_0 = 0$ on $[0, b]$, that is a sharp inequality.

We now move at the fractional level.

Let $t_0 \in [a, b]$, we rewrite (82) as follows

$$\left| \frac{1}{b-a} \int_a^b f\left(e^{it}\right) dt - f\left(e^{it_0}\right) \right| \leq$$

$$\left| \frac{1}{b-a} \int_a^b l_1(t)\, dt - l_1(t_0) \right| + \left| \frac{1}{b-a} \int_a^b l_2(t)\, dt - l_2(t_0) \right|. \quad (91)$$

By applying Theorem 18 to each of the last two summands we derive the following complex fractional Ostrowski inequality.

Theorem 20 *Let $f \in C(A, \mathbb{C})$, $t, t_0 \in [a, b] \subseteq [0, 2\pi)$; any $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha \geq 1$, $m = [\alpha]$, with $l_1, l_2 \in C^{\alpha}_{t_0-}([a, t_0])$ and $l_1, l_2 \in C^{\alpha}_{t_0}([t_0, b])$. Assume that $l_1^{(k)}(t_0) = l_2^{(k)}(t_0) = 0$, $k = 1, ..., m-1$, which is void when $1 \leq \alpha < 2$. Then*

$$\left| \frac{1}{b-a} \int_a^b f\left(e^{it}\right) dt - f\left(e^{it_0}\right) \right| \leq$$

$$\min \left\{ \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{t_0-} l_1 \right\|_{\infty,[a,t_0]} (t_0 - a)^{\alpha+1} + \left\| D^{\alpha}_{t_0} l_1 \right\|_{\infty,[t_0,b]} (b - t_0)^{\alpha+1} \right\}, \right.$$

$$\frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ \left\| D^{\alpha}_{t_0-} l_1 \right\|_{L_1([a,t_0])} (t_0 - a)^{\alpha} + \left\| D^{\alpha}_{t_0} l_1 \right\|_{L_1([t_0,b])} (b - t_0)^{\alpha} \right\},$$

$$\frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\left. \left\{ (t_0 - a)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{t_0-} l_1 \right\|_{L_q([a,t_0])} + (b - t_0)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{t_0} l_1 \right\|_{L_q([t_0,b])} \right\} \right\} +$$

$$\min \left\{ \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{t_0-} l_2 \right\|_{\infty,[a,t_0]} (t_0 - a)^{\alpha+1} + \left\| D^{\alpha}_{t_0} l_2 \right\|_{\infty,[t_0,b]} (b - t_0)^{\alpha+1} \right\}, \right.$$

$$\frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ \left\| D^{\alpha}_{t_0-} l_2 \right\|_{L_1([a,t_0])} (t_0 - a)^{\alpha} + \left\| D^{\alpha}_{t_0} l_2 \right\|_{L_1([t_0,b])} (b - t_0)^{\alpha} \right\},$$

$$\frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\left\{(t_0-a)^{\alpha+\frac{1}{p}}\left\|D^{\alpha}_{t_0-}l_2\right\|_{L_q([a,t_0])}+(b-t_0)^{\alpha+\frac{1}{p}}\left\|D^{\alpha}_{t_0}l_2\right\|_{L_q([t_0,b])}\right\}\right\}\leq \tag{92}$$

$$\min\left\{\left(\frac{(b-t_0)^{\alpha+1}+(t_0-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)}\right)\max\left\{\left\|D^{\alpha}_{t_0-}l_1\right\|_{\infty,[a,t_0]},\left\|D^{\alpha}_{t_0}l_1\right\|_{\infty,[t_0,b]}\right\},\right.$$

$$\left(\frac{(b-t_0)^{\alpha}+(t_0-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)}\right)\max\left\{\left\|D^{\alpha}_{t_0-}l_1\right\|_{L_1([a,t_0])},\left\|D^{\alpha}_{t_0}l_1\right\|_{L_1([t_0,b])}\right\},$$

$$\left(\frac{(b-t_0)^{\alpha+\frac{1}{p}}+(t_0-a)^{\alpha+\frac{1}{p}}}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right)\cdot$$

$$\left.\max\left\{\left\|D^{\alpha}_{t_0-}l_1\right\|_{L_q([a,t_0])},\left\|D^{\alpha}_{t_0}l_1\right\|_{L_q([t_0,b])}\right\}\right\}+$$

$$\min\left\{\left(\frac{(b-t_0)^{\alpha+1}+(t_0-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)}\right)\max\left\{\left\|D^{\alpha}_{t_0-}l_2\right\|_{\infty,[a,t_0]},\left\|D^{\alpha}_{t_0}l_2\right\|_{\infty,[t_0,b]}\right\},\right.$$

$$\left(\frac{(b-t_0)^{\alpha}+(t_0-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)}\right)\max\left\{\left\|D^{\alpha}_{t_0-}l_2\right\|_{L_1([a,t_0])},\left\|D^{\alpha}_{t_0}l_2\right\|_{L_1([t_0,b])}\right\},$$

$$\left(\frac{(b-t_0)^{\alpha+\frac{1}{p}}+(t_0-a)^{\alpha+\frac{1}{p}}}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right)\cdot$$

$$\left.\max\left\{\left\|D^{\alpha}_{t_0-}l_2\right\|_{L_q([a,t_0])},\left\|D^{\alpha}_{t_0}l_2\right\|_{L_q([t_0,b])}\right\}\right\}. \tag{93}$$

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