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Balanced Canavati type Fractional Opial Inequalities

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

Here we present L_p, $p > 1$, fractional Opial type inequalities subject to high order boundary conditions. They involve the right and left Canavati type generalised fractional derivatives. These derivatives are mixed together into the balanced Canavati type generalised fractional derivative. This balanced fractional derivative is introduced and activated here for the first time.

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1 Introduction

This article is inspired by the famous theorem of Z. Opial [10], 1960, which follows

Theorem 1 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{1}$$

In (1), the constant $\frac{h}{4}$ is the best possible. Inequality (1) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases}$$

where $c > 0$ is an arbitrary constant.

To prove easier Theorem 1, Beesack [4] proved the following well-known Opial type inequality which is used very commonly.

This is another insiration to our work.

Theorem 2 *Let $x(t)$ be absolutely continuous in $[0,a]$, and $x(0) = 0$. Then*

$$\int_0^a |x(t)\,x'(t)|\,dt \leq \frac{a}{2}\int_0^a (x'(t))^2\,dt. \tag{2}$$

Inequality (2) is sharp, it is attained by $x(t) = ct$, $c > 0$ is an arbitrary constant.

Opial type inequalities are used a lot in proving uniqueness of solutions to differential equations, also to give upper bounds to their solutions.

By themselves have made a great subject of intensive research and there exists a great literature about them.

Typical and great sources on them are the monographs [1], [2].

We define here the balanced Canavati type fractional derivative and we prove related Opial type inequalities subject to boundary conditions.

These have smaller constants than in other Opial inequalities when using traditional fractional derivatives.

2 Background

Let $\nu > 0$, $n := [\nu]$ (integral part of ν), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. The gamma function Γ is given by $\Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1}dt$. Here $[a,b] \subseteq \mathbb{R}$, $x, x_0 \in [a,b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a,b])$ and define the left Riemann-Liouville integral

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)}\int_{x_0}^x (x-t)^{\nu-1} f(t)\,dt, \tag{3}$$

$x_0 \leq x \leq b$. We define the subspace $C_{x_0}^\nu([a.b])$ of $C^n([a,b])$:

$$C_{x_0}^\nu([a,b]) := \left\{ f \in C^n([a,b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0,b]) \right\}. \tag{4}$$

For $f \in C_{x_0}^\nu([a,b])$, we define the left generalized ν-fractional derivative of f over $[x_0,b]$ as

$$D_{x_0}^\nu f := \left(J_{1-\alpha}^{x_0} f^{(n)}\right)', \tag{5}$$

see [2], p. 24, and Canavati derivative in [5].

Notice that $D_{x_0}^\nu f \in C([x_0,b])$.

We need the following generalization of Taylor's formula at the fractional level, see [2], pp. 8-10, and [5].

Theorem 3 *Let $f \in C_{x_0}^{\nu}([a,b])$, $x_0 \in [a,b]$ fixed.*

(i) If $\nu \geq 1$ then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + ... + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} \tag{6}$$

$$+ \left(J_{\nu}^{x_0} D_{x_0}^{\nu} f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0.$$

(ii) If $0 < \nu < 1$ we get

$$f(x) = \left(J_{\nu}^{x_0} D_{x_0}^{\nu} f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0 \tag{7}$$

We will use (6) and (7).

Furthermore we need:

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J)\, dJ, \tag{8}$$

$x \in [a,b]$, see also [3], [6], [7], [8], [11]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f := (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{10}$$

see [3]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [3], we need the following Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

1) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b_-)}{k!}(x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{11}$$

2) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x), \quad \forall\, x \in [a,b]. \tag{12}$$

We will use (11) and (12).

We introduce a new concept

Definition 5 *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a}^{\alpha}\left(\left[a, \frac{\alpha+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{13}$$

3 Main Result

We give our main result

Theorem 6 *Let $f \in C([a,b])$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. Assume further that*

$$f^{(k)}(a) = f^{(k)}(b) = 0, \quad k = 0, 1, ..., m-1; \tag{14}$$

$$p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \ \ \textit{and} \ \ \alpha > \frac{1}{q}.$$

(i) Case of $1 < q \leq 2$. Then

$$\int_a^b |f(\omega)| \, |D^{\alpha} f(\omega)| \, d\omega \leq \tag{15}$$

$$\frac{2^{-\left(\alpha + \frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b |D^{\alpha} f(\omega)|^q \, d\omega\right)^{\frac{2}{q}}.$$

(ii) Case of $q > 2$. Then

$$\int_a^b |f(\omega)| \, |D^{\alpha} f(\omega)| \, d\omega \leq \tag{16}$$

$$\frac{2^{-\left(\alpha + \frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b |D^{\alpha} f(\omega)|^q \, d\omega\right)^{\frac{2}{q}}.$$

(iii) When $p = q = 2$, $\alpha > \frac{1}{2}$, then

$$\int_a^b |f(\omega)| \, |D^{\alpha} f(\omega)| \, d\omega \leq \tag{17}$$

$$\frac{2^{-\left(\alpha + \frac{1}{2}\right)} (b-a)^{\alpha}}{\Gamma(\alpha) \left[\sqrt{2\alpha(2\alpha-1)}\right]} \left(\int_a^b |D^{\alpha} f(\omega)|^2 \, d\omega\right).$$

Remark 7 *Let us say that $\alpha = 1$, then by (17) we obtain*

$$\int_a^b |f(\omega)| \, |f'(\omega)| \, d\omega \leq \frac{(b-a)}{4} \left(\int_a^b (f'(\omega))^2 \, d\omega\right), \tag{18}$$

that is reproving and recovering Opial's inequality (1), see [10], see also Olech's result [9].

Proof. of Theorem 6. Let $x \in \left[a, \frac{a+b}{2}\right]$, we have by assumption $f^{(k)}(a) = 0$, $k = 0, 1, ..., m-1$ and Theorem 3 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_a^\alpha f(\tau) \, d\tau. \tag{19}$$

Let $x \in \left[\frac{a+b}{2}, b\right]$, we have by assumption $f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$ and Theorem 4 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} D_{b-}^\alpha f(\tau) \, d\tau. \tag{20}$$

Using Hölder's inequality on (19) we get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} |D_a^\alpha f(\tau)| \, d\tau \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x \left((x-\tau)^{\alpha-1} \right)^p d\tau \right)^{\frac{1}{p}} \left(\int_a^x |D_a^\alpha f(\tau)|^q \, d\tau \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x |D_a^\alpha f(\tau)|^q \, d\tau \right)^{\frac{1}{q}}. \tag{21}$$

Set

$$z(x) := \int_a^x |D_a^\alpha f(\tau)|^q \, d\tau, \quad (z(a) = 0).$$

Then

$$z'(x) = |D_a^\alpha f(x)|^q,$$

and

$$|D_a^\alpha f(x)| = (z'(x))^{\frac{1}{q}}, \quad \text{all } a \leq x \leq \frac{a+b}{2}.$$

Therefore by (21) we have

$$|f(\omega)| \, |D_a^\alpha f(\omega)| \leq \frac{1}{\Gamma(\alpha)} \frac{(\omega-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} (z(\omega) \, z'(\omega))^{\frac{1}{q}}, \tag{22}$$

all $a \leq \omega \leq x \leq \frac{a+b}{2}$.

Next working similarly with (20) we obtain

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} \left| D_{b-}^\alpha f(\tau) \right| d\tau \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_x^b \left((\tau-x)^{\alpha-1} \right)^p d\tau \right)^{\frac{1}{p}} \left(\int_x^b \left| D_{b-}^\alpha f(\tau) \right|^q d\tau \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)}\frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_x^b \left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau\right)^{\frac{1}{q}}. \tag{23}$$

Set

$$\lambda(x) := \int_x^b \left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau = -\int_b^x \left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau, \quad (\lambda(b)=0).$$

Then

$$\lambda'(x) = -\left|D_{b-}^{\alpha}f(x)\right|^q$$

and

$$\left|D_{b-}^{\alpha}f(x)\right| = \left(-\lambda'(x)\right)^{\frac{1}{q}}, \quad \text{all } \frac{a+b}{2} \le x \le b.$$

Therefore by (23) we have

$$|f(\omega)|\left|D_{b-}^{\alpha}f(\omega)\right| \le \frac{1}{\Gamma(\alpha)}\frac{(b-\omega)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left(-\lambda(\omega)\lambda'(\omega)\right)^{\frac{1}{q}}, \tag{24}$$

all $\frac{a+b}{2} \le x \le \omega \le b$.

Next we integrate (22) over $[a,x]$ to obtain

$$\int_a^x |f(\omega)|\left|D_a^{\alpha}f(\omega)\right| d\omega \le$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\int_a^x (\omega-a)^{\frac{p(\alpha-1)+1}{p}}\left(z(\omega)z'(\omega)\right)^{\frac{1}{q}} d\omega \le$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_a^x (\omega-a)^{p(\alpha-1)+1} d\omega\right)^{\frac{1}{p}}\left(\int_a^x z(\omega)z'(\omega)d\omega\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\frac{(x-a)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}}\frac{z(x)^{\frac{2}{q}}}{2^{\frac{1}{q}}} =$$

$$\frac{2^{-\frac{1}{q}}(x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^x \left|D_a^{\alpha}f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{25}$$

So we have proved

$$\int_a^x |f(\omega)|\left|D_a^{\alpha}f(\omega)\right| d\omega \le$$

$$\frac{2^{-\frac{1}{q}}(x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^x \left|D_a^{\alpha}f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}, \tag{26}$$

for all $a \le x \le \frac{a+b}{2}$.

By (26) we get

$$\int_a^{\frac{a+b}{2}} |f(\omega)|\left|D_a^{\alpha}f(\omega)\right| d\omega \le$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}}\, 2^{-\left[\frac{p(\alpha-1)+2}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^{\frac{a+b}{2}}\left|D_a^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{27}$$

Similarly we integrate (24) over $[x,b]$ to obtain

$$\int_x^b |f(\omega)|\left|D_{b-}^{\alpha} f(\omega)\right| d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\int_x^b (b-\omega)^{\frac{p(\alpha-1)+1}{p}}\left(-\lambda(\omega)\lambda'(\omega)\right)^{\frac{1}{q}} d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_x^b (b-\omega)^{p(\alpha-1)+1} d\omega\right)^{\frac{1}{p}}\left(\int_x^b -\lambda(\omega)\lambda'(\omega)\, d\omega\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\frac{(b-x)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}}\frac{(\lambda(x))^{\frac{2}{q}}}{2^{\frac{1}{q}}}. \tag{28}$$

We have proved that

$$\int_x^b |f(\omega)|\left|D_{b-}^{\alpha} f(\omega)\right| d\omega \leq$$

$$\frac{2^{-\frac{1}{q}}(b-x)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_x^b\left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}, \tag{29}$$

for all $\frac{a+b}{2} \leq x \leq b$.

By (29) we get

$$\int_{\frac{a+b}{2}}^b |f(\omega)|\left|D_{b-}^{\alpha} f(\omega)\right| d\omega \leq$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}}\, 2^{-\left[\frac{p(\alpha-1)+2}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_{\frac{a+b}{2}}^b\left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{30}$$

Adding (27) and (30) we get

$$\int_a^b |f(\omega)|\left|D^{\alpha} f(\omega)\right| d\omega \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\cdot$$

$$\left[\left(\int_a^{\frac{a+b}{2}}\left|D_a^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}+\left(\int_{\frac{a+b}{2}}^b\left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}\right] =: (*.) \tag{31}$$

Assume $1 < q \leq 2$, then $\frac{2}{q} \geq 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\cdot$$

$$\left[\int_a^{\frac{a+b}{2}} \left|D_a^{\alpha} f(\omega)\right|^q d\omega + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right]^{\frac{2}{q}} = \tag{32}$$

$$\frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b \left|D^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{33}$$

So for $1 < q \leq 2$ we have proved (15).

Assume now $q > 2$, then $0 < \frac{2}{q} < 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)} 2^{1-\frac{2}{q}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}.$$

$$\left[\int_a^{\frac{a+b}{2}} \left|D_a^{\alpha} f(\omega)\right|^q d\omega + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right]^{\frac{2}{q}} =$$

$$\frac{2^{-\left(\alpha+\frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b \left|D^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{34}$$

So when $q > 2$ we have established (16).

(iii) The case of $p = q = 2$, see (17), is obvious, it derives from (15) immediately. ■

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