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ERROR BOUNDS IN APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL OF C^{n+1}-CLASS INTEGRANDS AND NONSMOOTH INTEGRATORS

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Abstract. In the present paper we investigate the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(\lambda)\,du(\lambda)$ in the case when the integrand f is $(n+1)$-time differentiable $(n \geq 0)$ and the derivative $f^{(n+1)}$ is continuous on $[a,b]$, while the integrator u is Riemann integrable on $[a,b]$. A priory error bounds for different classes of functions are provided.

1. Introduction

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$, where $f, u : [a,b] \to \mathbb{R}$ are functions for which the above integral exists, S.S. Dragomir established in 2000, see [18], the following integral identity:

$$[u(b) - u(a)]\,f(x) - \int_a^b f(t)du(t) = \int_a^x [u(t) - u(a)]\,df(t) + \int_x^b [u(t) - u(b)]\,df(t), \quad x \in [a,b], \tag{1.1}$$

provided that the involved integrals exist. That happens, for instance when one of the functions is continuous and the second is of bounded variation or if one is Lipschitzian and the second is Riemann integrable on the interval $[a,b]$.

We observe that, in the particular case when $u(t) = t$, $t \in [a,b]$, the above identity reduces to the celebrated *Montgomery identity*(see [36, p. 565]) that has been extensively used by many authors in obtaining various *inequalities of Ostrowski type*.

For a comprehensive recent collection of works related to Ostrowski's inequality, see the book [30], the papers [2] – [11], [33], [39], [41] and [43].

It has been shown in [18] that, if $f : [a,b] \to \mathbb{R}$ is a function of *bounded variation* and $u : [a,b] \to \mathbb{R}$ is of *r-H-Hölder type*, i.e.,

$$|u(x) - u(y)| \leq H|x-y|^r \quad \text{for any} \quad x, y \in [a,b]\ , \tag{1.2}$$

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where $r \in (0,1]$ and $H > 0$ are given, then

$$\left|[u(b)-u(a)]\,f(x)-\int_a^b f(t)du(t)\right| \tag{1.3}$$
$$\leq H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right]$$
$$\leq H\times\begin{cases}[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]; \\ [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[(\bigvee_a^x(f))^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad\text{if}\quad p>1,\frac{1}{p}+\frac{1}{q}=1\ ; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f)\ ;\end{cases}$$

for any $x \in [a,b]$, where $\bigvee_c^d(f)$ denotes the *total variation*of f on $[c,d]$. Out of (1.3) we can obtain the following *mid-point inequality*

$$\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right| \leq \frac{H(b-a)^r}{2^r}\cdot\bigvee_a^b(f). \tag{1.4}$$

The dual result (see [20]), can be stated as follows:

If $u : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is of q-K-Hölder type, then

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)du(t)\right| \leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q\bigvee_a^b(u) \tag{1.5}$$

for any $x \in [a,b]$. In particular, for $x = \frac{a+b}{2}$, we get the mid-point inequality

$$\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)du(t)\right| \leq \frac{K(b-a)^q}{2^q}\cdot\bigvee_a^b(u). \tag{1.6}$$

In 2001, Dragomir et. al, see [26], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ in a different manner, considered the following *generalized trapezoid formula*

$$[u(b)-u(x)]\,f(b)+[u(x)-u(a)]\,f(a)\,,\quad x\in[a,b]\,.$$

They proved the error estimate

$$\left|\int_a^b f(t)\,du(t)-[u(b)-u(x)]\,f(b)-[u(x)-u(a)]\,f(a)\right| \tag{1.7}$$
$$\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f)$$

for any $x \in [a,b]$, provided that $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ and u is of r-H-Hölder type.

The case $x = \frac{a+b}{2}$ provides the simpler result

$$\left|\int_a^b f(t)\,du(t) - \left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(b) - \left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a)\right| \leq H \frac{1}{2^r} (b-a)^r \bigvee_a^b (f). \tag{1.8}$$

In [12], the following dual result has been obtained as well:

$$\begin{aligned}
&\left|\int_a^b f(t)\,du(t) - [u(b) - u(x)] f(b) - [u(x) - u(a)] f(a)\right| \\
&\leq K \left[(x-a)^q \bigvee_a^x (u) + (b-x)^q \bigvee_x^b (u)\right] \\
&\leq K \times \begin{cases}
[(x-a)^q + (b-x)^q] \left[\frac{1}{2}\bigvee_a^b (u) + \frac{1}{2}\left|\bigvee_a^x (u) - \bigvee_x^b (u)\right|\right]; \\
\left[(x-a)^{\alpha q} + (b-x)^{\beta q}\right]^{\frac{1}{\alpha}} \left[\left[\bigvee_a^x (u)\right]^\beta + \left[\bigvee_x^b (u)\right]^\beta\right]^{\frac{1}{\beta}} \\
\qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\
\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q \bigvee_a^b (u);
\end{cases}
\end{aligned} \tag{1.9}$$

for any $x \in [a,b]$, provided that f is of q-K-Hölder type and u is of bounded variation.

In particular we have

$$\left|\int_a^b f(t)\,du(t) - \left[u(b) - u\left(\frac{a+b}{2}\right)\right] f(b) - \left[u\left(\frac{a+b}{2}\right) - u(a)\right] f(a)\right| \leq K \frac{1}{2^q} (b-a)^q \bigvee_a^b (u) \tag{1.10}$$

For other inequalities of this type, see the recent papers [9], [6] and [15].

For some classical results concerning the approximation of the Riemann-Stieltjes integral, see the seminal paper due to Michael Tortorella from 1990, [42]. Earlier results in this direction, however, were provided by Dubuc and Todor in their 1984 and 1987 papers [31] and [32], respectively.

For recent results concerning the approximation of the Riemann-Stieltjes integral, see the work of Diethelm [16], Liu [34], Mercer [35], Munteanu [38], Mozyrska et al. [37] and the references therein. For other recent results obtained in the same direction by the first author and his colleagues from RGMIA, see [7], [6], [8], [15], [13], [14], [24] and [21]. A comprehensive list of preprints related to this subject may be found at http://rgmia.org.

Motivated by the above results, in the present paper we investigate the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(\lambda)\,du(\lambda)$ in the case when the integrand f is $(n+1)$-time differentiable $(n \geq 0)$ and the derivative $f^{(n+1)}$ is continuous on $[a,b]$, while the integrator u is Riemann integrable on $[a,b]$. A priory error bounds for different classes of functions are provided.

2. Some Approximation Rules

In this section we establish some representation results in the case when the integrand f is $(n+1)$-time differentiable $(n \geq 0)$ and the derivative $f^{(n+1)}$ is continuous on $[a,b]$, while the integrator u is Riemann integrable on $[a,b]$.

Theorem 1. *Assume that the function $f : I \to \mathbb{C}$ is of class $C^{n+1}$for $n \geq 1$, namely the derivative $f^{(n+1)}$ exists and is continuous on $\mathring{I}$, the interior of I. If $a, b \in \mathring{I}$ with $a < b, c \in [a,b]$ and $u : [a,b] \to \mathbb{C}$ is Riemann integrable on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(\lambda)\, du(\lambda)$ exists, we have the representation*

$$\int_a^b f(\lambda)\, du(\lambda) = S_n(f,u,a,c,b) + W_n(f,u,a,c,b), \tag{2.1}$$

where $S_n(f,u,a,c,b)$ is given by

$$S_n(f,u,a,c,b) := f(b)\,u(b) - f(a)\,u(a) - \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k+1)}(c) \int_a^b (\lambda - c)^k u(\lambda)\, d\lambda \tag{2.2}$$

and the reminder has the form

$$W_n(f,u,a,c,b) := -\frac{1}{(n-1)!} \int_a^b \left(\int_c^\lambda (\lambda - t)^{n-1} f^{(n+1)}(t)\, dt \right) u(\lambda)\, d\lambda, \tag{2.3}$$

where the integrals are taken in the Riemann sense.

Proof. Under the assumption of the theorem, we utilize the following Taylor's representation

$$f(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(\lambda - c)^k + \frac{1}{n!} \int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt \tag{2.4}$$

that holds for any $c \in [a,b]$ and $n \geq 0$. The integral in (2.4) is taken in the Riemann sense.

Further on, by integrating the identity (2.4) over $du(t)$ we get

$$\int_a^b f(\lambda)\, du(\lambda) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_a^b (\lambda - c)^k du(\lambda) + \frac{1}{n!} \int_a^b \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt \right) du(\lambda). \tag{2.5}$$

The Riemann-Stieltjes integrals in the right side of the equality (2.5) all exists since the integrands involved are clearly Lipschitzian on $[a,b]$ and the integrator is Riemann integrable on $[a,b]$. This implies that the Riemann-Stieltjes integral $\int_a^b f(\lambda)\, du(\lambda)$ exists and the representation (2.5) is valid as stated.

Utilizing the integration by parts formula for the Riemann-Stieltjes integral we have for $k \geq 1$ that

$$\begin{aligned} \int_a^b (\lambda - c)^k \, du(\lambda) &= (\lambda - c)^k u(\lambda)\Big|_a^b - k \int_a^b (\lambda - c)^{k-1} u(\lambda) \, d\lambda \\ &= (b-c)^k u(b) + (-1)^{k+1} (c-a)^k u(a) \\ &\quad - k \int_a^b (\lambda - c)^{k-1} u(\lambda) \, d\lambda. \end{aligned} \tag{2.6}$$

For $k = 0$ we have $\int_a^b du(\lambda) = u(b) - u(a)$.

Therefore, by (2.6) we get

$$\begin{aligned} &\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \int_a^b (\lambda - c)^k \, du(\lambda) \\ &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left[(b-c)^k u(b) + (-1)^{k+1} (c-a)^k u(a) \right] \\ &- \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k+1)}(c) \int_a^b (\lambda - c)^k u(\lambda) \, d\lambda. \end{aligned} \tag{2.7}$$

Now, since $f : I \to \mathbb{C}$ is of class C^{n+1} then we observe that integrating by parts in the Riemann-Stieltjes integral and utilizing Leibniz's rule for differentiation under the integral sign we have successively

$$\begin{aligned} &\int_a^b \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t) \, dt \right) du(\lambda) \\ &= \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t) \, dt \right) u(\lambda) \Bigg|_a^b \\ &- \int_a^b u(\lambda) \frac{d \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t) \, dt \right)}{d\lambda} d\lambda \\ &= \left(\int_c^b (b - t)^n f^{(n+1)}(t) \, dt \right) u(b) - \left(\int_c^a (a - t)^n f^{(n+1)}(t) \, dt \right) u(a) \\ &- n \int_a^b \left(\int_c^\lambda (\lambda - t)^{n-1} f^{(n+1)}(t) \, dt \right) u(\lambda) \, d\lambda. \end{aligned} \tag{2.8}$$

Since, by the representation (2.4) for the function $f : I \to \mathbb{C}$ that is of class C^{n+1}we have

$$\int_c^b (b - t)^n f^{(n+1)}(t) \, dt = \left[f(b) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (b - c)^k \right] n!$$

and

$$\int_c^a (a - t)^n f^{(n+1)}(t) \, dt = \left[f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) (a - c)^k \right] n!,$$

then we get from (2.8) that

$$\begin{aligned}
&\int_a^b \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt \right) du(\lambda) \\
&= n! \left[f(b) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(b-c)^k \right] u(b) \\
&- n! \left[f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(a-c)^k \right] u(a) \\
&- n \int_a^b \left(\int_c^\lambda (\lambda - t)^{n-1} f^{(n+1)}(t)\, dt \right) u(\lambda)\, d\lambda.
\end{aligned} \tag{2.9}$$

If we divide (2.9) by $n!$ we get for $n \geq 1$ that

$$\begin{aligned}
&\frac{1}{n!} \int_a^b \left(\int_c^\lambda (\lambda - t)^n f^{(n+1)}(t)\, dt \right) du(\lambda) \\
&= \left[f(b) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(b-c)^k \right] u(b) \\
&- \left[f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(a-c)^k \right] u(a) \\
&- \frac{1}{(n-1)!} \int_a^b \left(\int_c^\lambda (\lambda - t)^{n-1} f^{(n+1)}(t)\, dt \right) u(\lambda)\, d\lambda.
\end{aligned} \tag{2.10}$$

On making use of the identities (2.5), (2.7) and (2.10) we have

$$\begin{aligned}
\int_a^b f(\lambda)\, du(\lambda) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c) \left[(b-c)^k u(b) + (-1)^{k+1} (c-a)^k u(a) \right] \\
&- \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k+1)}(c) \int_a^b (\lambda - c)^k u(\lambda)\, d\lambda \\
&+ \left[f(b) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(b-c)^k \right] u(b) \\
&- \left[f(a) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(a-c)^k \right] u(a) \\
&- \frac{1}{(n-1)!} \int_a^b \left(\int_c^\lambda (\lambda - t)^{n-1} f^{(n+1)}(t)\, dt \right) u(\lambda)\, d\lambda \\
&= S_n(f, u, a, c, b) + W_n(f, u, a, c, b)
\end{aligned}$$

where $S_n(f, u, a, c, b)$ and $W_n(f, u, a, c, b)$ are given by (2.2) and (2.3). □

Remark 1. *Under the assumptions of Theorem 1 we have the representation*

$$\int_a^b f(\lambda)\, du(\lambda) =_d S_n(f, u, a, b) +_d W_n(f, u, a, b), \tag{2.11}$$

where ${}_{d}S_{n}(f,u,a,b)$ *is given by*

$$
\begin{aligned}
{}_{d}S_{n}(f,u,a,b) &:= S_{n}(f,u,a,a,b) = f(b)u(b)-f(a)u(a) \\
&\quad -\sum_{k=0}^{n-1}\frac{1}{k!}f^{(k+1)}(a)\int_{a}^{b}(\lambda-a)^{k}u(\lambda)\,d\lambda
\end{aligned}
\tag{2.12}
$$

and the reminder has the form

$$
\begin{aligned}
{}_{d}W_{n}(f,u,a,b) &:= W_{n}(f,u,a,a,b) \\
&= -\frac{1}{(n-1)!}\int_{a}^{b}\left(\int_{a}^{\lambda}(\lambda-t)^{n-1}f^{(n+1)}(t)\,dt\right)u(\lambda)\,d\lambda.
\end{aligned}
\tag{2.13}
$$

By taking $c=\frac{a+b}{2}$*, we have the representation*

$$
\int_{a}^{b}f(\lambda)\,du(\lambda) = {}_{M}S_{n}(f,u,a,b) + {}_{M}W_{n}(f,u,a,b),
\tag{2.14}
$$

where

$$
\begin{aligned}
{}_{M}S_{n}(f,u,a,b) &:= S_{n}\left(f,u,a,\frac{a+b}{2},b\right) = f(b)u(b)-f(a)u(a) \\
&\quad -\sum_{k=0}^{n-1}\frac{1}{k!}f^{(k+1)}\left(\frac{a+b}{2}\right)\int_{a}^{b}\left(\lambda-\frac{a+b}{2}\right)^{k}u(\lambda)\,d\lambda
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
{}_{M}W_{n}(f,u,a,b) &:= W_{n}\left(f,u,a,\frac{a+b}{2},b\right) \\
&= -\frac{1}{(n-1)!}\int_{a}^{b}\left(\int_{\frac{a+b}{2}}^{\lambda}(\lambda-t)^{n-1}f^{(n+1)}(t)\,dt\right)u(\lambda)\,d\lambda.
\end{aligned}
\tag{2.16}
$$

Finally, by choosing $c=b$ *we have*

$$
\int_{a}^{b}f(\lambda)\,du(\lambda) = {}_{u}S_{n}(f,u,a,b) + {}_{u}W_{n}(f,u,a,b)
\tag{2.17}
$$

where

$$
\begin{aligned}
{}_{u}S_{n}(f,u,a,b) &:= S_{n}(f,u,a,b,b) = f(b)u(b)-f(a)u(a) \\
&\quad +\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{k!}f^{(k+1)}(b)\int_{a}^{b}(b-\lambda)^{k}u(\lambda)\,d\lambda
\end{aligned}
\tag{2.18}
$$

and the new reminder has the form

$$
\begin{aligned}
{}_{u}W_{n}(f,u,a,b) &:= W_{n}(f,u,a,b,b) \\
&= \frac{(-1)^{n+1}}{(n-1)!}\int_{a}^{b}\left(\int_{\lambda}^{b}(t-\lambda)^{n-1}f^{(n+1)}(t)\,dt\right)u(\lambda)\,d\lambda.
\end{aligned}
\tag{2.19}
$$

3. Error Bounds

We use the following norm notations for integrable functions:

$$\|g\|_{[x,y],p} := \left|\int_x^y |g(s)|^p \, ds\right|^{1/p} \text{ if } p \geq 1, \; x, y \in [a,b] \text{ and } g \in L_p[a,b];$$

and for $g \in L_\infty [a,b]$ we denote

$$\|g\|_{[x,y],\infty} := \begin{cases} ess\sup_{s\in[x,y]} |g(s)| & \text{if } x < y \\ \\ ess\sup_{s\in[y,x]} |g(s)| & \text{if } y < x. \end{cases}$$

Theorem 2. *Assume that the function $f : I \to \mathbb{C}$ is of class $C^{n+1}$for $n \geq 1$, namely the derivative $f^{(n+1)}$ exists and is continuous on $\mathring{I}$, the interior of I. If $a, b \in \mathring{I}$ with $a < b, c \in [a,b]$ and $u : [a,b] \to \mathbb{C}$ is Riemann integrable on $[a,b]$, then the error $W_n(f,u,a,c,b)$ in the approximation rule (2.1) satisfies the bounds*

$$|W_n(f,u,a,c,b)| \leq B_{1,n}(f,u,a,c) + B_{2,n}(f,u,c,b) \tag{3.1}$$

where

$$B_{1,n}(f,u,a,c) := \frac{1}{(n-1)!} \int_a^c \left(\int_\lambda^c (t-\lambda)^{n-1} \left| f^{(n+1)}(t) \right| dt \right) |u(\lambda)| \, d\lambda \tag{3.2}$$

and

$$B_{2,n}(f,u,c,b) := \frac{1}{(n-1)!} \int_c^b \left(\int_c^\lambda (\lambda-t)^{n-1} \left| f^{(n+1)}(t) \right| dt \right) |u(\lambda)| \, d\lambda. \tag{3.3}$$

Moreover, we have

$$B_{1,n}(f,u,a,c) \tag{3.4}$$

$$\leq \begin{cases} \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}}{(n-1)![(n-1)q+1]^{1/q}} \int_a^c (c-\lambda)^{n-1+1/q} |u(\lambda)| \, d\lambda, & p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \\ \frac{1}{n!} \left\|f^{(n+1)}\right\|_{[a,c],\infty} \int_a^c (c-\lambda)^n |u(\lambda)| \, d\lambda \end{cases}$$

and

$$B_{2,n}(f,u,c,b) \tag{3.5}$$

$$\leq \begin{cases} \frac{\left\|f^{(n+1)}\right\|_{[c,b],r}}{(n-1)![(n-1)s+1]^{1/s}} \int_c^b (\lambda-c)^{n-1+1/s} |u(\lambda)| \, d\lambda, & r > 1, \frac{1}{r} + \frac{1}{s} = 1 \\ \\ \frac{1}{n!} \left\|f^{(n+1)}\right\|_{[c,b],\infty} \int_c^b (\lambda-c)^n |u(\lambda)| \, d\lambda \end{cases}$$

for any $c \in [a,b]$.

Proof. Utilizing the representation (2.3), then we have for each $c \in [a,b]$ that

$$
\begin{aligned}
(3.6)\qquad & \left|W_n(f,u,a,c,b)\right| \\
&\leq \frac{1}{(n-1)!}\left[\left|\int_a^c\left(\int_c^\lambda (\lambda-t)^{n-1} f^{(n+1)}(t)\,dt\right)u(\lambda)\,d\lambda\right|\right. \\
&\left.+\left|\int_c^b\left(\int_c^\lambda (\lambda-t)^{n-1} f^{(n+1)}(t)\,dt\right)u(\lambda)\,d\lambda\right|\right] \\
&\leq \frac{1}{(n-1)!}\left[\int_a^c\left|\int_\lambda^c (\lambda-t)^{n-1} f^{(n+1)}(t)\,dt\right|\left|u(\lambda)\right|d\lambda\right. \\
&\left.+\int_c^b\left|\int_c^\lambda (\lambda-t)^{n-1} f^{(n+1)}(t)\,dt\right|\left|u(\lambda)\right|d\lambda\right] \\
&\leq \frac{1}{(n-1)!}\left[\int_a^c\left(\int_\lambda^c (t-\lambda)^{n-1}\left|f^{(n+1)}(t)\right|dt\right)\left|u(\lambda)\right|d\lambda\right. \\
&\left.+\int_c^b\left(\int_c^\lambda (\lambda-t)^{n-1}\left|f^{(n+1)}(t)\right|dt\right)\left|u(\lambda)\right|d\lambda\right]
\end{aligned}
$$

and the inequality (3.6) is proved.

Now, on utilizing the Hölder integral inequality we have

$$
\begin{aligned}
\int_\lambda^c (t-\lambda)^{n-1}\left|f^{(n+1)}(t)\right|dt &\leq \left\|f^{(n+1)}\right\|_{[\lambda,c],p}\left[\int_\lambda^c\left[(t-\lambda)^{n-1}\right]^q dt\right]^{1/q} \\
&\leq \left\|f^{(n+1)}\right\|_{[a,c],p}\frac{(c-\lambda)^{n-1+1/q}}{\left[(n-1)q+1\right]^{1/q}},\; p>1,\; \frac{1}{p}+\frac{1}{q}=1
\end{aligned}
$$

and

$$
\begin{aligned}
\int_c^\lambda (\lambda-t)^{n-1}\left|f^{(n+1)}(t)\right|dt &\leq \left\|f^{(n+1)}\right\|_{[c,\lambda],r}\left[\int_c^\lambda\left[(\lambda-t)^{n-1}\right]^s dt\right]^{1/s} \\
&\leq \left\|f^{(n+1)}\right\|_{[c,b],r}\frac{(\lambda-c)^{n-1+1/s}}{\left[(n-1)s+1\right]^{1/s}},\; r>1,\; \frac{1}{r}+\frac{1}{s}=1.
\end{aligned}
$$

These imply that

$$
(3.7)\qquad B_{1,n}(f,u,a,c) \leq \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}}{(n-1)!\left[(n-1)q+1\right]^{1/q}}\int_a^c (c-\lambda)^{n-1+1/q}\left|u(\lambda)\right|d\lambda
$$

for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and

$$
(3.8)\qquad B_{2,n}(f,u,c,b) \leq \frac{\left\|f^{(n+1)}\right\|_{[c,b],r}}{(n-1)!\left[(n-1)s+1\right]^{1/s}}\int_c^b (\lambda-c)^{n-1+1/s}\left|u(\lambda)\right|d\lambda
$$

for $r>1$, $\frac{1}{r}+\frac{1}{s}=1$ and $c\in[a,b]$.

We also have that

$$\int_{\lambda}^{c}(t-\lambda)^{n-1}\left|f^{(n+1)}(t)\right|dt \leq \left\|f^{(n+1)}\right\|_{[\lambda,c],\infty}\int_{\lambda}^{c}(t-\lambda)^{n-1}dt$$
$$\leq \left\|f^{(n+1)}\right\|_{[a,c],\infty}\frac{(c-\lambda)^{n}}{n}$$

and

$$\int_{c}^{\lambda}(\lambda-t)^{n-1}\left|f^{(n+1)}(t)\right|dt \leq \left\|f^{(n+1)}\right\|_{[c,\lambda],\infty}\int_{c}^{\lambda}(\lambda-t)^{n-1}dt$$
$$\leq \left\|f^{(n+1)}\right\|_{[c,b],\infty}\frac{(\lambda-c)^{n}}{n}, r>1, \frac{1}{r}+\frac{1}{s}=1,$$

which produce the bounds

$$B_{1,n}(f,u,a,c) \leq \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[a,c],\infty}\int_{a}^{c}(c-\lambda)^{n}|u(\lambda)|\,d\lambda \tag{3.9}$$

and

$$B_{2,n}(f,u,c,b) \leq \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[c,b],\infty}\int_{c}^{b}(\lambda-c)^{n}|u(\lambda)|\,d\lambda \tag{3.10}$$

for any and $c\in[a,b]$.

On making use of (3.7)-(3.10) we deduce the desired bounds (3.4) and (3.5). □

If the p-norms of the integrator u are known, then we have the following error bounds involving these norms:

Corollary 1. *With the assumptions of Theorem 2 we have the error bounds*

$$B_{1,n}(f,u,a,c) \leq \begin{cases} C_{1,n,p}(f,u,a,c), & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \\ C_{1,n,\infty}(f,u,a,c); \end{cases} \tag{3.11}$$

where

$$\begin{aligned} &C_{1,n,p}(f,u,a,c) \\ &:= \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}}{(n-1)!\left[(n-1)q+1\right]^{1/q}}\int_{a}^{c}(c-\lambda)^{n-1+1/q}|u(\lambda)|\,d\lambda \\ &\leq \begin{cases} \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}\|u\|_{[a,c],\infty}}{(n+1/q)(n-1)![(n-1)q+1]^{1/q}}(c-a)^{n+1/q}; \\ \\ \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}\|u\|_{[a,c],\alpha}}{[(n-1+1/q)\beta+1]^{1/\beta}(n-1)![(n-1)q+1]^{1/q}}(c-a)^{n-1+1/q+1/\beta}, \\ \\ \alpha,\beta>1, \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}\|u\|_{[a,c],1}}{(n-1)![(n-1)q+1]^{1/q}}(c-a)^{n-1+1/q}; \end{cases} \end{aligned} \tag{3.12}$$

and

$$(3.13)\quad \begin{aligned} & C_{1,n,\infty}(f,u,a,c) \\ & := \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[a,c],\infty}\int_a^c (c-\lambda)^n |u(\lambda)|\,d\lambda \\ & \leq \begin{cases} \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{[a,c],\infty}\|u\|_{[a,c],\infty}(c-a)^{n+1} \\ \\ \frac{1}{n!(n\delta+1)^{1/\delta}}\left\|f^{(n+1)}\right\|_{[a,c],\infty}\|u\|_{[a,c],\gamma}(c-a)^{n+1/\delta} \\ \\ \quad \gamma,\delta>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1 \\ \frac{1}{n!}\|u\|_{[a,c],1}\left\|f^{(n+1)}\right\|_{[a,c],\infty}(c-a)^n. \end{cases} \end{aligned}$$

while

$$(3.14)\quad B_{2,n}(f,u,c,b) \leq \begin{cases} C_{2,n,p}(f,u,a,c), \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \\ C_{2,n,\infty}(f,u,a,c); \end{cases}$$

where

$$(3.15)\quad \begin{aligned} & C_{2,n,p}(f,u,a,c) \\ & := \frac{\left\|f^{(n+1)}\right\|_{[c,b],r}}{(n-1)!\left[(n-1)s+1\right]^{1/s}}\int_c^b (\lambda-c)^{n-1+1/s}|u(\lambda)|\,d\lambda \\ & \leq \begin{cases} \frac{\left\|f^{(n+1)}\right\|_{[c,b],r}\|u\|_{[c,b],\infty}}{(n+1/s)(n-1)!\left[(n-1)s+1\right]^{1/s}}(b-c)^{n+1/s}; \\ \\ \frac{\left\|f^{(n+1)}\right\|_{[c,b],r}\|u\|_{[c,b],\varepsilon}}{\left[(n-1+1/s)\zeta+1\right]^{1/\zeta}(n-1)!\left[(n-1)s+1\right]^{1/s}}(b-c)^{n-1+1/s+1/\zeta}, \\ \\ \quad \varepsilon,\zeta>1,\ \frac{1}{\varepsilon}+\frac{1}{\zeta}=1; \\ \frac{\left\|f^{(n+1)}\right\|_{[c,b],r}\|u\|_{[c,b],1}}{(n-1)!\left[(n-1)s+1\right]^{1/s}}(c-a)^{n-1+1/s}; \end{cases} \end{aligned}$$

and

$$(3.16)\quad \begin{aligned} & C_{2,n,\infty}(f,u,a,c) \\ & := \frac{1}{n!}\left\|f^{(n+1)}\right\|_{[c,b],\infty}\int_c^b (\lambda-c)^n |u(\lambda)|\,d\lambda \\ & \leq \begin{cases} \frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_{[c,b],\infty}\|u\|_{[c,b],\infty}(b-c)^{n+1}; \\ \\ \frac{1}{n!(n\theta+1)^{1/\theta}}\left\|f^{(n+1)}\right\|_{[c,b],\infty}\|u\|_{[c,b],\eta}(b-c)^{n+1/\theta}, \\ \\ \quad \eta,\theta>1,\ \frac{1}{\eta}+\frac{1}{\theta}=1; \\ \frac{1}{n!}\|u\|_{[c,b],1}\left\|f^{(n+1)}\right\|_{[c,b],\infty}(b-c)^n; \end{cases} \end{aligned}$$

for any $c \in [a,b]$.

Proof. Utilizing the Hölder integral inequality we have

$$\int_a^c (c-\lambda)^{n-1+1/q} |u(\lambda)|\, d\lambda$$
$$\leq \begin{cases} \|u\|_{[a,c],\infty} \int_a^c (c-\lambda)^{n-1+1/q}\, d\lambda \\ \|u\|_{[a,c],\alpha} \left\{\int_a^c \left[(c-\lambda)^{n-1+1/q}\right]^\beta d\lambda\right\}^{1/\beta} & \alpha,\beta>1, \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \|u\|_{[a,c],1} \sup_{\lambda\in[a,c]} (c-\lambda)^{n-1+1/q} \end{cases}$$
$$= \begin{cases} \frac{1}{n+1/q} \|u\|_{[a,c],\infty} (c-a)^{n+1/q} \\ \frac{1}{[(n-1+1/q)\beta+1]^{1/\beta}} \|u\|_{[a,c],\alpha} (c-a)^{n-1+1/q+1/\beta} & \alpha,\beta>1, \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \|u\|_{[a,c],1} (c-a)^{n-1+1/q} \end{cases}$$

then

$$\frac{\left\|f^{(n+1)}\right\|_{[a,c],p}}{(n-1)!\,[(n-1)q+1]^{1/q}} \int_a^c (c-\lambda)^{n-1+1/q} |u(\lambda)|\, d\lambda$$
$$\leq \frac{\left\|f^{(n+1)}\right\|_{[a,c],p}}{(n-1)!\,[(n-1)q+1]^{1/q}}$$
$$\times \begin{cases} \frac{1}{n+1/q} \|u\|_{[a,c],\infty} (c-a)^{n+1/q} \\ \frac{1}{[(n-1+1/q)\beta+1]^{1/\beta}} \|u\|_{[a,c],\alpha} (c-a)^{n-1+1/q+1/\beta} & \alpha,\beta>1, \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \|u\|_{[a,c],1} (c-a)^{n-1+1/q} \end{cases}$$
$$= \begin{cases} \frac{\left\|f^{(n+1)}\right\|_{[a,c],p} \|u\|_{[a,c],\infty}}{(n+1/q)(n-1)![(n-1)q+1]^{1/q}} (c-a)^{n+1/q} \\ \frac{\left\|f^{(n+1)}\right\|_{[a,c],p} \|u\|_{[a,c],\alpha}}{[(n-1+1/q)\beta+1]^{1/\beta}(n-1)![(n-1)q+1]^{1/q}} (c-a)^{n-1+1/q+1/\beta} & \alpha,\beta>1, \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \frac{\left\|f^{(n+1)}\right\|_{[a,c],p} \|u\|_{[a,c],1}}{(n-1)![(n-1)q+1]^{1/q}} (c-a)^{n-1+1/q}. \end{cases}$$

By the Hölder integral inequality we also have

$$\int_a^c (c-\lambda)^n |u(\lambda)| \, d\lambda$$
$$\leq \begin{cases} \|u\|_{[a,c],\infty} \int_a^c (c-\lambda)^n \, d\lambda \\ \\ \|u\|_{[a,c],\gamma} \left\{ \int_a^c \left[(c-\lambda)^n\right]^\delta d\lambda \right\}^{1/\delta} & \gamma, \delta > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1 \\ \\ \|u\|_{[a,c],1} \sup_{\lambda \in [a,c]} (c-\lambda)^n \end{cases}$$
$$= \begin{cases} \frac{1}{n+1} \|u\|_{[a,c],\infty} (c-a)^{n+1} \\ \\ \frac{1}{(n\delta+1)^{1/\delta}} \|u\|_{[a,c],\gamma} (c-a)^{n+1/\delta} & \gamma, \delta > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1 \\ \\ \|u\|_{[a,c],1} (c-a)^n \end{cases}$$

which implies

$$\frac{1}{n!} \left\| f^{(n+1)} \right\|_{[a,c],\infty} \int_a^c (c-\lambda)^n |u(\lambda)| \, d\lambda$$
$$\leq \begin{cases} \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{[a,c],\infty} \|u\|_{[a,c],\infty} (c-a)^{n+1} \\ \\ \frac{1}{n!(n\delta+1)^{1/\delta}} \left\| f^{(n+1)} \right\|_{[a,c],\infty} \|u\|_{[a,c],\gamma} (c-a)^{n+1/\delta} & \gamma, \delta > 1, \frac{1}{\gamma} + \frac{1}{\delta} = 1 \\ \\ \frac{1}{n!} \|u\|_{[a,c],1} \left\| f^{(n+1)} \right\|_{[a,c],\infty} (c-a)^n . \end{cases}$$

By a similar argument we obtain the remaining bounds and the details are omitted.

□

Remark 2. *In applications is easier to use the following bounds that can be obtained from the above:*

$$\begin{aligned} &|W_n(f,u,a,c,b)| \\ &\leq \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{[a,c],\infty} \|u\|_{[a,c],\infty} (c-a)^{n+1} \\ &+ \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{[c,b],\infty} \|u\|_{[c,b],\infty} (b-c)^{n+1} \\ &\leq \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{[a,b],\infty} \|u\|_{[a,b],\infty} \left[(c-a)^{n+1} + (b-c)^{n+1} \right] \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &|W_n(f,u,a,c,b)| \\ &\leq \frac{1}{n!} \|u\|_{[a,c],1} \left\| f^{(n+1)} \right\|_{[a,c],\infty} (c-a)^n \\ &+ \frac{1}{n!} \|u\|_{[c,b],1} \left\| f^{(n+1)} \right\|_{[c,b],\infty} (b-c)^n \\ &\leq \frac{1}{n!} \|u\|_{[a,b],1} \left\| f^{(n+1)} \right\|_{[a,b],\infty} \left[(c-a)^n + (b-c)^n \right] \end{aligned} \tag{3.18}$$

for any $c \in [a,b]$.

The case when $c=\frac{a+b}{2}$ *provides the following bounds for the error* $W_n\left(f,u,a,\frac{a+b}{2},b\right)$ *:*

$$\begin{aligned}\left|W_n\left(f,u,a,\frac{a+b}{2},b\right)\right| & \\ \leq \frac{1}{(n+1)!2^{n+1}}(b-a)^{n+1} & \\ \times\left[\left\|f^{(n+1)}\right\|_{\left[a,\frac{a+b}{2}\right],\infty}\|u\|_{\left[a,\frac{a+b}{2}\right],\infty}+\left\|f^{(n+1)}\right\|_{\left[\frac{a+b}{2},b\right],\infty}\|u\|_{\left[\frac{a+b}{2},b\right],\infty}\right] & \\ \leq \frac{1}{(n+1)!2^{n}}(b-a)^{n+1}\left\|f^{(n+1)}\right\|_{[a,b],\infty}\|u\|_{[a,b],\infty} &\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\left|W_n\left(f,u,a,\frac{a+b}{2},b\right)\right| & \\ \leq \frac{1}{n!2^{n}}(b-a)^{n} & \\ \times\left[\|u\|_{\left[a,\frac{a+b}{2}\right],1}\left\|f^{(n+1)}\right\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|u\|_{\left[\frac{a+b}{2},b\right],1}\left\|f^{(n+1)}\right\|_{\left[\frac{a+b}{2},b\right],\infty}\right] & \\ \leq \frac{1}{n!2^{n}}(b-a)^{n}\|u\|_{[a,b],1}\left\|f^{(n+1)}\right\|_{[a,b],\infty}. &\end{aligned} \tag{3.20}$$

4. Applications

We consider the following *finite Laplace-Stieltjes transform* defined by

$$\left(\mathcal{L}_{[a,b]}g\right)(s):=\int_a^b e^{-st}dg(t) \tag{4.1}$$

where a,b are real numbers with $a<b$, s is a complex number and $g:[a,b]\to\mathbb{C}$ is a Riemann integrable function on $[a,b]$.

Since the function $f_s:[a,b]\to\mathbb{C}$, $f_s(t):=e^{-st}$ is continuous for any $s\in\mathbb{C}$, the transform (4.1) is well defined for any $s\in\mathbb{C}$.

We observe that the function f_s has derivatives of all orders and

$$f_s^{(k)}(t)=(-1)^k s^k e^{-st} \text{ for any } s\in\mathbb{C},\ t\in[a,b] \text{ and } k\geq 0. \tag{4.2}$$

We also observe that

$$\begin{aligned}\left\|f_s^{(n+1)}\right\|_{[a,b],\infty} &:= \sup_{t\in[a,b]}\left|f_s^{(n+1)}(t)\right| = |s|^{n+1}\sup_{t\in[a,b]}\left|e^{-st}\right| \\ &= |s|^{n+1}\sup_{t\in[a,b]}e^{-t\operatorname{Re}s} = |s|^{n+1}\times\begin{cases} e^{-a\operatorname{Re}s} & \text{if } \operatorname{Re}s\geq 0,\\ e^{-b\operatorname{Re}s} & \text{if } \operatorname{Re}s<0.\end{cases}\end{aligned}$$

To simplify the notations, we denote by

$$\beta_{[a,b]}(s):=\begin{cases} e^{-a\operatorname{Re}s} & \text{if } \operatorname{Re}s\geq 0,\\ e^{-b\operatorname{Re}s} & \text{if } \operatorname{Re}s<0,\end{cases} \tag{4.3}$$

therefore we have

$$\left\|f_s^{(n+1)}\right\|_{[a,b],\infty}=|s|^{n+1}\beta_{[a,b]}(s)$$

for any $n\in\mathbb{N}$ a natural number and any complex number $s\in\mathbb{C}$.

Proposition 1. *With the above assumptions and notations we have the representation*

$$\left(\mathcal{L}_{[a,b]}g\right)(s)=\mathcal{S}_n(g,a,c,b)(s)+\mathcal{W}_n(g,a,c,b)(s) \tag{4.4}$$

for any $c\in[a,b]$, *where* $\mathcal{S}_n(g,a,c,b)$ *is given by*

$$\begin{aligned}\mathcal{S}_n(g,a,c,b)(s)&:=e^{-sb}g(b)-f(a)e^{-sa}\\&+\sum_{k=0}^{n-1}\frac{1}{k!}(-1)^k s^{k+1}e^{-sc}\int_a^b(\lambda-c)^k g(\lambda)\,d\lambda\end{aligned} \tag{4.5}$$

and the reminder has the form

$$\mathcal{W}_n(g,a,c,b)(s):=\frac{(-1)^k}{(n-1)!}\int_a^b\left(\int_c^\lambda(\lambda-t)^{n-1}s^{k+1}e^{-st}dt\right)g(\lambda)\,d\lambda, \tag{4.6}$$

where the integrals are taken in the Riemann sense, $n\in\mathbb{N}$, $n\geq1$ *and* $s\in\mathbb{C}$.

The proof follows by Theorem 1 in which we have chosen $f_s(t):=e^{-st}$, $t\in[a,b]$, $s\in\mathbb{C}$, $u=g$ and performed the required calculations.

Utilizing Theorem 2 and Corollary 1 one can get various bounds for the remainder $\mathcal{W}_n(g,a,c,b)$. However, we will restrict ourselves by giving only two bounds as incorporated in the corollary below:

Corollary 2. *With the assumptions in Proposition 1 we have the following bounds for the remainder:*

$$\begin{aligned}&|\mathcal{W}_n(g,a,c,b)(s)|\\&\leq\frac{1}{(n+1)!}|s|^{n+1}\beta_{[a,c]}(s)\|g\|_{[a,c],\infty}(c-a)^{n+1}\\&+\frac{1}{(n+1)!}|s|^{n+1}\beta_{[c,b]}(s)\|g\|_{[c,b],\infty}(b-c)^{n+1}\\&\leq\frac{1}{(n+1)!}|s|^{n+1}\beta_{[a,b]}(s)\|g\|_{[a,b],\infty}\left[(c-a)^{n+1}+(b-c)^{n+1}\right]\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&|\mathcal{W}_n(g,a,c,b)(s)|\\&\leq\frac{1}{n!}|s|^{n+1}\|g\|_{[a,c],1}\beta_{[a,c]}(s)(c-a)^n\\&+\frac{1}{n!}|s|^{n+1}\|g\|_{[c,b],1}\beta_{[c,b]}(s)(b-c)^n\\&\leq\frac{1}{n!}|s|^{n+1}\|g\|_{[a,b],1}\beta_{[a,b]}(s)\left[(c-a)^n+(b-c)^n\right]\end{aligned} \tag{4.8}$$

for any $c\in[a,b]$, $n\in\mathbb{N}$, $n\geq1$ *and* $s\in\mathbb{C}$.

The interested reader can obtain various representations for other *integral transforms* of interest such as the *finite Fourier-Stieltjes sine and cosine transforms* defined by

$$\left(\mathcal{F}_{s,[a,b]}g\right)(u):=\int_a^b\sin(ut)\,dg(t),\quad\left(\mathcal{F}_{c,[a,b]}g\right)(u):=\int_a^b\cos(ut)\,dg(t), \tag{4.9}$$

where a,b are real numbers with $a<b$, u is a real number and $g:[a,b]\to\mathbb{C}$ is a Riemann integrable function. However, the details are not presented here.

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