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SOME INEQUALITIES FOR POWER SERIES OF SELFADJOINT OPERATORS IN HILBERT SPACES VIA WIELANDT AND REVERSES OF SCHWARZ INEQUALITIES

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Abstract. In this paper we obtain some operator inequalities for functions defined by power series with complex coefficients and, more specifically, with nonnegative coefficients. In order to obtain these inequalities the classical Wielandt and some reverses of the Schwarz inequality for vectors in inner product spaces are utilized. Natural applications for some elementary functions of interest are also provided.

1. Introduction

Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be an analytic function defined by a power series with nonnegative coefficients a_n, $n \geq 0$ and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. As the most natural examples of such functions we have

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, z \in D(0,1) \text{ and } f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, z \in \mathbb{C}.$$

Other function as power series representations with nonnegative coefficients are, for instance

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \qquad z \in \mathbb{C};$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \qquad z \in \mathbb{C};$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \qquad z \in D(0,1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \qquad z \in D(0,1);$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \quad \alpha,\beta,\gamma > 0, z \in D(0,1).$$

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Now, by the help of the power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ with complex coefficients, we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z):=\sum_{n=0}^{\infty}|a_n|z^n$. It is obvious that this new power series will have the same radius of convergence as the original series.

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n = \ln\left(\frac{1}{1+z}\right), \qquad z\in D(0,1);\\ g(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n} = \cos z, \qquad z\in\mathbb{C};\\ h(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \sin z, \qquad z\in\mathbb{C};\\ l(z) &= \sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{1+z}, \qquad z\in D(0,1), \end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are obviously

$$f_A(z)=\ln\left(\frac{1}{1-z}\right), \qquad g_A(z)=\cosh z,$$
$$h_A(z)=\sinh z \quad \text{and} \quad l_A(z)=\frac{1}{1-z}$$

and they are defined on the same domains as the generating functions.

In the recent paper [6], the author has proved amongst other results the following inequalities concerning functions defined by power series of operators on complex Hilbert spaces $(H,\langle .,.\rangle)$:

Theorem 1. *Let consider the power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ with real coefficients a_n that is convergent on the open disk $D(0,R)$ with $R>0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T)\subset[m,M]$ and $0<m<M$ with $M<R$, then for any $x\in H$ with $\|x\|=1$ we have the inequality*

$$\max\{\|f(T)x\|,\|f_A(T)x\|\}\le\langle f_A(T)x,x\rangle+\frac{1}{2}[f_A(M)-f_A(m)]. \tag{1.1}$$

and

Theorem 2. *Let consider the power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ with real coefficients a_n that is convergent on the open disk $D(0,R)$ with $R>0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T)\subset[m,M]$ and $0<m<M$ with $M\sqrt{\frac{M}{m}}<R$, then for any $x\in H$ with $\|x\|=1$ we have the inequality*

$$\begin{aligned} &\max\{\|f(T)x\|,\|f_A(T)x\|\}\\ &\le\frac{1}{2}\left\langle\left[f_A\left(\sqrt{\frac{M}{m}}\cdot T\right)+f_A\left(\sqrt{\frac{m}{M}}\cdot T\right)\right]x,x\right\rangle. \end{aligned} \tag{1.2}$$

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [7] and the references therein. For other results, see [10], [11], [12] and [9].

The main aim of the present paper is to provide some operator inequalities for functions defined by power series with complex coefficients and, more specifically, with nonnegative coefficients. In order to obtain these inequalities we use the well known Wielandt inequality and some reverses for the Schwarz inequality. Natural applications for some elementary functions of interest are also considered.

2. Some Results Via Wielandt Inequality

For a selfadjoint operator $T : H \to H$ with the property that $0 < mI \leq T \leq MI$ the following inequality is well known in the literature as the *Wielandt inequality*

$$|\langle Tx, y\rangle|^2 \leq \left(\frac{M-m}{M+m}\right)^2 \langle Tx, x\rangle \langle Ty, y\rangle \tag{W}$$

for any $x, y \in H$ with $x \perp y$.

Theorem 3. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n that is convergent on the open disk $D(0, R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with $M\sqrt{\frac{M}{m}} < R$, then for any $x, y \in H$ with $x \perp y$ we have the inequality*

$$\begin{aligned} &\max\{|\langle f_A(T)x, y\rangle|, |\langle f(T)x, y\rangle|\} \\ &\leq \frac{1}{2}\left\langle\left[f_A\left(\sqrt{\frac{M}{m}}T\right) - f_A\left(\sqrt{\frac{m}{M}}T\right)\right]x, x\right\rangle^{1/2} \\ &\times \left\langle\left[f_A\left(\sqrt{\frac{M}{m}}T\right) - f_A\left(\sqrt{\frac{m}{M}}T\right)\right]y, y\right\rangle^{1/2}. \end{aligned} \tag{2.1}$$

Proof. Since $Sp(T) \subset [m, M]$ then we have $0 < m^k I \leq T^k \leq M^k I$ in the operator order of $B(H)$ and for any natural number $k \geq 0$, which implies that

$$\langle M^k x - T^k x, T^k x - m^k x\rangle \geq 0$$

for any $k \geq 0$.

If we write the Wielandt inequality (W) for T^k we have for any natural number $k \geq 0$

$$\left|\langle T^k x, y\rangle\right|^2 \leq \left(\frac{M^k - m^k}{M^k + m^k}\right)^2 \langle T^k x, x\rangle \langle T^k y, y\rangle \tag{2.2}$$

for any $x, y \in H$ with $x \perp y$. We observe that for $k = 0$ the inequality (2.2) reduces to an equality.

If we take the square root in (2.2) we get

$$
\begin{aligned}
\left|\left\langle T^{k}x,y\right\rangle\right| &\leq \frac{M^{k}-m^{k}}{M^{k}+m^{k}}\left\langle T^{k}x,x\right\rangle^{1/2}\left\langle T^{k}y,y\right\rangle^{1/2} \\
&\leq \frac{M^{k}-m^{k}}{2\sqrt{M^{k}m^{k}}}\left\langle T^{k}x,x\right\rangle^{1/2}\left\langle T^{k}y,y\right\rangle^{1/2} \\
&= \frac{1}{2}\left[\left(\sqrt{\frac{M}{m}}\right)^{k}-\left(\sqrt{\frac{m}{M}}\right)^{k}\right]\left\langle T^{k}x,x\right\rangle^{1/2}\left\langle T^{k}y,y\right\rangle^{1/2}
\end{aligned} \tag{2.3}
$$

for any $x, y \in H$ with $x \perp y$ and for any natural number $k \geq 0$.

If we multiply (2.3) by $|a_k|$ and sum over k from 0 to n then we get

$$
\begin{aligned}
&\sum_{k=0}^{n}|a_k|\left|\left\langle T^{k}x,y\right\rangle\right| \\
&\leq \frac{1}{2}\sum_{k=0}^{n}|a_k|\left[\left(\sqrt{\frac{M}{m}}\right)^{k}-\left(\sqrt{\frac{m}{M}}\right)^{k}\right]\left\langle T^{k}x,x\right\rangle^{1/2}\left\langle T^{k}y,y\right\rangle^{1/2}
\end{aligned} \tag{2.4}
$$

for any natural number n.

We use the Cauchy-Bunyakovsky-Schwarz weighted inequality

$$
\left(\sum_{k=0}^{n}m_k s_k t_k\right)^{2} \leq \sum_{k=0}^{n}m_k s_k^{2}\sum_{k=0}^{n}m_k t_k^{2},\ m_k,\ s_k,\ t_k \geq 0
$$

to get

$$
\begin{aligned}
&\sum_{k=0}^{n}|a_k|\left[\left(\sqrt{\frac{M}{m}}\right)^{k}-\left(\sqrt{\frac{m}{M}}\right)^{k}\right]\left\langle T^{k}x,x\right\rangle^{1/2}\left\langle T^{k}y,y\right\rangle^{1/2} \\
&\leq \left(\sum_{k=0}^{n}|a_k|\left[\left(\sqrt{\frac{M}{m}}\right)^{k}-\left(\sqrt{\frac{m}{M}}\right)^{k}\right]\left\langle T^{k}x,x\right\rangle\right)^{1/2} \\
&\times \left(\sum_{k=0}^{n}|a_k|\left[\left(\sqrt{\frac{M}{m}}\right)^{k}-\left(\sqrt{\frac{m}{M}}\right)^{k}\right]\left\langle T^{k}y,y\right\rangle\right)^{1/2}
\end{aligned} \tag{2.5}
$$

for any $x, y \in H$ with $x \perp y$ and for any natural number n.

By the generalized triangle inequality for modulus, we also have

$$
\max\left\{\left|\left\langle\sum_{k=0}^{n}|a_k|\,T^{k}x,y\right\rangle\right|,\left|\left\langle\sum_{k=0}^{n}a_k T^{k}x,y\right\rangle\right|,\right\} \leq \sum_{k=0}^{n}|a_k|\left|\left\langle T^{k}x,y\right\rangle\right| \tag{2.6}
$$

for any $x, y \in H$ with $x \perp y$ and for any natural number n.

Therefore, by (2.4)-(2.6) we have

$$\max\left\{\left|\left\langle\sum_{k=0}^{n}|a_k|\,T^k x,y\right\rangle\right|,\left|\left\langle\sum_{k=0}^{n}a_k T^k x,y\right\rangle\right|,\right\} \tag{2.7}$$
$$\leq \frac{1}{2}\left\langle\sum_{k=0}^{n}|a_k|\left(\sqrt{\frac{M}{m}}\right)^k T^k x-\sum_{k=0}^{n}|a_k|\left(\sqrt{\frac{m}{M}}\right)^k T^k x,x\right\rangle^{1/2}$$
$$\times\left\langle\sum_{k=0}^{n}|a_k|\left(\sqrt{\frac{M}{m}}\right)^k T^k y-\sum_{k=0}^{n}|a_k|\left(\sqrt{\frac{m}{M}}\right)^k T^k y,y\right\rangle^{1/2}$$

for any $x,y\in H$ with $x\perp y$ and for any natural number n.

Since

$$M\sqrt{\frac{m}{M}}\leq M\leq M\sqrt{\frac{M}{m}}<R$$

it follows that the series

$$\sum_{k=0}^{\infty}|a_k|\left(\sqrt{\frac{M}{m}}\right)^k T^k,\ \sum_{k=0}^{\infty}|a_k|\,T^k,\ \sum_{k=0}^{\infty}a_k T^k \text{ and } \sum_{k=0}^{\infty}|a_k|\left(\sqrt{\frac{m}{M}}\right)^k T^k$$

are convergent and taking the limit over $n\to\infty$ in (2.7) we deduce the desired result (2.1). □

The following result also holds:

Theorem 4. *Let consider the power series $f(z)=\sum_{n=0}^{\infty}a_n z^n$ with complex coefficients a_n that is convergent on the open disk $D(0,R)$ with $R>0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T)\subset[m,M]$ and $0<m<M$ with $M^2<R$, then for any $x,y\in H$ with $x\perp y$ we have the inequality*

$$\max\{|\langle[f_A(MT)+f_A(mT)]x,y\rangle|,|\langle[f(MT)+f(mT)]x,y\rangle|\} \tag{2.8}$$
$$\leq\langle[f_A(MT)-f_A(mT)]x,x\rangle^{1/2}\langle[f_A(MT)-f_A(mT)]y,y\rangle^{1/2}.$$

Proof. From the first inequality in (2.3) we have

$$\left(M^k+m^k\right)\left|\left\langle T^k x,y\right\rangle\right|\leq\left(M^k-m^k\right)\left\langle T^k x,x\right\rangle^{1/2}\left\langle T^k y,y\right\rangle^{1/2} \tag{2.9}$$

for any $x,y\in H$ with $x\perp y$ and for any natural number $k\geq 0$.

If we multiply (2.3) by $|a_k|$ and sum over k from 0 to n then we get

$$\sum_{k=0}^{n}|a_k|\left(M^k+m^k\right)\left|\left\langle T^k x,y\right\rangle\right| \tag{2.10}$$
$$\leq\sum_{k=0}^{n}|a_k|\left(M^k-m^k\right)\left\langle T^k x,x\right\rangle^{1/2}\left\langle T^k y,y\right\rangle^{1/2}$$

for any $x,y\in H$ with $x\perp y$ and for any natural number $n\geq 0$.

By the weighted Cauchy-Bunyakowsky-Schwarz inequality, we have

$$\begin{aligned}(2.11)\qquad &\sum_{k=0}^{n}|a_k|\left(M^k-m^k\right)\left\langle T^kx,x\right\rangle^{1/2}\left\langle T^ky,y\right\rangle^{1/2}\\ &\leq\left\langle\sum_{k=0}^{n}|a_k|\left(M^k-m^k\right)T^kx,x\right\rangle^{1/2}\left\langle\sum_{k=0}^{n}|a_k|\left(M^k-m^k\right)T^ky,y\right\rangle^{1/2}\end{aligned}$$

while by the generalized triangle inequality we also have

$$\begin{aligned}(2.12)\qquad &\max\left\{\left|\left\langle\sum_{k=0}^{n}|a_k|\left(M^k+m^k\right)T^kx,y\right\rangle\right|,\left|\left\langle\sum_{k=0}^{n}a_k\left(M^k+m^k\right)T^kx,y\right\rangle\right|\right\}\\ &\leq\sum_{k=0}^{n}|a_k|\left(M^k+m^k\right)\left|\left\langle T^kx,y\right\rangle\right|\end{aligned}$$

for any $x,y\in H$ with $x\perp y$ and for any natural number $n\geq 0$.

Therefore, by (2.10)-(2.12) we have

$$\begin{aligned}(2.13)\qquad &\max\left\{\left|\left\langle\sum_{k=0}^{n}|a_k|\left(M^k+m^k\right)T^kx,y\right\rangle\right|,\left|\left\langle\sum_{k=0}^{n}a_k\left(M^k+m^k\right)T^kx,y\right\rangle\right|\right\}\\ &\leq\left\langle\sum_{k=0}^{n}|a_k|\left(M^k-m^k\right)T^kx,x\right\rangle^{1/2}\left\langle\sum_{k=0}^{n}|a_k|\left(M^k-m^k\right)T^ky,y\right\rangle^{1/2}\end{aligned}$$

for any $x,y\in H$ with $x\perp y$ and for any natural number $n\geq 0$.

Since all the series whose partial sums are involved in the inequality (2.13) are convergent, then by taking the limit over $n\to\infty$ in this inequality, we deduce the desired result (2.8). □

Remark 1. *If we take in (2.8) $f(z)=z$, then we recapture the Wielandt inequality (W).*

Remark 2. *We observe that by the generalized Schwarz inequality for positive operators we have*

$$\begin{aligned}(2.14)\qquad &\left|\left\langle\left[f_A(MT)+f_A(mT)\right]x,y\right\rangle\right|\\ &\leq\left\langle\left[f_A(MT)+f_A(mT)\right]x,x\right\rangle^{1/2}\left\langle\left[f_A(MT)+f_A(mT)\right]y,y\right\rangle^{1/2}\end{aligned}$$

for all vectors $x,y\in H$ and positive operators T with $0<mI\leq T\leq MI$.

Since $f_A(mT)\geq 0$, we have

$$\left\langle\left[f_A(MT)-f_A(mT)\right]x,x\right\rangle\leq\left\langle\left[f_A(MT)+f_A(mT)\right]x,x\right\rangle$$

and hence the inequality (2.8) is an improvement of (2.14).

In the paper [8], Malamud obtained the following result of Wielandt type:

Lemma 1. *Let $T:H\to H$ a selfadjoint operator with the property that $0\leq mI\leq T\leq MI$. Then*

$$(2.15)\qquad \left|\left\langle Tx,y\right\rangle\right|^2\leq\left(\sqrt{M}-\sqrt{m}\right)^2\left\langle Ty,y\right\rangle$$

for any $x,y\in H$ with $x\perp y$.

We can state the following result:

Theorem 5. *Let consider the power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with complex coefficients* a_n *that is convergent on the open disk* $D(0,R)$ *with* $R > 0$. *If the selfadjoint operator* T *on the Hilbert space* H *has the spectrum* $Sp(T) \subset [m,M]$ *and* $0 < m < M$ *with* $M < R$, *then for any* $x, y \in H$ *with* $x \perp y$ *we have the inequality*

$$\max\{|\langle f_A(T)x,y\rangle|, |\langle f(T)x,y\rangle|\} \leq \left[f_A(M) - 2f_A\left(\sqrt{Mm}\right) + f_A(m)\right]^{1/2} \langle f_A(T)y,y\rangle^{1/2} \tag{2.16}$$

for any $x, y \in H$ *with* $x \perp y$.

Proof. Since $Sp(T) \subset [m,M]$ and $0 < m < M$, then we have $0 \leq m^k I \leq T^k \leq M^k I$ and by (2.15) we have

$$\left|\left\langle T^k x, y\right\rangle\right|^2 \leq \left(\sqrt{M^k} - \sqrt{m^k}\right)^2 \left\langle T^k y, y\right\rangle \tag{2.17}$$

for any $x, y \in H$ with $x \perp y$.

Taking the square root in (2.17) we have

$$\left|\left\langle T^k x, y\right\rangle\right| \leq \left(\sqrt{M^k} - \sqrt{m^k}\right) \sqrt{\langle T^k y, y\rangle} \tag{2.18}$$

for any $x, y \in H$ with $x \perp y$.

If we multiply (2.18) by $|a_k|$ and sum over k from 0 to n then we get

$$\begin{aligned} \sum_{k=0}^{n} |a_k| \left|\left\langle T^k x, y\right\rangle\right| &\leq \sum_{k=0}^{n} |a_k| \left(\sqrt{M^k} - \sqrt{m^k}\right) \sqrt{\langle T^k y, y\rangle} \\ &\leq \left[\sum_{k=0}^{n} |a_k| \left(\sqrt{M^k} - \sqrt{m^k}\right)^2\right]^{1/2} \left[\sum_{k=0}^{n} |a_k| \left\langle T^k y, y\right\rangle\right]^{1/2} \\ &= \left[\sum_{k=0}^{n} |a_k| \left(M^k - 2\sqrt{M^k m^k} + m^k\right)\right]^{1/2} \left\langle \sum_{k=0}^{n} |a_k| T^k y, y \right\rangle^{1/2} \end{aligned} \tag{2.19}$$

for any $x, y \in H$ with $x \perp y$.

By the generalized triangle inequality we also have

$$\max\left\{\left|\left\langle \sum_{k=0}^{n} |a_k| T^k x, y\right\rangle\right|, \left|\left\langle \sum_{k=0}^{n} a_k T^k x, y\right\rangle\right|\right\} \leq \sum_{k=0}^{n} |a_k| \left|\left\langle T^k x, y\right\rangle\right| \tag{2.20}$$

for any $x, y \in H$ with $x \perp y$ and for any natural number $n \geq 0$.

Therefore, by (2.19) and (2.20) we have

$$\begin{aligned} &\max\left\{\left|\left\langle \sum_{k=0}^{n} |a_k| T^k x, y\right\rangle\right|, \left|\left\langle \sum_{k=0}^{n} a_k T^k x, y\right\rangle\right|\right\} \\ &\leq \left[\sum_{k=0}^{n} |a_k| \left(M^k - 2\sqrt{M^k m^k} + m^k\right)\right]^{1/2} \left\langle \sum_{k=0}^{n} |a_k| T^k y, y \right\rangle^{1/2} \end{aligned} \tag{2.21}$$

for any $x, y \in H$ with $x \perp y$.

Since all the series whose partial sums are involved in the inequality (2.21) are convergent, then by letting $n \to \infty$ in this inequality, we get the desired result (2.16). □

Remark 3. *If we take in (2.16) $f(z) = z$, then we recapture the Malamud inequality (2.15).*

We also have

Theorem 6. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n that is convergent on the open disk $D(0,R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with p, $pM^2 < R$, then for any $x, y \in H$ with $x \perp y$ we have the inequality*

$$\max\{|\langle f_A(pT)x, y\rangle|, |\langle f(pT)x, y\rangle|\} \leq [f_A(p)]^{1/2} \left\langle \left[f_A(pMT) - 2f_A\left(p\sqrt{Mm}T\right) + f_A(pmT)\right]^{1/2} y, y\right\rangle^{1/2} \tag{2.22}$$

for any $x, y \in H$ with $x \perp y$.

Proof. If we multiply the inequality (2.17) by $|a_k| p^k$ and sum over k from 0 to n then we get

$$\begin{aligned} &\sum_{k=0}^{n} |a_k| p^k \left|\langle T^k x, y\rangle\right|^2 \\ &\leq \sum_{k=0}^{n} |a_k| p^k \left(\sqrt{M^k} - \sqrt{m^k}\right)^2 \langle T^k y, y\rangle \\ &= \sum_{k=0}^{n} |a_k| p^k \left(M^k - 2\sqrt{M^k m^k} + m^k\right) \langle T^k y, y\rangle \\ &= \left\langle \sum_{k=0}^{n} |a_k| p^k \left(M^k - 2\sqrt{M^k m^k} + m^k\right) T^k y, y\right\rangle \end{aligned} \tag{2.23}$$

for any $x, y \in H$ with $x \perp y$.

By the weighted Cauchy-Bunyakowsky-Schwarz inequality, we have

$$\left(\sum_{k=0}^{n} |a_k| p^k \left|\langle T^k x, y\rangle\right|\right)^2 \leq \sum_{k=0}^{n} |a_k| p^k \left|\langle T^k x, y\rangle\right|^2 \sum_{k=0}^{n} |a_k| p^k$$

and by (2.23) we have

$$\begin{aligned} &\left(\sum_{k=0}^{n} |a_k| p^k \left|\langle T^k x, y\rangle\right|\right)^2 \\ &\leq \sum_{k=0}^{n} |a_k| p^k \left\langle \sum_{k=0}^{n} |a_k| p^k \left(M^k - 2\sqrt{M^k m^k} + m^k\right) T^k y, y\right\rangle \end{aligned} \tag{2.24}$$

for any $x, y \in H$ with $x \perp y$.

The proof now follows by (2.24) in a similar way as shown above and the details are omitted. □

3. Other Inequalities

We start with the following lemma that is interest in itself, see for instance [8] or [7, p. 88]:

Lemma 2. *Let $T : H \to H$ a selfadjoint operator with the property that $0 \leq mI \leq T \leq MI$. Then*

$$0 \leq \|Tx\| - \langle Tx, x \rangle \leq \frac{1}{4} \cdot \frac{(M-m)^2}{m+M} \tag{3.1}$$

for any $x \in H, \|x\| = 1$.

Remark 4. *For an extension of the inequality (3.1) for vectors and complex numbers in a complex inner product space, see* [1].

Corollary 1. *For any positive operator on the Hilbert space H we have*

$$0 \leq \|Tx\| - \langle Tx, x \rangle \leq \frac{1}{4} \|T\| \tag{3.2}$$

for any $x \in H, \|x\| = 1$.

Proof. Follows from the above lemma on choosing $m = 0$ and $M = \|T\|$. □

Theorem 7. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n that is convergent on the open disk $D(0, R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with $M^2 < R$, then we have the inequality*

$$\begin{aligned} &\max\{\|[f_A(mT) + f_A(MT)]x\|, \|[f(mT) + f(MT)]x\|\} \\ &\leq \langle [f_A(mT) + f_A(MT)]x, x \rangle + \frac{1}{4}\left[f_A\left(M^2\right) - 2f_A(mM) + f_A\left(m^2\right)\right] \end{aligned} \tag{3.3}$$

for any $x \in H, \|x\| = 1$.

Moreover, if the operator T is positive and $\|T\| < R$, then we have the inequality

$$\max\{\|f_A(T)x\|, \|f(T)x\|\} \leq \langle f_A(T)x, x \rangle + \frac{1}{4} f_A(\|T\|) \tag{3.4}$$

for any $x \in H, \|x\| = 1$.

Proof. Since $Sp(T) \subset [m, M]$ then we have $0 < m^k I \leq T^k \leq M^k I$ in the operator order of $B(H)$ and for any natural number $k \geq 0$. Using the inequality (3.1) we have

$$\begin{aligned} \left(m^k + M^k\right)\left\|T^k x\right\| &\leq \left(m^k + M^k\right)\left\langle T^k x, x\right\rangle + \frac{1}{4}\left(M^k - m^k\right)^2 \\ &= \left(m^k + M^k\right)\left\langle T^k x, x\right\rangle + \frac{1}{4}\left(M^{2k} - 2M^k m^k + m^{2k}\right) \end{aligned}$$

which holds for any natural number $k \geq 0$.

Now utilizing a similar argument to the one in the proof of Theorem 4 we get the desired inequality (3.3).

The inequality (3.4) follows by (3.2) and the details are omitted. □

We can state now the following lemma that is of interest in its own right:

Lemma 3. *Let $T : H \to H$ a selfadjoint operator with the property that $0 < mI \leq T \leq MI$. Then*

$$0 \leq \|Tx\| - \langle Tx, x\rangle \leq \frac{1}{2}\left(\sqrt{M} - \sqrt{m}\right)^2 \tag{3.5}$$

for any $x \in H, \|x\| = 1$.

Proof. We put

$$F(x) := 2\left(\sqrt{x} - \sqrt{m}\right)^2 (x + m) - (x - m)^2$$

for all $x \geq m$.

Then we have $F(m) = 0$,

$$F(x) = x^2 + 6mx + m^2 - 4\sqrt{m}x^{3/2} - 4m\sqrt{m}x^{1/2}$$

and

$$F'(x) = 2\frac{\left(\sqrt{x} - \sqrt{m}\right)^3}{\sqrt{x}} \geq 0$$

for all $x \geq m$.

These imply that $F(x) \geq 0$ for all $x \geq m$ and therefore

$$\frac{1}{4} \cdot \frac{(M - m)^2}{m + M} \leq \frac{1}{2}\left(\sqrt{M} - \sqrt{m}\right)^2,$$

which together with the inequality (3.1) produces the desired result (3.5). □

Utilising this lemma and an argument similar to that employed to prove the theorems above, we can state the following result as well:

Theorem 8. *Let consider the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n that is convergent on the open disk $D(0, R)$ with $R > 0$. If the selfadjoint operator T on the Hilbert space H has the spectrum $Sp(T) \subset [m, M]$ and $0 < m < M$ with $M < R$, then we have the inequality*

$$\begin{aligned} &\max\left\{\|f_A(T)x\|, \|f(T)x\|\right\} \\ &\leq \langle f_A(T)x, x\rangle + \frac{1}{2}\left[f_A(M) - 2f_A\left(\sqrt{Mm}\right) + f_A(m)\right] \end{aligned} \tag{3.6}$$

for any $x \in H, \|x\| = 1$.

4. Some Examples

It is natural to state some of the above results for functions defined by power series as those provided in the introduction of this paper.

1. If we write the inequality (2.1) and (2.8) for the exponential function, then for the selfadjoint operator T on the Hilbert space H that has the spectrum $Sp(T) \subset [m, M]$ with $0 < m < M$ and for any $x, y \in H$ with $x \perp y$ we have the inequalities

$$\begin{aligned} |\langle \exp(T)x, y\rangle| \leq \frac{1}{2}&\left\langle\left[\exp\left(\sqrt{\frac{M}{m}}T\right) - \exp\left(\sqrt{\frac{m}{M}}T\right)\right]x, x\right\rangle^{1/2} \\ &\times\left\langle\left[\exp\left(\sqrt{\frac{M}{m}}T\right) - \exp\left(\sqrt{\frac{m}{M}}T\right)\right]y, y\right\rangle^{1/2} \end{aligned} \tag{4.1}$$

and

$$|\langle[\exp(MT)+\exp(mT)]x,y\rangle| \le \langle[\exp(MT)-\exp(mT)]x,x\rangle^{1/2}\langle[\exp(MT)-\exp(mT)]y,y\rangle^{1/2}. \tag{4.2}$$

Now, from (2.16) and from (2.22) we also have

$$|\langle\exp(T)x,y\rangle| \le \left[\exp(M)-2\exp\left(\sqrt{Mm}\right)+\exp(m)\right]^{1/2}\langle\exp(T)y,y\rangle^{1/2} \tag{4.3}$$

and for $p>0$,

$$|\langle\exp(pT)x,y\rangle| \le \exp\left(\frac{p}{2}\right)\left\langle\left[\exp(pMT)-2\exp\left(p\sqrt{Mm}T\right)+\exp(pmT)\right]^{1/2}y,y\right\rangle^{1/2} \tag{4.4}$$

for any $x,y\in H$ with $x\perp y$.

If we use the inequalities (3.3) and (3.6) for the operator T with $Sp(T)\subset[m,M]$, $0<m<M$, then

$$\|[\exp(mT)+\exp(MT)]x\| \le \langle[\exp(mT)+\exp(MT)]x,x\rangle+\frac{1}{4}\left[\exp\left(M^2\right)-2\exp(mM)+\exp\left(m^2\right)\right] \tag{4.5}$$

and

$$\|\exp(T)x\| \le \langle\exp(T)x,x\rangle+\frac{1}{2}\left[\exp(M)-2\exp\left(\sqrt{Mm}\right)+\exp(m)\right] \tag{4.6}$$

for any $x\in H, \|x\|=1$.

If T is a positive operator, then by (3.4) we have

$$\|\exp(T)x\| \le \langle\exp(T)x,x\rangle+\frac{1}{4}\exp(\|T\|) \tag{4.7}$$

for any $x\in H, \|x\|=1$.

2. If we use the function $f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln(1+z)$, $z\in D(0,1)$ for the selfadjoint operator T on the Hilbert space H that has the spectrum $Sp(T)\subset[m,M]$ and $0<m<M$ with $M\sqrt{\frac{M}{m}}<1$, then for any $x,y\in H$ with $x\perp y$ we have from (2.1) the inequality

$$\begin{aligned}\max&\left\{\left|\left\langle\ln(I-T)^{-1}x,y\right\rangle\right|,\left|\left\langle\ln(I+T)^{-1}x,y\right\rangle\right|\right\}\\ &\le\frac{1}{2}\left\langle\left[\ln\left(I-\sqrt{\frac{M}{m}}T\right)^{-1}-\ln\left(I-\sqrt{\frac{m}{M}}T\right)^{-1}\right]x,x\right\rangle^{1/2}\\ &\times\left\langle\left[\ln\left(I-\sqrt{\frac{M}{m}}T\right)^{-1}-\ln\left(I-\sqrt{\frac{m}{M}}T\right)^{-1}\right]y,y\right\rangle^{1/2},\end{aligned} \tag{4.8}$$

while from (2.8), we have

$$\begin{aligned}
&\max\left\{\left|\left\langle\left[\ln\left(I-MT\right)^{-1}+\ln\left(I-mT\right)^{-1}\right]x,y\right\rangle\right|,\right. \\
&\left.\left|\left\langle\left[\ln\left(I+MT\right)^{-1}+\ln\left(I+mT\right)^{-1}\right]x,y\right\rangle\right|\right\} \\
&\leq\left\langle\left[\ln\left(I-MT\right)^{-1}-\ln\left(I-mT\right)^{-1}\right]x,x\right\rangle^{1/2} \\
&\times\left\langle\left[\ln\left(I-MT\right)^{-1}-\ln\left(I-mT\right)^{-1}\right]y,y\right\rangle^{1/2}.
\end{aligned} \tag{4.9}$$

Now, from (2.16) and from (2.22) we also have, for $M<1$, that

$$\begin{aligned}
&\max\left\{\left|\left\langle\ln\left(I-T\right)^{-1}x,y\right\rangle\right|,\left|\left\langle\ln\left(I+T\right)^{-1}x,y\right\rangle\right|\right\} \\
&\leq\left[\ln\left(\frac{\left(1-\sqrt{Mm}\right)^{2}}{\left(1-M\right)\left(1-m\right)}\right)\right]^{1/2}\left\langle\ln\left(I-T\right)^{-1}y,y\right\rangle^{1/2}
\end{aligned} \tag{4.10}$$

and, for $0<p$, pM^2, $pM<1$ we have

$$\begin{aligned}
\max&\left\{\left|\left\langle\ln\left(I-pT\right)^{-1}x,y\right\rangle\right|,\left|\left\langle\ln\left(I+pT\right)^{-1}x,y\right\rangle\right|\right\} \\
&\leq\left[\ln\left(1-p\right)^{-1}\right]^{1/2} \\
&\times\left\langle\left[\ln\left(I-pMT\right)^{-1}-2\ln\left(I-p\sqrt{Mm}T\right)^{-1}+\ln\left(I-pmT\right)^{-1}\right]^{1/2}y,y\right\rangle^{1/2}
\end{aligned} \tag{4.11}$$

for any $x,y\in H$ with $x\perp y$.

If we use the inequalities (3.3) and (3.6) for the operator T with $Sp\left(T\right)\subset[m,M]$, $0<m<M<1$, then we have

$$\begin{aligned}
&\max\left\{\left\|\left[\ln\left(I-mT\right)^{-1}+\ln\left(I-MT\right)^{-1}\right]x\right\|,\right. \\
&\left.\left\|\left[\ln\left(I+mT\right)^{-1}+\ln\left(I+MT\right)^{-1}\right]x\right\|\right\} \\
&\leq\left\langle\left[\ln\left(I-mT\right)^{-1}+\ln\left(I-MT\right)^{-1}\right]x,x\right\rangle \\
&+\ln\left(\frac{\left(1-mM\right)^{1/2}}{\left(1-M^{2}\right)^{1/4}\left(1-m^{2}\right)^{1/4}}\right)
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&\max\left\{\left\|\ln\left(I-T\right)^{-1}x\right\|,\left\|\ln\left(I+T\right)^{-1}x\right\|\right\} \\
&\leq\left\langle\ln\left(I-T\right)^{-1}x,x\right\rangle+\ln\left(\frac{1-\sqrt{Mm}}{\left(1-M\right)^{1/2}\left(1-m\right)^{1/2}}\right)
\end{aligned} \tag{4.13}$$

for any $x\in H$, $\|x\|=1$.

Finally, if the operator T is positive and $\|T\|<1$, then we have the inequality

$$\begin{aligned}
&\max\left\{\left\|\ln\left(I-T\right)^{-1}x\right\|,\left\|\ln\left(I+T\right)^{-1}x\right\|\right\} \\
&\leq\left\langle\ln\left(I-T\right)^{-1}x,x\right\rangle+\frac{1}{4}\ln\left(1-\|T\|\right)^{-1}
\end{aligned} \tag{4.14}$$

for any $x \in H, \|x\| = 1$.

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