SOME GRÜSS TYPE INEQUALITIES FOR THE RIEMANN–STIELTJES INTEGRAL WITH LIPSCHITZIAN INTEGRATORS

M.W. ALOMARI[1] AND S.S. DRAGOMIR[2,3]

ABSTRACT. In this paper several new inequalities of Grüss' type for the Riemann–Stieltjes integral with Lipschitzian integrators are proved.

1. INTRODUCTION

The *Čebyšev functional*

$$(1.1) \qquad \mathcal{T}(f,g) = \frac{1}{b-a} \int_a^b f(t)\, g(t)\, dt - \frac{1}{b-a} \int_a^b f(t)\, dt \cdot \frac{1}{b-a} \int_a^b g(t)\, dt,$$

has interesting applications in the approximation of weighted integrals as one can has from the literature below.

Bounding Čebyšev functional has a long history, starting with Grüss inequality [14] in 1935, where Grüss had proved that for two integrable mappings f, g such that $\phi \leq f(x) \leq \Phi$ and $\gamma \leq f(x) \leq \Gamma$, the inequality

$$(1.2) \qquad |\mathcal{T}(f,g)| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma)$$

holds, and the constant $\frac{1}{4}$ is the best possible.

After that many authors have studied the functional (1.1) and several bounds under various assumptions for the functions involved have been obtained. For new results and generalizations the reader may refer to [2]–[15].

A generalization of (1.1) for Riemann–Stieltjes integral was considered by Dragomir in [10]. Namely, the author has introduced the following *Čebyšev functional for the Riemann–Stieltjes integral*:

$$(1.3) \quad \mathcal{T}(f,g;u) := \frac{1}{u(b) - u(a)} \int_a^b f(t)\, g(t)\, du(t)$$

$$- \frac{1}{u(b) - u(a)} \int_a^b f(t)\, du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(t)\, du(t)$$

under the assumptions that, f, g are continuous on $[a, b]$ and u is of bounded variation on $[a, b]$ with $u(b) \neq u(a)$.

By a simple computations with Riemann–Stieltjes integral, Dragomir [10] has obtained the identity,

$$(1.4) \quad \mathcal{T}(f, g; u) := \frac{1}{u(b) - u(a)} \int_a^b \left[f(t) - \frac{f(a) + f(b)}{2} \right]$$
$$\times \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right] du(t),$$

to obtain several sharp bounds of the Čebyšev functional for the Riemann–Stieltjes integral (1.3).

In this work, several inequalities of Grüss' type for the Riemann–Stieltjes integral with Lipschitzian integrators are proved.

2. The Results

We recall that a function $f : [a, b] \to \mathbb{C}$ is p–H_f–Holder continuous on $[a, b]$, if

$$|f(t) - f(s)| \leq H_f |t - s|^p$$

for all $t, s \in [a, b]$, where $p \in (0, 1]$ and $H_f > 0$ are given. If $p = 1$ we call f H_f–Lipschitzian.

We are ready to state our first result as follows:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a p–H_f–Hölder continuous on $[a, b]$, where $p \in (0, 1]$ and $H_f > 0$; are given. Let $g, u : [a, b] \to \mathbb{R}$ be such that g is Lebesgue integrable on $[a, b]$ and there exists the real numbers m, M such that $m \leq g(x) \leq M$ for all $x \in [a, b]$, and u is L_u–Lipschitzian on $[a, b]$ then*

$$(2.1) \qquad |\mathcal{T}(f, g; u)| \leq \frac{L_u^2 H_f}{p+1} \cdot \frac{(M-m)}{(u(b) - u(a))^2} (b-a)^{p+2}.$$

Proof. Taking the modulus in (1.4) and utilizing the triangle inequality, we get

$$|\mathcal{T}(f, g; u)| = \left| \frac{1}{u(b) - u(a)} \int_a^b \left[f(t) - \frac{f(a) + f(b)}{2} \right] \right.$$
$$\left. \times \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right] du(t) \right|$$
$$\leq \frac{L_u}{|u(b) - u(a)|} \cdot \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right|$$
$$\times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| dt$$
$$\leq \frac{L_u}{|u(b) - u(a)|} \cdot \sup_{t \in [a,b]} \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right|$$
$$\times \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| dt$$

$$\leq \frac{L_u}{|u(b) - u(a)|} \cdot \frac{L_u(M-m)}{|u(b) - u(a)|} (b-a) \cdot \frac{H_f}{2} \int_a^b [(t-a)^p + (b-t)^p] \, dt$$

$$= \frac{L_u^2 H_f}{p+1} \cdot \frac{(M-m)}{(u(b) - u(a))^2} (b-a)^{p+2},$$

since $m \leq g(x) \leq M$, for all $x \in [a,b]$, then

$$\left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| \leq \left| \frac{\int_a^b [g(t) - g(s)] \, du(s)}{u(b) - u(a)} \right|$$

$$\leq \frac{L_u}{|u(b) - u(a)|} \int_a^b |g(t) - g(s)| \, ds$$

$$(2.2) \qquad \leq \frac{L_u(M-m)}{|u(b) - u(a)|} (b-a),$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 1. *Let g, u be as in Theorem 1. If $f : [a,b] \to \mathbb{R}$ is L_f–Lipschitzian on $[a,b]$, then*

$$(2.3) \qquad |T(f,g;u)| \leq \frac{L_u^2 L_f(M-m)}{2(u(b) - u(a))^2} (b-a)^3.$$

Remark 1. *Under the assumptions of Theorem 1, we have*

$$(2.4) \qquad |\mathcal{T}(f,g)| \leq \frac{H_f}{(p+1)} (M-m) \cdot (b-a)^p.$$

In particular, if f is L_f–Lipschitzian, then

$$(2.5) \qquad |T(f,g)| \leq \frac{1}{2} L_f (b-a)(M-m).$$

Theorem 2. *Let g, u be as in Theorem 1. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, then we have*

$$(2.6) \qquad |\mathcal{T}(f,g;u)| \leq \frac{1}{2} \frac{L_u(M-m)}{(u(b) - u(a))^2} (b-a) \cdot \bigvee_a^b (f).$$

Proof. Since u is L_u–Lipschitzian on $[a,b]$, as in Theorem 1, we have

$$|\mathcal{T}(f,g;u)| \leq \frac{L_u}{|u(b) - u(a)|} \cdot \sup_{t \in [a,b]} \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right|$$

$$\times \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| dt$$

Since $m \leq g \leq M$, by (2.2) we have

$$(2.7) \quad \frac{1}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| du(t)$$

$$\leq \frac{L_u(M-m)}{(u(b) - u(a))^2} (b-a).$$

Now as f is of bounded variation on $[a, b]$, we have

$$\sup_{t \in [a,b]} \left| f(t) - \frac{f(a) + f(b)}{2} \right| = \sup_{t \in [a,b]} \left| \frac{f(t) - f(a) + f(t) - f(b)}{2} \right|$$

$$(2.8) \qquad\qquad\qquad \leq \frac{1}{2} \sup_{t \in [a,b]} \left[|f(t) - f(a)| + |f(t) - f(b)| \right] \leq \frac{1}{2} \bigvee_a^b (f),$$

for all $t \in [a, b]$. Finally, combining the inequalities (2.7)–(2.8), we obtain the required result (2.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3. *Let $g, u : [a, b] \to \mathbb{R}$ be such that g is of bounded variation on $[a, b]$, and u be L_u–Lipschitzian on $[a, b]$, then we have*

$$(2.9) \quad |\mathcal{T}(f, g; u)|$$

$$\leq \begin{cases} \frac{H_f L_u^2 (b-a)^{p+2}}{(p+1)(u(b)-u(a))^2} \cdot \bigvee_a^b (g), & \text{if } f \text{ is } H_f\text{-}p\text{-Hölder} \\[4mm] \frac{L_u^2 (b-a)^2}{2(u(b)-u(a))^2} \bigvee_a^b (g) \cdot \bigvee_a^b (f), & \text{if } f \text{ is of bounded variation} \end{cases}$$

where, $L_u, H_f > 0$ and $p \in (0, 1]$ are given.

Proof. Using (1.4) we may write

$$(2.10)$$

$$|\mathcal{T}(f, g; u)|$$

$$\leq \frac{L_u}{|u(b) - u(a)|} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) \, du(s) \right| dt$$

$$= \frac{L_u}{(u(b) - u(a))^2} \int_a^b \left| f(t) - \frac{f(a) + f(b)}{2} \right| \left| \int_a^b [g(t) - g(s)] \, du(s) \right| dt$$

$$\leq \frac{L_u^2}{(u(b) - u(a))^2} \int_a^b \left[\left| f(t) - \frac{f(a) + f(b)}{2} \right| \cdot \int_a^b |g(t) - g(s)| \, ds \right] dt,$$

but since g is of bounded variation then we have,

$$(2.11) \qquad \int_a^b |g(t) - g(s)| \, ds \leq \sup_{s \in [a,b]} |g(t) - g(s)| \cdot \int_a^b ds \leq (b - a) \bigvee_a^b (g).$$

Therefore, if f is of p–Hölder type, then we have

$$|\mathcal{T}(f, g; u)| \leq \frac{1}{2} \cdot \frac{L_u^2 (b - a)}{(u(b) - u(a))^2} \cdot \bigvee_a^b (g) \cdot \int_a^b \left[|f(t) - f(a)| + |f(t) - f(b)| \right] dt$$

$$\leq \frac{H_f}{2} \cdot \frac{L_u^2 (b - a)}{(u(b) - u(a))^2} \cdot \bigvee_a^b (g) \cdot \int_a^b \left[|t - a|^p + |t - b|^p \right] dt$$

$$= \frac{H_f}{(p + 1)} \cdot \frac{L_u^2 (b - a)^{p+2}}{(u(b) - u(a))^2} \cdot \bigvee_a^b (g),$$

which prove the first part of inequality (2.9).

To prove the second part of (2.9), assume that f is of bounded variation, then by (2.10) we have

$$|\mathcal{T}(f,g;u)| \leq \frac{L_u^2}{(u(b)-u(a))^2} \int_a^b \left[\left| f(t) - \frac{f(a)+f(b)}{2} \right| \cdot \int_a^b |g(t)-g(s)|\, ds \right] dt$$

$$\leq \frac{L_u^2(b-a)^2}{2(u(b)-u(a))^2} \bigvee_a^b (g) \cdot \bigvee_a^b (f),$$

and thus the theorem is proved. $\qquad\square$

Remark 2. *Under the assumptions of Theorem 3, we have*

$$(2.12) \quad |\mathcal{T}(f,g)|$$

$$\leq \begin{cases} \frac{H_f}{p+1}(b-a)^p \cdot \bigvee_a^b (g), & \text{if } f \text{ is } H_f\text{-}p\text{-Hölder} \\[2ex] \frac{1}{2} \bigvee_a^b (g) \cdot \bigvee_a^b (f), & \text{if } f \text{ is of bounded variation} \end{cases}$$

where, $H_f, > 0$ and $p \in (0,1]$ are given.

An improvement for the first inequality in (2.9) may be stated as follows:

Corollary 2. *Let g, u be as in Theorem 3 and $f : [a,b] \to \mathbb{R}$ be of $p\text{-}H_f$–Holder type on $[a,b]$, then*

$$(2.13) \qquad |\mathcal{T}(f,g;u)| \leq \frac{L_u^2 H_f (b-a)^{p+2}}{2^p (u(b)-u(a))^2} \bigvee_a^b (g).$$

Proof. By Theorem 3 we have

$$|\mathcal{T}(f,g;u)| \leq \frac{L_u^2}{(u(b)-u(a))^2} \int_a^b \left[\left| f(t) - \frac{f(a)+f(b)}{2} \right| \cdot \int_a^b |g(t)-g(s)|\, ds \right] dt$$

$$\leq \frac{L_u^2(b-a)}{(u(b)-u(a))^2} \bigvee_a^b (g) \int_a^b \left| f(t) - \frac{f(a)+f(b)}{2} \right| dt$$

$$\leq \frac{L_u^2(b-a)^2}{(u(b)-u(a))^2} \bigvee_a^b (g) \cdot \sup_{t \in [a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right|$$

$$\leq \frac{L_u^2(b-a)^2}{(u(b)-u(a))^2} \bigvee_a^b (g) \cdot H_f \left(\frac{b-a}{2} \right)^p,$$

and since f is of p–H_f–Holder type on $[a,b]$, we have

$$\left| f(t) - \frac{f(a)+f(b)}{2} \right| = \left| \frac{f(t)-f(a)+f(t)-f(b)}{2} \right| \leq \frac{1}{2}|f(t)-f(a)| + \frac{1}{2}|f(t)-f(b)|$$

$$\leq \frac{H_f}{2} [(t-a)^p + (b-t)^p],$$

it follows that

$$(2.14) \qquad \sup_{t \in [a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| \leq H_f \left(\frac{b-a}{2} \right)^p.$$

which completes the proof. $\qquad\square$

Remark 3. *Under the assumptions of Corollary 2, we have*

$$(2.15) \qquad |\mathcal{T}(f,g)| \leq \frac{1}{2^p} H_f (b-a)^p \bigvee_a^b (g),$$

which improves the first inequality in (2.12), where $H_f > 0$ and $p \in (0,1]$ are given.

Theorem 4. *Let $g, u : [a,b] \to \mathbb{R}$ be such that g is of q-H_g–Hölder type on $[a,b]$, and and u be L_u–Lipschitzian on $[a,b]$, then we have*

$(2.16) \quad |\mathcal{T}(f,g;u)|$

$$\leq L_u^2 H_g \cdot \begin{cases} \frac{(b-a)^{q+2}}{(q+1)(q+2)(u(b)-u(a))^2} \cdot \bigvee_a^b (f), & \text{if } f \text{ is of bounded variation} \\[2ex] \frac{H_f (b-a)^{p+q+2}}{2^p (q+1)(q+2)(u(b)-u(a))^2}, & \text{if } f \text{ is } H_f\text{-}p\text{-Hölder} \end{cases}$$

where, $L_u, H_g, H_f > 0$ and $p, q \in (0,1]$ are given.

Proof. Assume that g is of q-H_g–Hölder type on $[a,b]$ and f is of bounded variation on $[a,b]$. Using (1.4), then we may write

$$|\mathcal{T}(f,g;u)| \leq \frac{L_u}{|u(b)-u(a)|} \int_a^b \left| f(t) - \frac{f(a)+f(b)}{2} \right|$$

$$\times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s)\, du(s) \right| dt$$

$$= \frac{L_u}{(u(b)-u(a))^2} \int_a^b \left| f(t) - \frac{f(a)+f(b)}{2} \right| \left| \int_a^b [g(t) - g(s)]\, du(s) \right| dt$$

$$\leq \frac{L_u^2}{(u(b)-u(a))^2} \int_a^b \left[\left| f(t) - \frac{f(a)+f(b)}{2} \right| \cdot \int_a^b |g(t) - g(s)|\, ds \right] dt$$

(2.17)

$$\leq \frac{L_u^2}{(u(b)-u(a))^2} \cdot \sup_{t \in [a,b]} \left| f(t) - \frac{f(a)+f(b)}{2} \right| \cdot \int_a^b \left[\int_a^b |g(t) - g(s)|\, ds \right] dt$$

$$\leq \frac{L_u^2 H_g}{2(u(b)-u(a))^2} \cdot \bigvee_a^b (f) \cdot \int_a^b \left[\int_a^b |t-s|^q\, ds \right] dt.$$

$$= \frac{L_u^2 H_g}{2(u(b)-u(a))^2} \cdot \bigvee_a^b (f) \cdot \int_a^b \left[\int_a^t (s-a)^q\, ds + \int_t^b (b-s)^q\, ds \right] dt$$

$$= \frac{L_u^2 H_g}{2(u(b)-u(a))^2} \cdot \bigvee_a^b (f) \cdot \int_a^b \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1} \right] dt$$

$$= \frac{L_u^2 H_g}{(u(b)-u(a))^2} \cdot \bigvee_a^b (f) \cdot \frac{(b-a)^{q+2}}{(q+1)(q+2)},$$

which proves the first inequality in (2.16).

To prove the second inequality in (2.16), assume that f is of p–H_f–Hölder type on $[a, b]$, from (2.17) we have

$$(2.18) \qquad \sup_{t \in [a,b]} \left| f(t) - \frac{f(a) + f(b)}{2} \right| \leq H_f \left(\frac{b-a}{2} \right)^p$$

which together with (2.17) proves the second part of (2.16), and thus the proof is established. $\qquad \square$

Corollary 3. *Let $g, u : [a, b] \to \mathbb{R}$ be respectively; L_g, L_u–Lipschitzian on $[a, b]$, then we have*

$$(2.19) \quad |\mathcal{T}(f, g; u)|$$

$$\leq L_u^2 L_g \cdot \begin{cases} \frac{(b-a)^3}{6(u(b)-u(a))^2} \cdot \bigvee_a^b (f), & \text{if } f \text{ is of bounded variation} \\[2ex] \frac{H_f (b-a)^{p+3}}{2^{p+1} \cdot 3 (u(b)-u(a))^2}, & \text{if } f \text{ is } H_f\text{-}p\text{-Holder} \end{cases}$$

where, $H_g, H_f > 0$ and $p, q \in (0, 1]$ are given.

Remark 4. *Under the assumptions of Theorem 4, we have*

$$(2.20) \quad |\mathcal{T}(f, g)| \leq H_g \cdot \begin{cases} \frac{(b-a)^q}{(q+1)(q+2)} \cdot \bigvee_a^b (f), & \text{if } f \text{ is of bounded variation} \\[2ex] \frac{H_f (b-a)^{p+q}}{2^p (q+1)(q+2)}, & \text{if } f \text{ is } H_f\text{-}p\text{-Holder} \end{cases}$$

where, $H_f > 0$ and $p, q \in (0, 1]$ are given. In particular, if g is L_g–Lipschitzian, then

$$(2.21) \quad |\mathcal{T}(f, g)| \leq L_g \cdot \begin{cases} \frac{1}{6}(b-a) \cdot \bigvee_a^b (f), & \text{if } f \text{ is of bounded variation} \\[2ex] \frac{1}{12} L_f (b-a)^2, & \text{if } f \text{ is } L_f\text{-Lipschitzian} \end{cases}$$

References

[1] M.W. Alomari and S.S. Dragomir, New Grüss type inequalities for the Riemann–Stieltjes integral with monotonic integrators and applications, submitted. Avalibale at: http://ajmaa.org/RGMIA/papers/v15/v15a57.pdf.

[2] M.W. Alomari, Some Grüss type inequalities for Riemann-Stieltjes integral and applications, *Acta Mathematica Universitatis Comenianae*, (N.S.) 81 (2012), no. 2, 211–220.

[3] G.A. Anastassiou, Grüss type inequalities for the Stieltjes integral. *Nonlinear Funct. Anal. Appl.* **12** (2007), no. 4, 583–593.

[4] G.A. Anastassiou, Chebyshev-Grüss type and comparison of integral means inequalities for the Stieltjes integral. *Panamer. Math. J.* **17** (2007), no. 3, 91–109.

[5] P. Cerone and S.S. Dragomir, New bounds for the Čebyšev functional, *Appl. Math. Lett.* 18 (2005) 603–611.

[6] P. Cerone and S.S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* 38 (1) (2007) 37–49.

[7] S.S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.* 26 (2004) 89–112.

[8] S.S. Dragomir, New Grüss' type inequalities for functions of bounded variation and applications, *Appl. Math. Lett.* In press, http://dx.doi.org/10.1016/j.aml.2011.12.027, (2012).

[9] S.S. Dragomir, New estimates of the Čebyšev functional for Stieltjes integrals and applications, *J. Korean Math. Soc.*, 41(2) (2004), 249–264.

[10] S.S. Dragomir, Sharp bounds of Čebyšev functional for Stieltjes integrals and applications, *Bull. Austral. Math. Soc.*, 67(2) (2003), 257–266.

[11] S.S. Dragomir, Some integral inequality of Grüss type, Indian *J. Pure Appl. Math.*, 31(4) (2000), 397–415.

[12] S.S. Dragomir, Grüss type integral inequality for mappings of r-Hölder type and applications for trapezoid formula, *Tamkang J. Math.*, 31(1) (2000), 43–47.

[13] S.S. Dragomir and I. Fedotov, An inequality of Gruss type for Riemann–Stieltjes integral and applications for special means, *Tamkang J. Math.*, 29(4) (1998), 287–292.

[14] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$, *Math. Z.* 39 (1935) 215–226.

[15] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral. *Soochow J. Math.* **30** (2004), no. 4, 483–489.

^{1}DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, JERASH UNIVERSITY, 26150 JERASH, JORDAN.

E-mail address: mwomath@gmail.com

^{2} MATHEMATICS, SCHOOL OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.

^{3}SCHOOL OF COMPUTATIONAL & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG, SOUTH AFRICA.

E-mail address: sever.dragomir@vu.edu.au