Received 20/06/13

# INEQUALITIES FOR THE RIEMANN-STIELTJES INTEGRAL OF MIDPOINT SEPARATED FUNCTIONS WITH APPLICATIONS

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Abstract. For $\lambda \in \mathbb{R}$, we say that the function $f:[a,b]\to\mathbb{R}$ is $\lambda$-separated in the midpoint $\frac{a+b}{2}$ if

$$f(s)-f\left(\frac{a+b}{2}\right)\geq \lambda\left(s-\frac{a+b}{2}\right)$$

for any $s\in[a,b]$.

In this paper we show amongst other that

$$\int_a^b f(t)\,du(t)\geq f\left(\frac{a+b}{2}\right)[u(b)-u(a)] + \lambda\left[\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(t)\,dt\right]$$

provided that $f:[a,b]\to\mathbb{R}$ is $\lambda$-separated in the midpoint and $u$ is monotonic nondecreasing on $[a,b]$.

Some particular cases for the weighted integrals in connection with the Fejér inequalities are also provided.

## 1. Introduction

The following inequality holds for any *convex function* $f$ defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)\leq\int_a^b f(x)dx\leq(b-a)\frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a,b\in\mathbb{R}$ with $a<b$.

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [22]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [25].

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [22]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [25].

For related results, see for instance the research papers [1], [3]-[15], [17], [19], [20], [24], [23], [26], [27], [28], the monograph online [14] and the references therein.

In 1906, Fejér [16], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

2000 *Mathematics Subject Classification.* 26D15, 26D10.

*Key words and phrases.* Fejér inequality, Monotonic functions, Riemann-Stieltjes integral, Midpoint inequality, Trapezoid inequality.

**Theorem 1.** *Consider the integral $\int_a^b h(x) w(x) dx$, where $h$ is a convex function in the interval $(a,b)$ and $w$ is a positive function in the same interval such that*

$$w(a+t) = w(b-t), \quad 0 \le t \le \frac{1}{2}(a+b),$$

*i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right) \int_a^b w(x) dx \le \int_a^b h(x) w(x) dx \le \frac{h(a)+h(b)}{2} \int_a^b w(x) dx. \tag{1.2}$$

*If $h$ is concave on $(a,b)$, then the inequalities reverse in (1.2).*

Clearly, for $w(x) \equiv 1$ on $[a,b]$ we get 1.1.

Motivated by these classical results and their impact in the literature, it is natural to ask when inequalities for the Riemann-Stieltjes integral of the following type

$$f\left(\frac{a+b}{2}\right)[u(b) - u(a)] \le \int_a^b f(t) du(t) \tag{1.3}$$

holds.

In order to address this question, we have introduced in this paper the concept of functions $\lambda$-*separated in the midpoint point* $\frac{a+b}{2}$ on a closed interval $[a,b]$, which generalizes the concept of convex function on $[a,b]$ and established some fundamental inequalities for the Riemann-Stieltjes integral for various classes of integrands and integrators. Some particular cases for the weighted integrals in connection with the Fejér first inequality (1.2) are provided.

## 2. Trapezoid and Midpoint Type Functions

Following the recent paper [11], we consider the class of function defined as follows:

**Definition 1.** *We say that the Lebesgue integrable function $f : [a,b] \to \mathbb{R}$ is sub-trapezoidal if*

$$\frac{f(a)+f(b)}{2}(b-a) \ge \int_a^b f(t) dt. \tag{2.1}$$

*We denote this by $f \in \mathcal{T}_{Sub}[a,b]$.*

As above, we observe that $\mathcal{T}_{Sub}[a,b]$ is a closed convex cone in the uniform convergence topology of the space of all Lebesgue integrable functions defined on $[a,b]$ denoted, as usual, by $\mathcal{L}[a,b]$.

As in the case of convex-concave functions, we can say that $f$ is *super-trapezoidal* if $-f \in \mathcal{T}_{Sub}[a,b]$. We denote this by $f \in \mathcal{T}_{Sup}[a,b]$. Moreover, we say that $f$ is *trapezoidal* if $f$ and $-f \in \mathcal{T}_{Sub}[a,b]$, i.e.

$$\frac{f(a)+f(b)}{2}(b-a) = \int_a^b f(t) dt. \tag{2.2}$$

We denote this by $f \in \mathcal{T}[a,b]$. We observe that $\mathcal{T}[a,b]$ is a closed linear subspace of $\mathcal{L}[a,b]$ with the uniform convergence topology.

If we denote by $\mathcal{C}_v[a,b]$ the closed convex cone of all convex functions defined on $[a,b]$ then we can state the following result:

**Proposition 1.** *We have the strict inclusion*

$$\mathcal{C}_v[a,b] \subsetneqq \mathcal{T}_{Sub}[a,b]. \tag{2.3}$$

*Proof.* If $f$ is convex on $[a,b]$ then for any $\lambda \in [0,1]$ and $x, y \in [a,b]$ we have

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y). \tag{2.4}$$

Taking $x = a$ and $y = b$ in (2.4) and integrating over $\lambda \in [0,1]$ we have

$$\frac{f(a)+f(b)}{2} \geq \int_a^b f(\lambda a + (1-\lambda) b)\, d\lambda.$$

Changing the variable $t := \lambda a + (1-\lambda) b$, $\lambda \in [0,1]$ we have

$$\int_a^b f(\lambda a + (1-\lambda) b)\, d\lambda = \frac{1}{b-a}\int_a^b f(t)\, dt$$

and the inequality is proved.

Consider the function

$$f_1 : [0,2\pi] \to \mathbb{R},\ f_1(t) = \sin t,$$

then we observe that $f_1 \in \mathcal{T}[0,2\pi]$ and *a fortiori* $f_1 \in \mathcal{T}_{Sub}[0,2\pi]$, but it is easy to see that $f_1$ is not convex on the interval $[0,2\pi]$. □

**Definition 2.** *We say that the function $f : [a,b] \to \mathbb{R}$ is symmetric (or anti-symmetric) on the interval $[a,b]$ if*

$$f(t) = f(a+b-t)\ (or - f(a+b-t))$$

*for any $t \in [a,b]$. We denote this by $f \in \mathcal{S}_y[a,b]$ (or $f \in \mathcal{A}_s[a,b]$).*

The following result holds:

**Proposition 2.** *We have the strict inclusion*

$$\mathcal{A}_s[a,b] \cap \mathcal{L}[a,b] \subsetneqq \mathcal{T}[a,b]. \tag{2.5}$$

*Proof.* If $f \in \mathcal{A}_s[a,b] \cap \mathcal{L}[a,b]$ then obviously $f(a) = -f(b)$ and $\int_a^b f(t)\, dt = 0$ and the equality (2.2) is trivially satisfied.

Now, if we consider the function $f_0 : [-2\pi, 2\pi] \to \mathbb{R}$ defined by

$$f_0(t) = \begin{cases} 0 & \text{if } t \in [-2\pi, 0] \\ \sin t & \text{if } t \in (0, 2\pi], \end{cases}$$

then we observe that $f_0 \in \mathcal{T}[-2\pi, 2\pi]$ but $f_0$ is not anti-symmetric on $[-2\pi, 2\pi]$. □

**Proposition 3.** *Let $w : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function. Define $f : [a,b] \to \mathbb{R}$ by*

$$f(t) = \int_a^t w(s)\, ds - \frac{1}{2}\int_a^b w(s)\, ds = \frac{1}{2}\left(\int_a^t w(s)\, ds - \int_t^b w(s)\, ds\right). \tag{2.6}$$

*If $w \in \mathcal{S}_y[a,b]$ then $f \in \mathcal{A}_s[a,b]$.*

*Proof.* Let $t\in[a,b]$. We have by the definition of $f$ that

$$f(a+b-t)=\int_a^{a+b-t}w(s)\,ds-\frac{1}{2}\int_a^b w(s)\,ds. \tag{2.7}$$

If we make the change of variable $u=a+b-s$, then we have

$$\int_a^{a+b-t}w(s)\,ds=-\int_b^t w(a+b-u)\,du=\int_t^b w(a+b-u)\,du. \tag{2.8}$$

Since $w\in\mathcal{S}_y[a,b]$, then

$$\int_t^b w(a+b-u)\,du=\int_t^b w(u)\,du \tag{2.9}$$

for any $t\in[a,b]$.

On making use of (2.7)-(2.9) we have

$$\begin{aligned}f(a+b-t)&=\int_t^b w(u)\,du-\frac{1}{2}\int_a^b w(s)\,ds\\&=\frac{1}{2}\left(\int_t^b w(s)\,ds-\int_a^t w(s)\,ds\right)=-f(t)\end{aligned}$$

for any $t\in[a,b]$.

The proof is complete. □

The following result also holds:

**Proposition 4.** *Let $w:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function. Define $f:[a,b]\to\mathbb{R}$ by*

$$f(t)=\int_a^t w(s)\,ds. \tag{2.10}$$

*The following statements are equivalent:*

(i) $f\,(or\ -f)\in\mathcal{T}_{Sub}[a,b]$;
(ii) *We have the inequality:*

$$\int_a^b tw(t)\,dt\geq(or\ \leq)\frac{a+b}{2}\int_a^b w(t)\,dt. \tag{2.11}$$

*Proof.* Utilising the integration by parts for the Riemann integral we have:

$$\begin{aligned}&\frac{f(b)+f(a)}{2}(b-a)-\int_a^b f(t)\,dt\\&=\frac{1}{2}(b-a)\int_a^b w(t)\,dt-\int_a^b\left(\int_a^t w(s)\,ds\right)dt\\&=\frac{1}{2}(b-a)\int_a^b w(t)\,dt-\left[\left(\int_a^t w(s)\,ds\right)t\Big|_a^b-\int_a^b tw(t)\,dt\right]\\&=\frac{1}{2}(b-a)\int_a^b w(t)\,dt-\left[\left(\int_a^b w(s)\,ds\right)b-\int_a^b tw(t)\,dt\right]\\&=\int_a^b tw(t)\,dt-\frac{a+b}{2}\int_a^b w(t)\,dt,\end{aligned}$$

which proves the desired statement. □

**Remark 1.** *We observe that, by Proposition 4 we have* $f \in \mathcal{T}[a,b]$, *where* $f$ *is defined by (2.10), if and only if*

$$\int_a^b tw(t)\,dt = \frac{a+b}{2}\int_a^b w(t)\,dt. \tag{2.12}$$

We denote in the following the closed convex cone of monotonic nondecreasing functions defined on $[a,b]$ by $\mathcal{M}^{\nearrow}[a,b]$ and by $\mathcal{C}[a,b]$ the Banach space of continuous functions on the interval $[a,b]$.

We have the following result:

**Corollary 1.** *If* $w$ *(or* $-w$*)* $\in \mathcal{M}^{\nearrow}[a,b]$, *then the function* $f$ $(-f)$ *defined by (2.10) belongs to* $\mathcal{T}_{Sub}[a,b]$.

*Proof.* We use the Čebyšev inequality that state that

$$\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt \geq (\leq)\,\frac{1}{b-a}\int_a^b F(t)\,dt\frac{1}{b-a}\int_a^b G(t)\,dt$$

provided $F$ and $G$ have the same (opposite) monotonicity on $[a,b]$.

Writing this inequality for $F(t)=t$ and $G(t)=w(t)$ we obtain the desired result. □

**Definition 3.** *We say that the Lebesgue integrable function* $f:[a,b]\to\mathbb{R}$ *is of sub(supper)-midpoint type if*

$$\int_a^b f(t)\,dt \geq (\leq)\, f\left(\frac{a+b}{2}\right)(b-a). \tag{2.13}$$

*We denote this by* $f \in \mathcal{M}_{Sub(Sup)}[a,b]$.

Moreover, we say that $f$ is of *midpoint* type if

$$f \in \mathcal{M}_{Sub}[a,b] \cap \mathcal{M}_{Sup}[a,b],$$

i.e.

$$\int_a^b f(t)\,dt = f\left(\frac{a+b}{2}\right)(b-a). \tag{2.14}$$

We denote this by $f \in \mathcal{M}[a,b]$. We observe that if $f \in \mathcal{A}_s[a,b]$ then obviously $f \in \mathcal{M}[a,b]$ and there are functions which are of midpoint type but not anti-symmetric. Indeed, if we consider the function $f_0:[-2\pi,2\pi]\to\mathbb{R}$ defined by

$$f_0(t) = \begin{cases} 0 & \text{if } t\in[-2\pi,0] \\ \\ \sin t & \text{if } t\in(0,2\pi], \end{cases}$$

then we observe that $f_0 \in \mathcal{M}[-2\pi,2\pi]$ but $f_0$ is not anti-symmetric on $[-2\pi,2\pi]$.

It is obvious that $\mathcal{M}_{Sub}[a,b]$ is a closed convex cone and it contains strictly the convex cone of convex functions defined on $[a,b]$, i.e.

$$\mathcal{C}_v[a,b] \subsetneq \mathcal{M}_{Sub}[a,b].$$

**Proposition 5.** *Let* $w:[a,b]\to\mathbb{R}$ *be a Lebesgue integrable function. Define* $f:[a,b]\to\mathbb{R}$ *by*

$$f(t) = \int_a^t w(s)\,ds.$$

*The following statements are equivalent:*

(i) $f\,(or\ -f) \in \mathcal{M}_{Sub}[a,b]$;
(ii) *We have the inequality:*

$$\int_a^b tw(t)\,dt \le (or\ \ \ge)\,(b-1)\int_a^{\frac{a+b}{2}} w(s)\,ds + b\int_{\frac{a+b}{2}}^b w(s)\,ds. \tag{2.15}$$

*Proof.* Utilising the integration by parts for the Riemann integral we have:

$$\begin{aligned}
&\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\
&= \int_a^b \left(\int_a^t w(s)\,ds\right)dt - \int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= \left[\left(\int_a^t w(s)\,ds\right)t\Big|_a^b - \int_a^b tw(t)\,dt\right] - \int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= \left(\int_a^b w(s)\,ds\right)b - \int_a^b tw(t)\,dt - \int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= b\int_a^{\frac{a+b}{2}} w(s)\,ds + b\int_{\frac{a+b}{2}}^b w(s)\,ds - \int_a^b tw(t)\,dt - \int_a^{\frac{a+b}{2}} w(t)\,dt \\
&= (b-1)\int_a^{\frac{a+b}{2}} w(s)\,ds + b\int_{\frac{a+b}{2}}^b w(s)\,ds - \int_a^b tw(t)\,dt,
\end{aligned}$$

which proves the desired result. □

We can introduce now the following class of functions:

**Definition 4.** *For $\lambda \in \mathbb{R}$, we say that the function $f:[a,b]\to\mathbb{R}$ is $\lambda$-separated in the midpoint point $\frac{a+b}{2}$ and we write that as $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ if*

$$\frac{f\left(\frac{a+b}{2}\right) - f(\tau)}{\frac{a+b}{2} - \tau} \le \lambda \le \frac{f(t) - f\left(\frac{a+b}{2}\right)}{t - \frac{a+b}{2}} \tag{2.16}$$

*for any $a \le \tau < \frac{a+b}{2} < t \le b$, or equivalently*

$$f(s) - f\left(\frac{a+b}{2}\right) \ge \lambda\left(s - \frac{a+b}{2}\right) \tag{2.17}$$

*for any $s \in [a,b]$.*

We observe that if $f_1, f_2 \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ and $\alpha \in [0,1]$ then $\alpha f_1 + (1-\alpha) f_2 \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ which shows that $\mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ is a convex subset in the space of all functions defined on $[a,b]$. It is also closed in the uniform topology.

**Proposition 6.** *If $f$ is convex on $[a,b]$ then $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ for any*

$$\lambda \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right)\right],$$

*where $f'_-$ and $f'_+$ are the left and right derivatives of the convex function $f$. There are functions in $\mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ for some $\lambda \in \mathbb{R}$ which are not convex on $[a,b]$.*

*Proof.* Since $f$ is convex, then for any $x, y \in (a,b)$ we have the gradient inequality

$$f(x) - f(y) \geq \gamma_y (x - y) \tag{2.18}$$

where $\gamma_y \in \partial_y = \left[f'_-(y), f'_+(y)\right]$, the subdifferential of the function $f$ in the point $y$.

Taking $x = s$ and $y = \frac{a+b}{2}$ in (2.18) we conclude that $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ with $\lambda = \gamma_{\frac{a+b}{2}}$.

Take $\lambda = 0$, $a = -1$, $b = 1$ and $f_0(t) = \sqrt{|t|}$. We observe that

$$f_0(s) - f_0(0) = \sqrt{|s|} \geq 0 = 0\left(s - \frac{a+b}{2}\right)$$

for any $s \in [-1,1]$, which shows that $f_0 \in \mathcal{S}_{0,0}[-1,1]$. However, it is clear that $f_0$ is not convex on $[-1,1]$. □

**Proposition 7.** *If there exists $\lambda \in \mathbb{R}$ such that $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$, then $f \in \mathcal{M}_{Sub}[a,b]$.*

*Proof.* If $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ then

$$f(s) - f\left(\frac{a+b}{2}\right) \geq \lambda\left(s - \frac{a+b}{2}\right)$$

for any $s \in [a,b]$.

Integrating on $[a,b]$ we have

$$\int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right)(b-a) \geq \lambda \int_a^b \left(s - \frac{a+b}{2}\right) ds = 0$$

and the statement is proved. □

**Remark 2.** *Let $w : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a,b]$. Define $f : [a,b] \to \mathbb{R}$ by $f(t) = \int_a^t w(s)\,ds$. We observe that, by (2.16) we have $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ if and only if*

$$\frac{1}{\frac{a+b}{2} - \tau} \int_\tau^{\frac{a+b}{2}} w(s)\,ds \leq \lambda \leq \frac{1}{t - \frac{a+b}{2}} \int_{\frac{a+b}{2}}^t w(s)\,ds \tag{2.19}$$

*for any $a \leq \tau < \frac{a+b}{2} < t \leq b$.*

## 3. Midpoint Type Inequalities for the Riemann-Stieltjes Integral

The following result holds:

**Theorem 2.** *Let $\lambda \in \mathbb{R}$. If $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ and $u \in \mathcal{M}^{\nearrow}[a,b]$, then*

$$\int_a^b f(t)\,du(t) \geq f\left(\frac{a+b}{2}\right)[u(b) - u(a)] + \lambda\left[\frac{u(a) + u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right] \tag{3.1}$$

*or, equivalently*

$$\int_a^b u(t)\,df(t) \leq \left[f(b) - f\left(\frac{a+b}{2}\right)\right] u(b) + \left[f\left(\frac{a+b}{2}\right) - f(a)\right] u(a) + \lambda\left[\int_a^b u(t)\,dt - \frac{u(a)+u(b)}{2}(b-a)\right]. \tag{3.2}$$

*Proof.* Since $f$ is continuous and $u$ monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists.

Since $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b]$ then

$$f(s) - f\left(\frac{a+b}{2}\right) \geq \lambda\left(s - \frac{a+b}{2}\right)$$

for any $s \in [a,b]$.

Integrating over the nondecreasing integrator $u$ on the interval $[a,b]$, we have

$$\begin{aligned} &\int_a^b f(t)\,du(t) - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \\ &= \int_a^b \left[f(s) - f\left(\frac{a+b}{2}\right)\right] du(t) \geq \lambda \int_a^b \left(s - \frac{a+b}{2}\right) du(t) \\ &= \lambda\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right], \end{aligned} \tag{3.3}$$

which is equivalent with (3.1).

Integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} &\int_a^b \left[f(s) - f\left(\frac{a+b}{2}\right)\right] du(t) \\ &= \left[f(s) - f\left(\frac{a+b}{2}\right)\right] u(t)\Big|_a^b - \int_a^b u(t)\,df(t) \\ &= \left[f(b) - f\left(\frac{a+b}{2}\right)\right] u(b) + \left[f\left(\frac{a+b}{2}\right) - f(a)\right] u(a) \\ &- \int_a^b u(t)\,df(t). \end{aligned}$$

Utilising the inequality (3.3) we deduce the desired result (3.2). □

**Corollary 2.** *Let $f \in \mathcal{C}_v[a,b]$ and $u \in \mathcal{M}^{\nearrow}[a,b]$, then*

$$\int_a^b f(t)\,du(t) \geq f\left(\frac{a+b}{2}\right)[u(b)-u(a)] + \lambda_{\frac{a+b}{2}}\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right] \tag{3.4}$$

*or, equivalently*

$$\int_a^b u(t)\,df(t) \le \left[f(b) - f\left(\frac{a+b}{2}\right)\right] u(b) + \left[f\left(\frac{a+b}{2}\right) - f(a)\right] u(a) + \lambda_{\frac{a+b}{2}} \left[\int_a^b u(t)\,dt - \frac{u(a)+u(b)}{2}(b-a)\right], \tag{3.5}$$

*where* $\lambda_{\frac{a+b}{2}} \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right)\right]$.

The following midpoint type inequality for the Riemann-Stieltjes integral holds:

**Corollary 3.** *Let* $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ *for some* $\lambda \in \mathbb{R}$ *and* $u \in \mathcal{M}^{\nearrow}[a,b] \cap \mathcal{T}[a,b]$ *then*

$$\int_a^b f(t)\,du(t) \ge f\left(\frac{a+b}{2}\right)[u(b) - u(a)] \tag{3.6}$$

*or, equivalently*

$$\int_a^b u(t)\,df(t) \le \left[f(b) - f\left(\frac{a+b}{2}\right)\right] u(b) + \left[f\left(\frac{a+b}{2}\right) - f(a)\right] u(a). \tag{3.7}$$

**Remark 3.** *Let* $w : [a,b] \to \mathbb{R}$ *be a Lebesgue integrable function. If* $w \ge 0$ *and*

$$\int_a^b t w(t)\,dt = \frac{a+b}{2} \int_a^b w(t)\,dt,$$

*then the function* $u(t) := \int_a^t w(s)\,ds$ *is in* $\mathcal{M}^{\nearrow}[a,b] \cap \mathcal{T}[a,b]$ *and by (3.6) we deduce that*

$$\int_a^b f(t) w(t)\,dt \ge f\left(\frac{a+b}{2}\right) \int_a^b w(t)\,dt, \tag{3.8}$$

*where* $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ *for some* $\lambda \in \mathbb{R}$*, which provides a generalization of the first Fejér's inequality in (1.2).*

The following corollary also holds:

**Corollary 4.** *Let* $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ *for some* $\lambda \in \mathbb{R}$ *and* $u \in \mathcal{M}^{\nearrow}[a,b]$*. If* $\lambda > 0$ *and* $u \in \mathcal{T}_{Sub}[a,b]$ *or* $\lambda < 0$ *and* $u \in \mathcal{T}_{Sup}[a,b]$ *then (3.6) or, equivalently (3.7) holds true.*

**Remark 4.** *Let* $w : [a,b] \to \mathbb{R}$ *be a Lebesgue integrable function with* $w \ge 0$ *and* $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ *for some* $\lambda \in \mathbb{R}$*. If* $\lambda > 0$ *and*

$$\int_a^b t w(t)\,dt \ge \frac{a+b}{2} \int_a^b w(t)\,dt,$$

*or* $\lambda < 0$ *and*

$$\int_a^b t w(t)\,dt \le \frac{a+b}{2} \int_a^b w(t)\,dt,$$

*then (3.8) holds true.*

The second general result is as follows:

**Theorem 3.** *Let* $\lambda \in \mathbb{R}$. *If* $f \in \mathcal{S}_{\lambda, \frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ *and* $u \in \mathcal{M}^{\nearrow}[a,b]$, *then*

$$f(b)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] + f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \geq \lambda\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right] \tag{3.9}$$

*and*

$$f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] + f(a)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \geq \lambda\left[\int_a^b u(t)\,dt - \frac{u(a)+u(b)}{2}(b-a)\right]. \tag{3.10}$$

*Proof.* If $f \in \mathcal{S}_{\lambda, \frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$, then

$$f(b) - f\left(\frac{a+b}{2}\right) \geq \frac{1}{2}\lambda(b-a) \tag{3.11}$$

and

$$f(a) - f\left(\frac{a+b}{2}\right) \geq -\frac{1}{2}\lambda(b-a). \tag{3.12}$$

If we multiply the inequality (3.11) by $t-a$ and (3.12) by $b-t$ and add the obtained inequalities, then we get

$$f(b)(t-a) + f(a)(b-t) - f\left(\frac{a+b}{2}\right)(b-a) \geq \lambda(b-a)\left(t - \frac{a+b}{2}\right)$$

for any $t \in [a,b]$.

Integrating over the nondecreasing integrator $u$ on the interval $[a,b]$, we have

$$f(b)\int_a^b (t-a)\,du(t) + f(a)\int_a^b (b-t)\,du(t) - f\left(\frac{a+b}{2}\right)(b-a)[u(b)-u(a)] \geq \lambda(b-a)\int_a^b \left(t - \frac{a+b}{2}\right)du(t). \tag{3.13}$$

Utilising the integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^b (t-a)\,du(t) = (b-a)u(b) - \int_a^b u(t)\,dt,$$

$$\int_a^b (b-t)\,du(t) = \int_a^b u(t)\,dt - (b-a)u(a)$$

and

$$\int_a^b \left(t - \frac{a+b}{2}\right) du(t) = \frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt.$$

Therefore by (3.13) we get the desired result (3.9).

Further, if we multiply the inequality (3.11) by $b-t$ and (3.12) by $t-a$ and add the obtained inequalities, then we get

$$\begin{aligned} & f(b)(b-t) + f(a)(t-a) - f\left(\frac{a+b}{2}\right)(b-a) \\ & \geq -\lambda(b-a)\left(t - \frac{a+b}{2}\right) \end{aligned}$$

for any $t \in [a,b]$.

Integrating over the nondecreasing integrator $u$ on the interval $[a,b]$, we deduce the desired result (3.10). □

**Corollary 5.** *Let $f \in \mathcal{C}_v[a,b]$ and $u \in \mathcal{M}^{\nearrow}[a,b]$, then*

$$\begin{aligned} & f(b)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] + f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] \\ & - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \\ & \geq \lambda_{\frac{a+b}{2}}\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right] \end{aligned} \tag{3.14}$$

*and*

$$\begin{aligned} & f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] + f(a)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] \\ & - f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \\ & \geq \lambda_{\frac{a+b}{2}}\left[\int_a^b u(t)\,dt - \frac{u(a)+u(b)}{2}(b-a)\right], \end{aligned} \tag{3.15}$$

*where $\lambda_{\frac{a+b}{2}} \in \left[f'_-\left(\frac{a+b}{2}\right), f'_+\left(\frac{a+b}{2}\right)\right]$.*

**Corollary 6.** *Let $f \in \mathcal{S}_{\lambda,\frac{a+b}{2}}[a,b] \cap \mathcal{C}[a,b]$ for some $\lambda \in \mathbb{R}$ and $u \in \mathcal{M}^{\nearrow}[a,b]$. If $\lambda > 0$ and $u \in \mathcal{T}_{Sub}[a,b]$ or $\lambda < 0$ and $u \in \mathcal{T}_{Sup}[a,b]$, then*

$$\begin{aligned} & f(b)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] + f(a)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] \\ & \geq f\left(\frac{a+b}{2}\right)[u(b)-u(a)]. \end{aligned} \tag{3.16}$$

*If* $\lambda < 0$ *and* $u \in \mathcal{T}_{Sub}[a,b]$ *or* $\lambda > 0$ *and* $u \in \mathcal{T}_{Sup}[a,b]$, *then*

$$f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] + f(a)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] \geq f\left(\frac{a+b}{2}\right)[u(b)-u(a)] \tag{3.17}$$

**Remark 5.** *Similar weighted inequalities may be stated. However the details are not presented here.*

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