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Some Inequalities of Jensen Type for Arg-square Convex Functions of Unitary Operators in Hilbert Spaces

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Abstract. Some inequalities of Jensen type for Arg-square-convex functions of unitary operators in Hilbert spaces are given.

1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. We recall that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [4, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \le E_\mu$ for $0 \le \lambda \le \mu \le 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \le \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of *Riemann-Stieltjes* type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{1.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f(U)x = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda x, \tag{1.2}$$

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{1.3}$$

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and

$$\|f(U)x\|^2=\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2, \tag{1.4}$$

for any $x,y\in H$.

For $z\in\mathbb{C}\setminus\{0\}$ we call the *principal value* of $log(z)$ the complex number

$$Log(z):=\ln|z|+iArg(z)$$

where $0\le Arg(z)<2\pi$.

We observe that for $t\in[0,2\pi)$ we have

$$Log\left(e^{it}\right)=it.$$

If we extend this equality by continuity in the point $t=2\pi$, then we can define the operator $Log(U):H\to H$ as follows:

$$Log(U)x:=\int_0^{2\pi}Log\left(e^{i\lambda}\right)dE_\lambda x=\int_0^{2\pi}(i\lambda)\,dE_\lambda x,\ x\in H.$$

In what follows we establish some results connecting this operator with the function of operator $f(U)$ for a class of function we call *Arg-square-convex* such that a Jensen type inequality and related results can be derived.

2. The Results

The function $f:\mathcal{C}(0,1)\to\mathbb{C}$ will be called *Arg-square-convex* if the composite function $\varphi:[0,2\pi]\to[0,\infty)$,

$$\varphi(t)=\begin{cases}\left|f\left(e^{it}\right)\right|^2, & t\in[0,2\pi)\\ \\ \lim_{s\to 2\pi-}\left|f\left(e^{is}\right)\right|^2, & t=2\pi\end{cases}$$

is continuous and convex on $[0,2\pi]$.

To make the distinction between the value $\varphi(0)=\left|f\left(e^{i0}\right)\right|^2=|f(1)|^2$ and the value $\varphi(2\pi)=\lim_{s\to 2\pi-}\left|f\left(e^{is}\right)\right|^2$ we denote by $f_c(1):=\lim_{s\to 2\pi-}f\left(e^{is}\right)$. With this notation we have $\varphi(2\pi)=|f_c(1)|^2$.

The function $f_n:\mathcal{C}(0,1)\to\mathbb{C},\ f_n(z)=[Log(z)]^n$, where n is a positive integer, is Arg-square-convex. We have

$$\varphi_n(t)=\left|f_n\left(e^{it}\right)\right|^2=\left|\left[Log\left(e^{it}\right)\right]^n\right|^2=|it|^{2n}=t^{2n},t\in[0,2\pi),$$

and

$$\varphi_n(2\pi)=\lim_{s\to 2\pi-}\left|f_n\left(e^{is}\right)\right|^2=|f_{n,c}(1)|^2=(2\pi)^{2n}.$$

For $q\ge\frac{1}{2}$ define the function $f_q:\mathcal{C}(0,1)\to[0,\infty)$ by $f_q(z)=|Log(z)|^q$. We have

$$\varphi_q(t)=\left|f_q\left(e^{it}\right)\right|^2=\left|Log\left(e^{it}\right)\right|^{2q}=|it|^{2q}=t^{2q},t\in[0,2\pi)$$

and

$$\varphi_q(2\pi)=\lim_{s\to 2\pi-}\left|f_q\left(e^{is}\right)\right|^2=|f_{q,c}(1)|^2=(2\pi)^{2q}.$$

The function f_q for $q\ge\frac{1}{2}$ is an Arg-square-convex function.

If $g:[0,2\pi]\to[0,\infty)$ is continuous and convex on $[0,2\pi]$, then the composite function $f:\mathcal{C}(0,1)\to[0,\infty)$ defined by

$$f(z):=[g(|Log(z)|)]^{1/2}$$

is an Arg-square-convex function on $\mathcal{C}(0,1)$.

The following Jensen's type result holds:

THEOREM 1. *Let $U\in B(H)$ be a unitary operator on the Hilbert space H and $f:\mathcal{C}(0,1)\to\mathbb{C}$ an Arg-square-convex function on $\mathcal{C}(0,1)$. Then we have*

$$\left[\frac{|f(1)|^2(\langle[2\pi 1_H-|Log(U)|]x,x\rangle)+|f_c(1)|^2\langle|Log(U)|x,x\rangle}{2\pi}\right]^{1/2} \tag{2.1}$$
$$\geq\|f(U)x\|\geq\left|f\left(e^{\langle LogUx,x\rangle}\right)\right|,$$

for any $x\in H,\|x\|=1,$ where $f_c(1):=\lim_{s\to 2\pi-}f(e^{is})$.

PROOF. Since f is continuous on $\mathcal{C}(0,1)$ and U is a unitary operator, then by (1.4) we have

$$\|f(U)x\|^2=\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2d\|E_\lambda x\|^2=\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2d\langle E_\lambda x,x\rangle \tag{2.2}$$

for any $x\in H,\|x\|=1,$ where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

Now, since $\left|f\left(e^{i\cdot}\right)\right|^2$ is continuous convex on $[0,2\pi]$, then by Jensen's integral inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators we have

$$\frac{\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2d\langle E_\lambda x,x\rangle}{\int_0^{2\pi}d\langle E_\lambda x,x\rangle}\geq\left|f\left(\exp\left(i\frac{\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle}{\int_0^{2\pi}d\langle E_\lambda x,x\rangle}\right)\right)\right|^2 \tag{2.3}$$

for any $x\in H,\|x\|=1.$

Since

$$\int_0^{2\pi}d\langle E_\lambda x,x\rangle=\|x\|^2=1$$

and

$$\int_0^{2\pi}(i\lambda)\,d\langle E_\lambda x,x\rangle=\int_0^{2\pi}Log\left(e^{i\lambda}\right)d\langle E_\lambda x,x\rangle=\langle LogUx,x\rangle$$

for any $x\in H,\|x\|=1,$ then we get from (2.3) the second inequality in (2.1).

Now, if $\varphi:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$ then for any $\lambda\in[a,b]$ we have the inequality

$$\frac{(b-\lambda)\varphi(a)+(\lambda-a)\varphi(b)}{b-a}\geq\varphi(\lambda).$$

If we write this inequality for the continuous convex function $\varphi(t)=\left|f\left(e^{it}\right)\right|^2$ on the interval $[0,2\pi]$, then we have

$$\frac{(2\pi-\lambda)|f(1)|^2+\lambda|f_c(1)|^2}{2\pi}\geq\left|f\left(e^{i\lambda}\right)\right|^2$$

for any $\lambda\in[0,2\pi]$.

Integrating on $[0,2\pi]$ over the monotonic nondecreasing integrator $\langle E_\lambda x,x\rangle$ we get

$$\frac{|f(1)|^2\left(2\pi-\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle\right)+|f_c(1)|^2\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle}{2\pi}$$
$$\geq\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x,x\rangle$$

for any $x\in H,\|x\|=1.$

Now, observe that the Riemann-Stieltjes integral $\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle$ exists and can be written as:

$$\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle=\int_0^{2\pi}\left|Log\left(e^{i\lambda}\right)\right|d\langle E_\lambda x,x\rangle=\langle|Log(U)|x,x\rangle$$

for any $x\in H,\|x\|=1.$

The proof is completed. □

The following result also holds:

THEOREM 2. *Let $U\in B(H)$ be a unitary operator on the Hilbert space H and $f:\mathcal{C}(0,1)\to\mathbb{C}$ an Arg-square-convex function on $\mathcal{C}(0,1)$. Then we have*

$$\begin{aligned}(2.4)\qquad &\frac{1}{\pi}\left[\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\right]\\&\times\langle[\pi 1_H-|Log(U)-i\pi 1_H|]x,x\rangle\\&\leq\frac{|f(1)|^2(\langle[2\pi 1_H-|Log(U)|]x,x\rangle)+|f_c(1)|^2\langle|Log(U)|x,x\rangle}{2\pi}\\&-\|f(U)x\|^2\\&\leq\frac{1}{\pi}\left[\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\right]\\&\times\langle[\pi 1_H+|Log(U)-i\pi 1_H|]x,x\rangle,\end{aligned}$$

for any $x\in H,\|x\|=1.$

PROOF. First of all, we recall the following result obtained by the author in [**1**] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}(2.5)\qquad &n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right]\\&\leq\frac{1}{P_n}\sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\\&\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n\Phi(x_i)-\Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right],\end{aligned}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n:=\sum_{i=1}^n p_i>0.$

For $n=2$ we deduce from (2.5) that

$$\begin{aligned}(2.6)\qquad & 2\min\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\ &\leq t\Phi(x)+(1-t)\Phi(y)-\Phi(tx+(1-t)y)\\ &\leq 2\max\{t,1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $x,y\in C$ and $t\in[0,1]$.

Now, if $\varphi:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then for any $\lambda\in[a,b]$ we have the inequality

$$\begin{aligned}(2.7)\qquad & 2\min\left\{\frac{b-\lambda}{b-a},\frac{\lambda-a}{b-a}\right\}\left[\frac{\varphi(a)+\varphi(b)}{2}-\varphi\left(\frac{a+b}{2}\right)\right]\\ &\leq\frac{(b-\lambda)\varphi(a)+(\lambda-a)\varphi(b)}{b-a}-\varphi(\lambda)\\ &\leq 2\max\left\{\frac{b-\lambda}{b-a},\frac{\lambda-a}{b-a}\right\}\left[\frac{\varphi(a)+\varphi(b)}{2}-\varphi\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

If we write the inequality (2.7) for the continuous convex function $\varphi(t)=\left|f\left(e^{it}\right)\right|^2$ on the interval $[0,2\pi]$, then we get

$$\begin{aligned}(2.8)\qquad & \frac{1}{\pi}\min\{2\pi-\lambda,\lambda\}\left[\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\right]\\ &\leq\frac{(2\pi-\lambda)|f(1)|^2+\lambda|f_c(1)|^2}{2\pi}-\left|f\left(e^{i\lambda}\right)\right|^2\\ &\leq\frac{1}{\pi}\max\{2\pi-\lambda,\lambda\}\left[\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\right]\end{aligned}$$

for any $\lambda\in[0,2\pi]$.

Let $x\in H$ with $\|x\|=1$. Integrating on $[0,2\pi]$ over the monotonic nondecreasing integrator $\langle E_\lambda x,x\rangle$ we get

$$\begin{aligned}(2.9)\qquad & \frac{1}{\pi}\left[\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\right]\int_0^{2\pi}\min\{2\pi-\lambda,\lambda\}\,d\langle E_\lambda x,x\rangle\\ &\leq\frac{|f(1)|^2\left(2\pi-\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle\right)+|f_c(1)|^2\int_0^{2\pi}\lambda d\langle E_\lambda x,x\rangle}{2\pi}\\ &-\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2d\langle E_\lambda x,x\rangle\\ &\leq\frac{1}{\pi}\left[\frac{|f(1)|^2+|f_c(1)|^2}{2}-|f(-1)|^2\right]\int_0^{2\pi}\max\{2\pi-\lambda,\lambda\}\,d\langle E_\lambda x,x\rangle\end{aligned}$$

and since

$$\begin{aligned}
(2.10)\quad & \int_0^{2\pi} \min\{2\pi-\lambda,\lambda\}\, d\langle E_\lambda x,x\rangle \\
&= \int_0^{2\pi} [\pi - |\lambda-\pi|]\, d\langle E_\lambda x,x\rangle = \pi - \int_0^{2\pi} |\lambda-\pi|\, d\langle E_\lambda x,x\rangle \\
&= \pi - \int_0^{2\pi} |\lambda-\pi|\, d\langle E_\lambda x,x\rangle = \pi - \int_0^{2\pi} |i\lambda-i\pi|\, d\langle E_\lambda x,x\rangle \\
&= \pi - \int_0^{2\pi} \left|Log\left(e^{it}\right) - i\pi\right| d\langle E_\lambda x,x\rangle = \pi - \langle |Log(U) - i\pi 1_H|\, x,x\rangle \\
&= \langle [\pi 1_H - |Log(U) - i\pi 1_H|]\, x,x\rangle
\end{aligned}$$

and, similarly,

$$(2.11)\quad \int_0^{2\pi} \max\{2\pi-\lambda,\lambda\}\, d\langle E_\lambda x,x\rangle = \langle [\pi 1_H + |Log(U) - i\pi 1_H|]\, x,x\rangle$$

then by (2.9)-(2.11) we get the desired result (2.4). □

In the following, an upper bound for the nonnegative difference

$$\|f(U)x\|^2 - \left|f\left(e^{\langle LogUx,x\rangle}\right)\right|^2$$

where $x \in H$ with $\|x\| = 1$, is also provided:

Theorem 3. *Let $U \in B(H)$ be a unitary operator on the Hilbert space H and $f : \mathcal{C}(0,1) \to \mathbb{C}$ an Arg-square-convex function on $\mathcal{C}(0,1)$. Then we have*

$$\begin{aligned}
(2.12)\quad 0 &\le \|f(U)x\|^2 - \left|f\left(e^{\langle LogUx,x\rangle}\right)\right|^2 \\
&\le \frac{1}{\pi}\max\{\langle (2\pi 1_H - |Log(U)|)\, x,x\rangle, \langle |Log(U)|\, x,x\rangle\} \\
&\quad\times \left[\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\right] \\
&\le 2\left[\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\right]
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$.

PROOF. By the convexity of the function $\varphi(t) = \left|f\left(e^{it}\right)\right|^2$ on the interval $[0, 2\pi]$, we have

$$
\begin{aligned}
&\|f(U)x\|^2 - \left|f\left(e^{\langle LogUx,x\rangle}\right)\right|^2 \\
&= \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle - \left|f\left(\exp\left(i\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&= \int_0^{2\pi} \left|f\left(e^{i\left[\frac{2\pi-\lambda}{2\pi}\cdot 0 + \frac{\lambda}{2\pi}\cdot 2\pi\right]}\right)\right|^2 d\langle E_\lambda x, x\rangle \\
&- \left|f\left(\exp\left(i\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&\leq \int_0^{2\pi} \left[\frac{2\pi-\lambda}{2\pi}|f(1)|^2 + \frac{\lambda}{2\pi}|f_c(1)|^2\right] d\langle E_\lambda x, x\rangle \\
&- \left|f\left(\exp\left(i\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&= \frac{|f(1)|^2\left(2\pi - \int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right) + |f_c(1)|^2 \int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle}{2\pi} \\
&- \left|f\left(\exp\left(i\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right)\right)\right|^2
\end{aligned}
\tag{2.13}
$$

for any $x \in H, \|x\| = 1$.

Applying the second inequality from (2.7) for the convex function $\varphi(t) = \left|f\left(e^{it}\right)\right|^2$ on the interval $[0, 2\pi]$ and for the intermediate point $\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle \in [0, 2\pi]$ we can write that

$$
\begin{aligned}
&\frac{|f(1)|^2\left(2\pi - \int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right) + |f_c(1)|^2 \int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle}{2\pi} \\
&- \left|f\left(\exp\left(i\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle\right)\right)\right|^2 \\
&\leq 2\max\left\{\frac{2\pi - \int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle}{2\pi}, \frac{\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle}{2\pi}\right\} \\
&\times \left[\frac{|f(1)|^2 + |f_c(1)|^2}{2} - |f(-1)|^2\right]
\end{aligned}
\tag{2.14}
$$

for any $x \in H, \|x\| = 1$.

Since, as above,

$$
\int_0^{2\pi} \lambda d\langle E_\lambda x, x\rangle = \langle |Log(U)| x, x\rangle,
$$

for any $x \in H, \|x\| = 1$, then we deduce from (2.13) and (2.14) the desired result (2.12). □

3. Examples

Let $U \in B(H)$ be a unitary operator on the Hilbert space H. Then for $n \geq 1$, a natural number, we have

$$(2\pi)^{n-1/2} \langle |Log(U)| x, x\rangle^{1/2} \geq \| [Log(U)]^n x\| \geq |\ln |\langle LogUx, x\rangle| + iArg(\langle LogUx, x\rangle)|^n, \tag{3.1}$$

for any $x \in H, \|x\| = 1$.

It follows from (2.1) applied for the function $f_n : \mathcal{C}(0,1) \to \mathbb{C}$, $f_n(z) = [Log(z)]^n$.

If we apply the same inequality for $f_q : \mathcal{C}(0,1) \to [0,\infty)$, $f_q(z) = |Log(z)|^q$, then we get

$$(2\pi)^{q-1/2} \langle |Log(U)| x, x\rangle^{1/2} \geq \| |Log(U)|^q x\| \geq |\ln |\langle LogUx, x\rangle| + iArg(\langle LogUx, x\rangle)|^q, \tag{3.2}$$

for any $x \in H, \|x\| = 1$ and $q \geq \frac{1}{2}$.

Now, if we use the inequality (2.4) for the function $f_n(z) = [Log(z)]^n$, then we get

$$\begin{aligned}
&\left(2^{2n-1} - 1\right)\pi^{2n-1} \langle [\pi 1_H - |Log(U) - i\pi 1_H|] x, x\rangle \\
&\leq (2\pi)^{2n-1} \langle |Log(U)| x, x\rangle - \| [Log(U)]^n x\|^2 \\
&\leq \left(2^{2n-1} - 1\right)\pi^{2n-1} \langle [\pi 1_H + |Log(U) - i\pi 1_H|] x, x\rangle,
\end{aligned} \tag{3.3}$$

for any $x \in H, \|x\| = 1$, where n is a natural number with $n \geq 1$.

The same inequality applied for $f_q(z) = |Log(z)|^q$ provides

$$\begin{aligned}
&\left(2^{2q-1} - 1\right)\pi^{2q-1} \langle [\pi 1_H - |Log(U) - i\pi 1_H|] x, x\rangle \\
&\leq (2\pi)^{2q-1} \langle |Log(U)| x, x\rangle - \| |Log(U)|^q x\|^2 \\
&\leq \left(2^{2q-1} - 1\right)\pi^{2q-1} \langle [\pi 1_H + |Log(U) - i\pi 1_H|] x, x\rangle,
\end{aligned} \tag{3.4}$$

for any $x \in H, \|x\| = 1$ and $q \geq \frac{1}{2}$.

Finally, if we use the first inequality from (2.12) we also get:

$$\begin{aligned}
0 &\leq \| [Log(U)]^n x\|^2 - |\ln |\langle LogUx, x\rangle| + iArg(\langle LogUx, x\rangle)|^q \\
&\leq \left(2^{2n-1} - 1\right)\pi^{2n-1} \max\{\langle (2\pi 1_H - |Log(U)|) x, x\rangle, \langle |Log(U)| x, x\rangle\} \\
&\leq 2\left(2^{2n-1} - 1\right)\pi^{2n}
\end{aligned} \tag{3.5}$$

for any $x \in H, \|x\| = 1$, where n is a natural number with $n \geq 1$.

If $q \geq \frac{1}{2}$, then we have

$$\begin{aligned}
0 &\leq \| |Log(U)|^q x\|^2 - |\ln |\langle LogUx, x\rangle| + iArg(\langle LogUx, x\rangle)|^{2n} \\
&\leq \left(2^{2q-1} - 1\right)\pi^{2q-1} \max\{\langle (2\pi 1_H - |Log(U)|) x, x\rangle, \langle |Log(U)| x, x\rangle\} \\
&\leq 2\left(2^{2q-1} - 1\right)\pi^{2q}
\end{aligned} \tag{3.6}$$

for any $x \in H, \|x\| = 1$.

If $g : [0, 2\pi] \to [0,\infty)$ is continuous and convex on $[0, 2\pi]$, then the composite function $f : \mathcal{C}(0,1) \to [0,\infty)$ defined by

$$f(z) := [g(|Log(z)|)]^{1/2}$$

is an Arg-square-convex function on $\mathcal{C}(0,1)$.

As examples of such functions we have

$$f_{\alpha}(z) := \exp(\alpha |Log(z)|)$$

which are Arg-square-convex functions on $\mathcal{C}(0,1)$ for any real number $\alpha \neq 0$.

We also notice that the family of functions $f_{m,n} : \mathcal{C}(0,1) \to \mathbb{C}$, $f_{m,n}(z) = z^m [Log(z)]^n$, where $m \neq 0$ is an integer and n is a positive integer, are Arg-square-convex functions.

The reader may apply the above inequalities for these functions as well. However, the details are omitted.

For Jensen's type inequalities for functions of selfadjoint operators see the recent book [**2**]. For related results see, [**3**].

References

[1] S.S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 471-476.

[2] S.S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[3] S.S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type*, Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[4] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

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