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INTEGRAL FORMS FOR SOME INEQUALITIES

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ABSTRACT. In this paper several inequalities will be given using as method the power series method and then some integral forms for them will be given. Also the integral forms of several classical inequalities and of Radon's inequality were presented.

1. INTRODUCTION

It is necessary to recall the inequality of J. Radon which was published in [9].

For every real numbers $p > 0$, $x_k \geq 0$, $a_k > 0$ for $1 \leq k \leq n$, we have the following inequality:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n})^{p+1}}{(\sum_{k=1}^{n} a_k)^p}, \quad p > 0.$$

According to [4], the reverse of previous inequality is true in case $p \in (-1, 0)$, see for example [9]:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \leq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}, \ p \in (-1, 0).$$

In [10], the authors consider two n-tuples $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_1, ..., b_n)$ where $ab = (a_1b_1, a_2b_2, ..., a_nb_n)$ and $a^m = (a_1^m, a_2^m, ..., a_n^m)$, for any real number m. Then $a > 0$ and $b > 0$ if $a_i > 0$ and $b_i > 0$ for every $1 < i < n$. They consider the expression:

$$\Delta_n^{[p]}(a; b) := \sum_{i=1}^{n} \frac{a_i^p}{b_i^{p-1}} - \frac{(\sum_{i=1}^{n} a_i)^p}{(\sum_{i=1}^{n} b_i)^{p-1}}, \tag{1.1}$$

for real number $p > 1$ and for n-tuples $a \geq 0$ and $b > 0$. Radon proved in [9] that

$$\Delta_n^{[p]}(a; b) \geq 0.$$

We study the sign of this expression for real number $p \in (0, 1)$ and for n-tuples $a \geq 0$ and $b > 0$.

Then the well-known Radon's inequality can be written as:

$$\Delta_n^{[p]}(a; b) \leq 0$$

for real number $p \in (0, 1)$, $n \geq 2$ and for n-tuples $a \geq 0$ and $b > 0$.

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It is necessary to recall the following two results which were established by [6] and [1] when the number $p > 0$.

Theorem 1. *([6]) For $a_k,\ x_k > 0,\ p \geq 1,\ k \in \{1,2,...,n\},\ n \in N$ and $n \geq 2$ the inequality takes place,*

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p} + \max_{1\leq i<j\leq n}\left(\frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i+x_j)^{p+1}}{(a_i+a_j)^p}\right)$$

with equality if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = ... = \frac{x_n}{a_n}$.

Theorem 2. *([1]) If $n \in N,\ x_k \geq 0, y_k > 0,\ k \in \{1,2,...,n\}$ and $m \geq p \geq 0$,*

$$\frac{x_1^{m+1}}{y_1^p} + \frac{x_2^{m+1}}{y_2^p} + ... + \frac{x_n^{m+1}}{y_n^p} \geq n^{p-m}\frac{(x_1+x_2+...+x_n)^{m+1}}{(y_1+y_2+...+y_n)^p}$$

with equality if and only if $x_1 = x_2 = ... = x_n$ and $y_1 = y_2 = ... = y_n$.

In the case when $p \geq 0$ the integral form of the inequality from Theorem 2.4, see [1] was given by Theorem 2.5.

Theorem 3. *([1])*
Ifa, $b \in R$ with $a < b,\ m \geq p \geq 0,\ f,g : f,g : [a,b] \to [0,\infty)$ are integrable functions on $[a,b]$ with $g(x) > 0,\ (\forall)\ x \in [a,b]$ then we have:

$$\int_a^b \frac{(f(x))^{m+1}}{(g(x))^p}dx \geq (b-a)^{p-m}\frac{(\int_a^b f(x)dx)^{m+1}}{(\int_a^b g(x)dx)^p}.$$

2. The results

When $p \in (-1,0)$ the inequality of Radon can be also written as, below:

Theorem 4. *For $a_k,\ x_k > 0,\ k \in \{1,2,...,n\}$ and $n \geq 2$ the following inequality takes place:*

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \leq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p} + \min_{1\leq i<j\leq n}\left(\frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i+x_j)^{p+1}}{(a_i+a_j)^p}\right).$$

Proof. As in the proof of a theorem from [6] because $d_1 = 0$, remains significant the inequality $d_n \leq d_2,\ (\forall)n \in N,\ n \geq 2$, where

$$d_n = \frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + ... + \frac{x_n^{p+1}}{a_n^p} - \frac{(x_1+x_2+...+x_n)^{p+1}}{(a_1+a_2+...+a_n)^p}.$$

Using the expression of d_2 we have:

$$d_n \leq \frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} - \frac{(x_1+x_2)^{p+1}}{(a_1+a_2)^p}$$

and by symmetry relatively to a_i and x_j when $i,j \in \{1,2,...,n\}$ we obtain

$$d_n \leq \frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i+x_j)^{p+1}}{(a_i+a_j)^p},\ (\forall)n \in N,\ n \geq 2,\ (\forall)i,j \in \{1,2,...,n\}.$$

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Remark 1. *If $p \in (-1,0)$ and we take $x_k = 1$ and a_k will be replaced by x_k then for every natural number $n \geq 2$ and $x_1, x_2, ..., x_n > 0$ we have:*

$$\frac{1}{x_1^p} + \frac{1}{x_2^p} + ... + \frac{1}{x_n^p} \leq \frac{n^{p+1}}{(x_1 + x_2 + ... + x_n)^p} + \min_{1 \leq i < j \leq n} \left(\frac{1}{x_i^p} + \frac{1}{x_j^p} - \frac{2^{p+1}}{(x_i + x_j)^p} \right).$$

We give the reverse of an inequality presented in Theorem 2.4, from [1] when $p \geq 0$.

Theorem 5. *If $n \in N$, $x_k \geq 0, y_k > 0$, $k \in \{1, 2, ..., n\}$ and $p \in (-1,0)$, $m \in (-1,0)$ and $m \leq p$ then we have,*

$$\frac{x_1^{m+1}}{y_1^p} + \frac{x_2^{m+1}}{y_2^p} + ... + \frac{x_n^{m+1}}{y_n^p} \leq n^{p-m} \frac{(x_1 + x_2 + ... + x_n)^{m+1}}{(y_1 + y_2 + ... + y_n)^p}.$$

Proof. As in [1], Theorem 2.4, we denote $X_n = x_1 + x_2 + ... + x_n$ and $Y_n = y_1 + y_2 + ... + y_n$ and write

$$\sum_{k=1}^{n} \frac{x_k^{m+1}}{y_k^p} = Y_n \sum_{k=1}^{n} \frac{y_k}{Y_n} t_k^{p+1},$$

where $t = \frac{x_k^{\frac{m+1}{p+1}}}{y_k}$ and $k \in \{1, 2, ..., n\}$. Then we consider the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(t) = t^{p+1}$, $t \in (0, \infty)$ which is concave on $(0, \infty)$ when $p \in (-1, 0)$ and therefore

$$\sum_{k=1}^{n} \frac{y_k}{Y_n} t_k^{p+1} \leq \left(\sum_{k=1}^{n} \frac{y_k}{Y_n} t_k \right)^{p+1}$$

or

$$\sum_{k=1}^{n} \frac{y_k}{Y_n} t_k^{p+1} \leq \frac{1}{Y_n^{p+1}} \left(\sum_{k=1}^{n} x_k^{\frac{m+1}{p+1}} \right)^{p+1}.$$

Now if we consider the function, $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = x^{\frac{m+1}{p+1}}$, $x \in (0, \infty)$ this is concave when $m < p$ and then

$$\sum_{k=1}^{n} x_k^{\frac{m+1}{p+1}} \leq n^{\frac{p-m}{p+1}} X_n^{\frac{m+1}{p+1}}.$$

From here we obtain the inequality of the theorem.

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As a consequence of Theorem 5 we obtain the integral form of inequality from previous theorem and this is also the reverse of inequality from Theorem 2.5, see [1].

Theorem 6. *Let a, $b \in R$ with $a < b$, $p \in (-1,0)$, $m \in (-1,0)$ and $m \leq p$. If $f, g : [a,b] \to \mathbb{R}_+$ are two integrable functions on $[a,b]$ with $g(x) > 0$, $(\forall)$ $x \in [a,b]$ then we have:*

$$\int_a^b \frac{(f(x))^{m+1}}{(g(x))^p} dx \leq (b-a)^{p-m} \frac{(\int_a^b f(x)dx)^{m+1}}{(\int_a^b g(x)dx)^p}.$$

Proof. We use the same technique as in [1]. Let $n \in N$ and $x_k = a + k\frac{b-a}{n},\ k \in \{0, 1, ..., n\}$. We will take in Theorem 5 instead of x_k and y_k, $f(x_k)$ and $g(x_k)$ and then the inequality becomes:

$$\sum_{k=1}^{n} \frac{(f(x_k))^{m+1}}{(g(x_k))^p} \leq n^{p-m} \frac{(\sum_{k=1}^{n} f(x_k))^{m+1}}{(\sum_{k=1}^{n} g(x_k))^p}$$

where $\Delta_n = (x_1, x_2, ..., x_n)$ is a division of the interval $[a, b]$.

Multiplying by $\frac{b-a}{n}$ last inequality we obtain:

$$\frac{b-a}{n} \sum_{k=1}^{n} \frac{(f(x_k))^{m+1}}{(g(x_k))^p} \leq (b-a)^{p-m} \frac{(\sum_{k=1}^{n} \frac{b-a}{n} f(x_k))^{m+1}}{(\sum_{k=1}^{n} \frac{b-a}{n} g(x_k))^p}.$$

It results that

$$\sigma\left(\frac{f^{m+1}}{g^p}, \Delta_n, x_k\right) \leq (b-a)^{p-m} \frac{(\sigma(f, \Delta_n, x_k))^{m+1}}{(\sigma(g, \Delta_n, x_k))^p},$$

where $\sigma\left(\frac{f^{m+1}}{g^p},\ \Delta_n, x_k\right)$ is the corresponding Riemann sum of function $\frac{f^{m+1}}{g^p}$, $\Delta_n = (x_1, x_2, ..., x_n)$ division, and the intermediate points x_k.

When n tends to infinity, in previous inequality the limits become:

$$\int_a^b \frac{(f(x))^{m+1}}{(g(x))^p} dx \leq (b-a)^{p-m} \frac{(\int_a^b f(x)dx)^{m+1}}{(\int_a^b g(x)dx)^p}.$$

∎

In the following we will use the following inequality, see [7]:

Theorem 7. *([7]) If $\{x_1, x_2, ..., x_p\},\ x_i \in \mathbb{R}^+$ and p are real, positive numbers and $m \in \mathbb{N}$ then we have:*

$$\sum_{i=1}^{p} x_i^m - (p-1)a^m \leq \left(\sum_{i=1}^{p} x_i - (p-1)a\right)^m.$$

In fact using the method of power series, see [3] in previous inequality we obtain:

Theorem 8. *If $\{x_1, x_2, ..., x_p\},\ x_i \in \mathbb{R}^+$ are p real, positive numbers with $0 < x_i < 1,\ i \in \{1, ..., p\}$ and $\sum_{i=1}^{p} x_i < (p-1)a + 1$ then we have:*

$$\sum_{i=1}^{p} \frac{1}{1-x_i} + \frac{1-p}{1-a} \leq \frac{1}{pa - \sum_{i=1}^{p} x_i}.$$

Proof. Using the inequality,

$$\sum_{i=1}^{p} x_i^m - (p-1)a^m \leq \left(\sum_{i=1}^{p} x_i - (p-1)a\right)^m,$$

and summing then like below,

$$\sum_{k=0}^{m} \left(\sum_{i=1}^{p} x_i^k - (p-1)a^k\right) \leq \sum_{k=0}^{m} \left(\sum_{i=1}^{p} x_i - (p-1)a\right)^k$$

we obtain

$$\sum_{i=1}^{p}\sum_{k=0}^{m} x_i^k - (p-1)\sum_{k=0}^{m} a^k \leq \sum_{k=0}^{m}\left(\sum_{i=1}^{p} x_i - (p-1)a\right)^k$$

and then when m tends to infinity we have:

$$\sum_{i=1}^{p}\frac{1}{1-x_i} - \frac{p-1}{1-a} \leq \frac{1}{pa - \sum_{i=1}^{p} x_i}.$$

∎

Using the same techniques as in previous theorem, see [1] we can give an integral form of the inequality from Theorem 8.

Theorem 9. *Let $a_1,\ b_1 \in R$ with $a_1 < b_1$, $f : [a_1, b_1] \to \mathbb{R}_+$ is an integrable function on $[a_1, b_1]$(or continuous function on $[a_1, b_1]$). If $0 < f(x) < 1$ and $\sum_{i=1}^{p} f(x_i) < (p-1)a+1$, $(\forall)\ i \in \{1, ..., p\}$ and $(\forall)\ p \in N$ where $a = \min_{x\in[a_1,b_1]} f(x)$ or ($\int_{a_1}^{b_1} f(x)dx < (b_1 - a_1)\min_{x\in[a_1,b_1]} f(x)$) then the following inequality holds:*

$$\int_{a_1}^{b_1}\frac{1}{1-f(x)}dx - \frac{b_1 - a_1}{1-\min_{x\in[a_1,b_1]} f(x)} \leq \frac{1}{(b_1 - a_1)\min_{x\in[a_1,b_1]} f(x) - \int_{a_1}^{b_1} f(x)dx}.$$

Proof. Using inequality

$$\sum_{i=1}^{p}\frac{1}{1-x_i} + \frac{1-p}{1-a} \leq \frac{1}{pa - \sum_{i=1}^{p} x_i}.$$

from Theorem 8, and multiplying it by $\frac{b_1-a_1}{p}$ we obtain,

$$\frac{b_1 - a_1}{p}\sum_{i=1}^{p}\frac{1}{1-x_i} + \frac{b_1 - a_1}{p}\frac{1-p}{1-a} \leq \frac{b_1 - a_1}{p}\frac{1}{pa - \sum_{i=1}^{p} x_i}.$$

Let $p \in N$ and $x_k = a_1 + k\frac{b_1-a_1}{p}$, $k \in \{0, 1, ..., p\}$. We will take in Theorem 8 instead of x_k, $f(x_k)$ and then the inequality becomes:

$$\frac{b_1 - a_1}{p}\sum_{i=1}^{p}\frac{1}{1-f(x_i)} + \frac{b_1 - a_1}{p}\frac{1-p}{1-a} \leq \frac{b_1 - a_1}{p}\frac{1}{pa - \sum_{i=1}^{p} f(x_i)}.$$

where $\Delta_p = (x_1, x_2, ..., x_p)$ is a division of the interval $[a_1, b_1]$.

It results that

$$\sigma\left(\frac{1}{1-f}, \Delta_p, x_k\right) + \frac{1-p}{p}\frac{b_1 - a_1}{1-a} \leq (b_1 - a_1)\frac{1}{p^2\left(a - \frac{1}{b_1 - a_1}\sigma(f, \Delta_p, x_k)\right)},$$

where $\sigma\left(\frac{1}{1-f},\ \Delta_p, x_k\right)$ is the corresponding Riemann sum of function $\frac{1}{1-f}$, $\Delta_p = (x_1, x_2, ..., x_p)$ division, and the intermediate points x_k.

When p tends to infinity, we consider the following

$$\sigma\left(\frac{1}{1-f}, \Delta_p, x_k\right) + \frac{1-p}{p}\frac{b_1 - a_1}{1-a} \leq (b_1 - a_1)\frac{1}{(b_1 - a_1)^2\left(a - \frac{1}{b_1 - a_1}\sigma(f, \Delta_p, x_k)\right)},$$

and then we obtain the inequality.

∎

Following the method used by C. Mortici, see [5] and [3] for proving and discovering a class of inequalities using infinite series, we give the following result:

Theorem 10. *If* $p \in \mathbb{N},\ y_i > 0,\ i \in \{1, 2, ..., p\},\ 0 \le r \le 1$ *and* $0 \le x_i < 1,\ i \in \{1, 2, ..., p\}$ *then the following inequality takes place:*

$$\sum_{i=1}^{p} \frac{x_i}{y_i^r(1-x_i)} \ge \frac{p^{r+1}}{(\sum_{i=1}^{p} y_i)^r} \frac{\sum_{i=1}^{p} x_i}{p - \sum_{i=1}^{p} x_i}$$

Proof. Using inequality from Theorem 2 with r instead of p and p instead of n we have

$$\frac{x_1^{m+1}}{y_1^r} + \frac{x_2^{m+1}}{y_2^r} + ... + \frac{x_p^{m+1}}{y_p^r} \ge p^{r-m} \frac{\left(\sum_{i=1}^{p} x_i\right)^{m+1}}{\left(\sum_{i=1}^{p} y_i\right)^r}$$

and summing when $k \in \{1, 2, ..., m\}$ we we obtain,

$$\sum_{k=0}^{m} \sum_{i=1}^{p} \frac{x_i^{k+1}}{y_i^r} \ge \sum_{k=0}^{m} p^{r-k} \frac{\left(\sum_{i=1}^{p} x_i\right)^{k+1}}{\left(\sum_{i=1}^{p} y_i\right)^r}$$

or

$$\sum_{i=1}^{p} \frac{1}{y_i^r} \sum_{k=0}^{m} x_i^{k+1} \ge \frac{p^{r+1}}{\left(\sum_{i=1}^{p} y_i\right)^r} \sum_{k=0}^{m} \frac{\left(\sum_{i=1}^{p} x_i\right)^{k+1}}{p^{k+1}}.$$

Taking now into account the hypothesis, $x_i \in [0, 1),\ i \in \{1, 2, ..., p\}$ which means that

$$0 \le \frac{\sum_{i=1}^{p} x_i}{p} < 1$$

, when m tends to infinity previous inequality becomes

$$\sum_{i=1}^{p} \frac{1}{y_i^r} \left(\frac{1}{1-x_i} - 1 \right) \ge p^{r+1} \frac{1}{\left(\sum_{i=1}^{p} y_i\right)^r} \left(\frac{1}{1 - \frac{\sum_{i=1}^{p} x_i}{p}} - 1 \right).$$

∎

Now we can think which is an integral form of previous inequality.

Consequence 1. *Let* $a,\ b \in R$ *with* $a < b,$. *If* $f, g : [a, b] \to \mathbb{R}_+$ *are two integrable functions,* $g(x) > 0,\ (\forall)\ x \in [a, b]$ *and* $f(x) \in (0, 1),\ (\forall) x \in [a, b]$ *then the following inequality holds:*

$$\int_a^b \frac{f(x)}{g^r(x)(1 - f(x))} \ge \frac{(b-a)^r}{\left(\int_a^b g(x)dx\right)^r} \frac{\int_a^b f(x)dx}{1 - \frac{1}{b-a}\int_a^b f(x)dx},$$

where $r \in [0, 1)$.

Proof. Let $p \in N$ and $x_k = a + k\frac{b-a}{p},\ k \in \{0, 1, ..., p\}$. We will take in Theorem 5 instead of x_k and y_k, $f(x_k)$ and $g(x_k)$ and then multiplying by $\frac{b-a}{p}$ the inequality we obtain,

$$\sum_{i=1}^{p} \frac{b-a}{p} \frac{f(x_i)}{g(x_i)^r(1 - f(x_i))} \ge \frac{p^r}{\left(\sum_{i=1}^{p} g(x_i)\right)^r} \frac{\sum_{i=1}^{p} \frac{b-a}{p} f(x_i)}{1 - \frac{1}{b-a}\sum_{i=1}^{p} \frac{b-a}{p} f(x_i)}$$

where $\Delta_p = (x_1, x_2, ..., x_p)$ is a division of the interval $[a, b]$. Then when p tends to infinity, taking into account that $p^r \geq (b-a)^r$ we have

$$\sigma\left(\frac{f}{g^r(1-f)}, \Delta_p, x_i\right) \geq \frac{(b-a)^r}{(\sigma(g, \Delta_p, x_i))^r} \frac{\sigma(f, \Delta_p, x_i)}{(1 - \frac{1}{b-a}\sigma(f, \Delta_p, x_i))},$$

where $\sigma\left(\frac{f}{g^r(1-f)},\ \Delta_p, x_i\right)$ is the corresponding Riemann sum of function $\frac{f}{g^r(1-f)}$, $\Delta_p = (x_1, x_2, ..., x_p)$ division, and the intermediate points x_i.

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