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SOME WEIGHTED INTEGRAL INEQUALITIES FOR DIFFERENTIABLE PREINVEX AND PREQUASIINVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we present weighted integral inequalities of Hermite-Hadamard type for differentiable preinvex and prequasiinvex functions. Our results, on one hand give a weighted generalization of recent results for preinvex functions and on the other hand extend several results connected with the Hermite-Hadamard type integral inequalities. Applications of the obtained results are provided as well.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities in (1.1) hold in reversed direction if f is concave. The inequalities (1.1) are famous in mathematical literature due to its rich geometrical significance and applications and are known as the Hermite-Hadamard inequalities (see [26]).

For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [2, 7, 8, 9], [11]-[15], [24, 25], [28]-[33].

In [7], Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in (1.1):

Theorem 1. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f' \in L(a, b)$. If $\left|f'\right|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 2. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$,and $f' \in L(a, b)$. If $\left|f'\right|^{\frac{p}{p-1}}$ is convex function on $[a, b]$, then the*

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following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}\right], \tag{1.3}$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

In [24], Pearce and J. Pečarić gave an improvement and simplification of the constant in Theorem 2 and consolidated this results with Theorem 1. The following is the main result from [24]:

Theorem 3. [24] *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, *where* a, $b \in I$ *with* $a<b$, *and* $f' \in L(a,b)$. *If* $\left|f'\right|^q$ *is convex function on* $[a,b]$, *for some* $q \geq 1$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

If $\left|f'\right|^q$ *is concave on* $[a,b]$, *for some* $q \geq 1$. *Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{1.5}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f:[a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x; y \in [a; b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex, (see [11]).

Recently, Ion [11] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follows:

Theorem 4. [11] *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, *where* a, $b \in I$ *with* $a<b$. *If* $\left|f'\right|$ *is quasi-convex function on* $[a,b]$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\} \tag{1.6}$$

Theorem 5. [11] *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, *where* a, $b \in I$ *with* $a<b$. *If* $\left|f'\right|^p$ *is quasi-convex function on* $[a,b]$, *for some* $p>1$, *then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\frac{1}{b-a}\int_a^b f(x)g(x)dx\right| \\ &\qquad \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}, \end{aligned} \tag{1.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In [2], Alomari, Darus and Kirmaci established Hermite-Hadamard-type inequalities for quasi-convex functions which give refinements of those given above in Theorem 4 and Theorem 5.

Theorem 6. [2] *Let* $f : I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f^{'} \in L[a,b]$, *where* a, $b \in I$ *with* $a<b$. *If the mapping* $\left|f^{'}\right|$ *is quasi-convex function on* $[a,b]$, *then the following inequality holds:*

$$\begin{aligned}(1.8)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{8}\left[\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right\}+\sup\left\{\left|f^{'}(b)\right|,\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right\}\right].\end{aligned}$$

Theorem 7. [2] *Let* $f : I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f^{'} \in L[a,b]$, *where* a, $b \in I$ *with* $a<b$. *If* $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is quasi-convex function on* $[a,b]$, *for* $p>1$, *then the following inequality holds:*

$$\begin{aligned}(1.9)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\qquad \left.+\left(\sup\left\{\left|f^{'}(b)\right|^{\frac{p}{p-1}},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned}$$

Theorem 8. [2] *Let* $f : I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f^{'} \in L[a,b]$, *where* a, $b \in I$ *with* $a<b$. *If* $\left|f^{'}\right|^{q}$ *is quasi-convex function on* $[a,b]$, *for* $q\geq 1$, *then the following inequality holds:*

$$\begin{aligned}(1.10)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{b-a}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \\ &\qquad \left.+\left(\sup\left\{\left|f^{'}(b)\right|^{q},\left|f^{'}\left(\frac{a+b}{2}\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right].\end{aligned}$$

In [9], D. -Y. Hwang established the following results for convex and quasi-convex functions those results provide a weighted generalization of the results given in Theorem 1, Theorem 3, Theorem 6 and Theorem 8.

Theorem 9. [9] *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, *where* a, $b \in I$ *with* $a<b$ *and let* $g : [a,b] \to [0,\infty)$ *be continuous positive mapping and*

symmetric to $\frac{a+b}{2}$*. If* $\left|f^{'}\right|$ *is convex function on* $[a,b]$*, then the following inequality holds:*

$$\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right| \leq \frac{b-a}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1\int_{L(a,b,t)}^{U(a,b,t)} g(x)dxdt, \tag{1.11}$$

where $U(a,b,t)=\frac{1-t}{2}a+\frac{1+t}{2}b$ *and* $L(a,b,t)=\frac{1+t}{2}a+\frac{1-t}{2}b$.

Theorem 10. [9] *Suppose the assumptions of Theorem 9 are satisfied and* $q\geq 1$*. If* $\left|f^{'}\right|^q$ *is convex function on* $[a,b]$*, then the following inequality holds:*

$$\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right| \leq \frac{b-a}{2}\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1\int_{L(a,b,t)}^{U(a,b,t)} g(x)dxdt, \tag{1.12}$$

where $U(a,b,t)$ *and* $L(a,b,t)$ *are as defined in Theorem 9.*

Theorem 11. [9] *Suppose the assumptions of Theorem 9 are satisfied. If* $\left|f^{'}\right|$ *is quasi-convex function on* $[a,b]$*, then the following inequality holds:*

$$\begin{aligned}&\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right| \\ &\leq \frac{b-a}{4}\left[\max\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f^{'}(b)\right|,\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right\}\right] \\ &\qquad\times\int_0^1\int_{L(a,b,t)}^{U(a,b,t)} g(x)dxdt,\end{aligned} \tag{1.13}$$

where $U(a,b,t)$ *and* $L(a,b,t)$ *are as defined in Theorem 9.*

Theorem 12. [9] *Suppose the assumptions of Theorem 9 are satisfied and* $q\geq 1$*. If* $\left|f^{'}\right|^q$ *is quasi-convex function on* $[a,b]$*, then the following inequality holds:*

$$\begin{aligned}&\left|\left[\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right| \\ &\leq \frac{b-a}{4}\left[\left(\max\left\{\left|f^{'}(a)\right|^q,\left|f^{'}\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f^{'}(b)\right|^q,\left|f^{'}\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]\int_0^1\int_{L(a,b,t)}^{U(a,b,t)} g(x)dxdt,\end{aligned} \tag{1.14}$$

where $U(a,b,t)$ *and* $L(a,b,t)$ *are as defined in Theorem 9.*

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [10], Ben-Israel and Mond [5], Pini [23], M. A. Noor [20, 21], Yang and Li [35] and Weir [34]. Mond [5], Weir [33] and Noor [19, 20], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [10], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4], gave the concept of preinvex function which is special case of invexity. Pini [23], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning preinvexity and prequasiinvexity.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x + t\eta(y,x) \in K, \forall x, y \in K, t \in [0,1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1. [34] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v,u)) \leq (1-t) f(u) + tf(v), \forall u, v \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse is not true see for instance [33].

Definition 2. [22] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v,u)) \leq \max\{f(u), f(v)\}, \forall u, v \in K, t \in [0,1].$$

Also Every quasi-convex function is a prequasiinvex with respect to the map $\eta(v,u)$ but the converse does not holds, see for example [36].

In the recent paper, Noor [18] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 13. [18]*Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $a < a + \eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.15}$$

Barani, Ghazanfari and Dragomir in [4], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 14. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then,* *for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right). \tag{1.16}$$

Theorem 15. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f^{'}\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f^{'}(a)\right|^{\frac{p}{p-1}}+\left|f^{'}(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.17}$$

In [3], Barani, Ghazanfari and Dragomir gave similar results for prequasiinvex functions as follows:

Theorem 16. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f^{'}\right|$ is prequasiinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}(b)\right|\right\}. \tag{1.18}$$

Theorem 17. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f^{'}\right|^{\frac{p}{p-1}}$ is prequasiinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \tag{1.19}$$

M. A. Latif [16], proved the following results which give a refinement of the results given in Theorem 14-Theorem 17.

Theorem 18. [16] *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L([a, a+\eta(b,a)])$. If $\left|f^{'}\right|$ is prequasiinvex on K, then* *we have the following inequality:*

$$\begin{aligned}(1.20)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \\ &\leq \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\right. \\ &\quad \left.+\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right].\end{aligned}$$

Theorem 19. [16] *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta : K\times K\to\mathbb{R}$ and $a,\, b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $\left|f'\right|^p$ is prequasiinvex on K from some $p>1$, then we have the following inequality:*

$$\begin{aligned}(1.21)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \\ &\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\quad \left.+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},\left|f'(a+\eta(b,a))\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned}$$

Theorem 20. [16] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta : K\times K\to\mathbb{R}$ and $a,\, b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $\left|f'\right|^q$ for $q\geq 1$, is prequasiinvex on K, then for every $a,b\in K$ with $\eta(b,a)>0$ we have the following inequality:*

$$\begin{aligned}(1.22)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \\ &\leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f'(a)\right|^{q},\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}\right].\end{aligned}$$

For several new results on inequalities for preinvex and prequasiinvex functions we refer the interested reader to [4, 22, 27] and the references therein.

In the present paper we give new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are preinvex and prequasiinvex. Our results extend those results presented in a very recent results from [2], [7], [9], [11] and [25] and generalize those results from [3], [4] and [17].

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $h : [a, a+\eta(b,a)] \to [0,\infty)$ be a differentiable mapping, then the following equality holds:*

$$\frac{1}{2}\left[\left(h(a+\eta(b,a)) - 2h(a)\right) f(a) + h(a+\eta(b,a)) f(a+\eta(b,a))\right] - \int_a^{a+\eta(b,a)} f(x) h'(x)\,dx = \frac{\eta(b,a)}{4}$$
$$\times \left\{\int_0^1 \left[2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right] f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) dt\right.$$
$$\left. + \int_0^1 \left[2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right] f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) dt\right\}. \tag{2.1}$$

Proof. It suffices to note that

$$\begin{aligned}
I_1 &= \int_0^1 \left[2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right] f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\
&= -2\left.\frac{\left[2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right] f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)}{\eta(b,a)}\right|_0^1 \\
&\quad - 2\int_0^1 h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\
&= \frac{-2\left[2h(a) - h(a+\eta(b,a))\right] f(a)}{\eta(b,a)} \\
&\quad + \frac{2\left[2h\left(a+\frac{1}{2}\eta(b,a)\right) - h(a+\eta(b,a))\right] f\left(a+\frac{1}{2}\eta(b,a)\right)}{\eta(b,a)} \\
&\quad - 2\int_0^1 h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) dt
\end{aligned}$$

Setting $x = a + \left(\frac{1-t}{2}\right)\eta(b,a)$ and $dx = -\frac{\eta(b,a)}{2}dt$, which gives

$$I_1 = \frac{2\left[h(a+\eta(b,a)) - 2h(a)\right] f(a)}{\eta(b,a)} - \frac{4}{\eta(b,a)}\int_a^{a+\frac{1}{2}\eta(b,a)} h'(x) f(x)\,dt + \frac{2\left[2h\left(a+\frac{1}{2}\eta(b,a)\right) - h(a+\eta(b,a))\right] f\left(a+\frac{1}{2}\eta(b,a)\right)}{\eta(b,a)} \tag{2.2}$$

Similarly, we also have

$$\begin{aligned}
I_2 &= \int_0^1 \left[2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right] f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) dt \\
&= \frac{2h(a+\eta(b,a)) f(a+\eta(b,a))}{\eta(b,a)} - \frac{4}{\eta(b,a)}\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} h'(x) f(x)\,dt \\
&\quad - \frac{2\left[2h\left(a+\frac{1}{2}\eta(b,a)\right) - h(a+\eta(b,a))\right] f\left(a+\frac{1}{2}\eta(b,a)\right)}{\eta(b,a)}
\end{aligned} \tag{2.3}$$

Thus from (2.2) and (2.3), we have

$$\begin{aligned}&\frac{\eta(b,a)}{4}[I_1+I_2]\\&=\frac{1}{2}[(h(a+\eta(b,a))-2h(a))f(a)+h(a+\eta(b,a))f(a+\eta(b,a))]\\&-\int_a^{a+\eta(b,a)}f(x)h^{'}(x)\,dx.\end{aligned}$$

which is the required result. □

Remark 1. *If we take $\eta(b,a)=b-a$, then Lemma 1 reduces to Lemma 2.1 from [9].*

Now using Lemma 1, we shall propose some new upper bounds for the difference between the rightmost and middle terms of weighted version of the Hadamard's inequality (1.15) using preinvex and prequasiinvex mappings. Our results provide a weighted generalization of those results given in [3, 4] and [16].

In what follows we use the notations $L^{'}(a,b,t)=a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $U^{'}(a,b,t)=a+\left(\frac{1+t}{2}\right)\eta(b,a)$.

Theorem 21. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f^{'}\right|$ is preinvex on K, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|\\&\qquad\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dxdt.\end{aligned}\tag{2.4}$$

Proof. Let $h(t)=\int_a^t w(t)\,dt$ for all $t\in[a,a+\eta(b,a)]$ in Lemma 1, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(t)\,dt-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|\\&\leq\frac{\eta(b,a)}{4}\left\{\int_0^1\left|2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|\right.\\&\qquad\times\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt\\&\left.+\int_0^1\left|2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right\}.\end{aligned}\tag{2.5}$$

Since $w(x)$ is symmetric to $a+\frac{1}{2}\eta(b,a)$, we have

$$\left|2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|=\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\tag{2.6}$$

and

$$\left|2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|=\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\tag{2.7}$$

for all $t \in [0,1]$. Using (2.6) and (2.7) in (2.5), we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(t)\,dt-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{4}\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left[\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|\right. \\
&\qquad\left.+\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|\right]dt
\end{aligned} \tag{2.8}$$

Since $\left|f'\right|$ is preinvex on K, hence for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\begin{aligned}
&\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|+\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right| \\
&\leq\left(\frac{1+t}{2}\right)\left|f'(a)\right|+\left(\frac{1-t}{2}\right)\left|f'(b)\right|+\left(\frac{1-t}{2}\right)\left|f'(a)\right|+\left(\frac{1+t}{2}\right)\left|f'(b)\right| \\
&=\left|f'(a)\right|+\left|f'(b)\right|
\end{aligned} \tag{2.9}$$

Using (2.9) in (2.8), we get the required inequality. This completes the proof of the theorem. □

Remark 2. *In Theorem 21, if we take $w(x) = \frac{1}{\eta(b,a)}$ for all $x \in [a, a+\eta(b,a)]$, then (2.4) becomes the inequality (1.16).*

Remark 3. *If $\eta(b,a) = b-a$ in Theorem 21, then (2.4) reduces to the inequality (1.11) from* [9].

Theorem 22. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K and $w : [a, a+\eta(b,a)] \to [0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is preinvex on K for $q > 1$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \\
&\leq\frac{\eta(b,a)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}},
\end{aligned} \tag{2.10}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Continuing from inequality (2.8) in the proof of Theorem 21 and using the well known Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(t)\,dt-\int_a^{a+\eta(b,a)}f(x)\,w(x)\,dx\right| \tag{2.11}$$
$$\leq\frac{\eta(b,a)}{4}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

By the power-mean inequality $t^r+s^r<2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$, and by the the preinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality

$$\left(\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \tag{2.12}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt+\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1\left\{\left(\frac{1+t}{2}\right)\left|f'(a)\right|^q+\left(\frac{1-t}{2}\right)\left|f'(b)\right|^q\right.\right.$$
$$\left.\left.+\left(\frac{1-t}{2}\right)\left|f'(a)\right|^q+\left(\frac{1+t}{2}\right)\left|f'(b)\right|^q\right\}dt\right]^{\frac{1}{q}}$$
$$=2^{1-\frac{1}{q}}\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

Using the last inequality (2.12) in (2.11), we get the desired inequality. This completes the proof of the theorem as well. □

Remark 4. *In Theorem 22 if we take $w(x)=\frac{1}{\eta(b,a)}$ for all $x\in[a,a+\eta(b,a)]$ with $\eta(b,a)>0$, then (2.10) reduces to the inequality (1.17).*

Remark 5. *If we take $\eta(b,a)=b-a$ in Theorem 22, then (2.10) reduces to the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b f(x)\,w(x)\,dx\right| \tag{2.13}$$
$$\leq\frac{b-a}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L(a,b,t)}^{U(a,b,t)}w(x)\,dx\right]^p dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $L(a,b,t)=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b$, $U(a,b,t)=\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b$ and $h(t)=\int_a^t w(t)\,dt$, $t\in[a,b]$.

A similar result may be stated as follows:

Theorem 23. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K and $w : [a, a+\eta(b,a)] \to [0,\infty)$ be continuous and symmetric to $a + \frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is preinvex on K for $q \geq 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$ we have the following inequality:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \leq \frac{\eta(b,a)}{2}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}} \int_0^1 \int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt. \tag{2.14}$$

Proof. Continuing from inequality (2.8) in the proof of Theorem 21 and using the well known Hölder's integral inequality, we have

$$\begin{aligned} &\left|\frac{f(a) + f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(t)\,dt - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \\ &\leq \frac{\eta(b,a)}{4}\left[\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)dt\right]^{1-\frac{1}{q}} \\ &\times\left[\left\{\left(\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)\right\}^{\frac{1}{q}}\right. \\ &\left.+\left\{\left(\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a + \left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)\right\}^{\frac{1}{q}}\right]. \end{aligned} \tag{2.15}$$

By the power-mean inequality $t^r + s^r < 2^{1-r}(t+s)^r$ for $t > 0$, $s > 0$ and $r < 1$, and by the the preinvexity of $\left|f'\right|^q$ on K for $q > 1$, we have for every $a, b \in K$ with $\eta(b,a) > 0$ the following inequality

$$\begin{aligned} &\left\{\left(\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)\right\}^{\frac{1}{q}} \\ &+\left\{\left(\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a + \left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)\right\}^{\frac{1}{q}} \\ &\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)dt\right]^{\frac{1}{q}}\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}}. \end{aligned} \tag{2.16}$$

Utilizing inequality (2.16) in (2.15), we get the inequality (2.14). This completes the proof of the theorem. $\square$

Corollary 1. *Suppose all the assumptions of Theorem 23 are satisfied and if* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$, *then we have the following inequality:*
(2.17)

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 6. *If we take* $\eta(b,a) = b-a$ *in Theorem 23, then the inequality reduces to the inequality (1.12) from* [9].

Remark 7. *For* $q=1$, *(2.17) reduces to the inequality proved in Theorem 14. If* $q = \frac{p}{p-1}$ $(p>1)$, *we have* $4^p > p+1$ *for* $p>1$ *and accordingly*

$$\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}.$$

This reveals that the inequality (2.17) is better than the one given by (1.17) in Theorem 15 from [4].

Now we give our results for prequasiinvex functions.

Theorem 24. *Let* $K \subseteq [0,\infty)$ *be an open invex subset with respect to* $\eta : K\times K \to \mathbb{R}$ *and* $a, b \in K$ *with* $a < a+\eta(b,a)$. *Suppose* $f : K \to \mathbb{R}$ *is a differentiable mapping on* K *and* $w : [a, a+\eta(b,a)] \to [0,\infty)$ *be continuous and symmetric to* $a+\frac{1}{2}\eta(b,a)$. *If* $\left|f'\right|$ *is prequasiinvex on* K, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$ *we have the following inequality:*

$$\begin{aligned}(2.18)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)w(x)\,dx\right| \\ &\leq \frac{\eta(b,a)}{4}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\right. \\ &\left.+\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\right]\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt.\end{aligned}$$

Proof. We continue the inequality (2.8) in the proof of Theorem 21. Since $\left|f'\right|$ is prequasiinvex on K hence for every $t \in [0,1]$, we obtain,

$$(2.19)\qquad \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right| \leq \sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}$$

and
(2.20)

$$\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right| \leq \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}.$$

A combination of (2.8), (2.19) and (2.20) gives the required inequality (2.18). □

Corollary 2. *Suppose all the conditions of Theorem 24 are satisfied. Moreover*

(1) *If* $\left|f^{'}\right|$ *is non-decreasing, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)\,w(x)\,dx\right| \tag{2.21}$$
$$\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right]\int_0^1\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dxdt.$$

and

(2) *If* $\left|f^{'}\right|$ *is non-increasing, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)\,w(x)\,dx\right| \tag{2.22}$$
$$\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]\int_0^1\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dxdt.$$

Remark 8. [16] *If in Theorem 24, we take* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$*, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.23}$$
$$\leq\frac{\eta(b,a)}{8}\left[\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\right.$$
$$\left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f^{'}(a+\eta(b,a))\right|\right\}\right].$$

The inequality (2.23) represents a new refinement of the inequality (1.16) for prequasiinvex functions and hence for preinvex functions. Moreover,

(1) *If* $\left|f^{'}\right|$ *is non-decreasing, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.24}$$
$$\leq\frac{\eta(b,a)}{8}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right]$$

and

(2) *If* $\left|f^{'}\right|$ *is non-increasing, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.25}$$
$$\leq\frac{\eta(b,a)}{8}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].$$

Remark 9. *If* $\eta(b,a)=b-a$ *in Theorem 24, then (2.18) reduces to the inequality (1.13) established in Theorem 11from* [9] *and the inequalities (2.24) and (2.25) recapture the related inequalities given in the corollary of Theorem 11.*

Remark 10. *If $\eta(b,a) = b - a$ in Remark 8, then (2.23) becomes the inequality (1.8) of Theorem 6 from* [2] *and the inequalities (2.24) and (2.25) recapture the related inequalities of corollary of Theorem 6.*

Theorem 25. *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K and $w : [a, a+\eta(b,a)] \to [0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is prequasiinvex on K for $q > 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$ we have the following inequality:*

(2.26)
$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)w(x)\,dx\right| \le \frac{\eta(b,a)}{4}$$
$$\times\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}\left[\left(\sup\left\{\left|f'(a)\right|^q, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, \left|f'(a+\eta(b,a))\right|^q\right\}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We continue inequality (2.11) in the proof of Theorem 22. By the prequasiinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $t \in [0,1]$

$$\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q \le \sup\left\{\left|f'(a)\right|^q, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\} \tag{2.27}$$

and

(2.28)
$$\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q \le \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q, \left|f'(a+\eta(b,a))\right|^q\right\}.$$

A combination of (2.11), (2.27) and (2.28) gives us the required inequality (2.26). This completes the proof of the Theorem. □

Corollary 3. *Suppose all the conditions of Theorem 25 are satisfied. Moreover*

(1) *If $\left|f'\right|^q$ is non-decreasing for $q>1$, then the following inequality holds:*

(2.29)
$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)w(x)\,dx\right|$$

(2.30)
$$\le \frac{\eta(b,a)}{4}\left[\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right| + \left|f'(a+\eta(b,a))\right|\right]\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}$$

and

(2) *If* $\left|f^{'}\right|^{q}$ *is non-increasing for* $q>1$*, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)}w(x)\,dx-\int_{a}^{a+\eta(b,a)}f(x)\,w(x)\,dx\right| \tag{2.31}$$
$$\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]\left(\int_{0}^{1}\left[\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\right]^{p}dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 11. [16] *If in Theorem 25, we take* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$*, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.32}$$
$$\leq\frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\right.$$
$$\left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f^{'}(a+\eta(b,a))\right|\right\}\right].$$

The inequality (2.32) represents a new refinement of the inequality (1.19) for pre-quasiinvex functions and hence for preinvex functions. Moreover,

(1) *If* $\left|f^{'}\right|$ *is non-decreasing, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.33}$$
$$\leq\frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right]$$

and

(2) *If* $\left|f^{'}\right|$ *is non-increasing, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.34}$$
$$\leq\frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 12. *If we take* $\eta(b,a)=b-a$ *in Remark 11, then (2.32) becomes the inequality (1.9) of Theorem 7 from* [2] *and the inequalities (2.33) and (2.34) become the related inequalities given in the corollary of Theorem 7.*

Theorem 26. *Let* $K\subseteq[0,\infty)$ *be an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$ *and* $a,b\in K$ *with* $a<a+\eta(b,a)$*. Suppose* $f:K\to\mathbb{R}$ *is a differentiable mapping on* K *and* $w:[a,a+\eta(b,a)]\to[0,\infty)$ *be continuous and symmetric to* $a+\frac{1}{2}\eta(b,a)$.

If $\left|f^{'}\right|^{q}$ *is prequasiinvex on* K *for* $q\geq 1$*, then for every* $a,b\in K$ *with* $\eta(b,a)>0$ *we have the following inequality:*

(2.35)
$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)}w(x)\,dx-\int_{a}^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|\leq\frac{\eta(b,a)}{4}$$
$$\times\left(\int_{0}^{1}\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dxdt\right)\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}\right)^{\frac{1}{q}}\right].$$

Proof. We continue inequality (2.15) in the proof of Theorem 23. By the prequasiinvex of $\left|f^{'}\right|^{q}$ on K for $q\geq 1$, we have for every $t\in[0,1]$

$$\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}\leq\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\tag{2.36}$$

and

(2.37)
$$\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}\leq\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}.$$

A combination of (2.15), (2.36) and (2.37) gives us the required inequality (2.35). This completes the proof of the Theorem. □

Corollary 4. *Suppose all the conditions of Theorem 26 are satisfied. Moreover*

(1) *If* $\left|f^{'}\right|^{q}$ *is non-decreasing for* $q\geq 1$*, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)}w(x)\,dx-\int_{a}^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|\tag{2.38}$$
$$\leq\frac{\eta(b,a)}{4}\left(\int_{0}^{1}\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dxdt\right)\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right].$$

and

(2) *If* $\left|f^{'}\right|^{q}$ *is non-increasing for* $q\geq 1$*, then the the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)}w(x)\,dx-\int_{a}^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|\tag{2.39}$$
$$\leq\frac{\eta(b,a)}{4}\left(\int_{0}^{1}\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dxdt\right)\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right].$$

Remark 13. [16] *If in Theorem 26, we take* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$*, then we have the inequality (1.22). Moreover,*

(1) *If* $\left|f^{'}\right|^{q}$ *is non-decreasing, then the following inequality (2.24) holds:*
and

(2) *If* $\left|f^{'}\right|^{q}$ *is non-increasing, then the following inequality (2.25) holds:*

Remark 14. *If* $\eta(b,a)=b-a$ *in Theorem 26, then (2.35) reduces to the inequality (1.14) established in Theorem 12 from* [9] *and the inequalities (2.38) and (2.39) recapture the related inequalities established in the corollary of Theorem 12.*

Remark 15. *If* $\eta(b,a)=b-a$ *in Remark 13, then (1.22) becomes the inequality (1.10) of Theorem 8 from* [2] *and the inequalities (2.24) and (2.25) recapture the related inequalities of corollary of Theorem 8.*

3. Applications to Special Means

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3. [6]*A function* $M:\mathbb{R}_{+}^{2}\to\mathbb{R}_{+}$, *is called a Mean function if it has the following properties:*

1. *Homogeneity:* $M(ax,ay)=aM(x,y)$, *for all* $a>0$,
2. *Symmetry :* $M(x,y)=M(y,x)$,
3. *Reflexivity :* $M(x,x)=x$,
4. *Monotonicity: If* $x\leq x^{'}$ *and* $y\leq y^{'}$, *then* $M(x,y)\leq M(x^{'},y^{'})$,
5. *Internality:* $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α, β (see for instance [6]).

1. The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}$$
2. The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}$$
3. The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
4. The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\ r\geq 1$$
5. The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha\neq\beta\\ \alpha, & \alpha=\beta\end{cases}$$
6. The logarithmic mean:
$$L:=L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|},\ \ |\alpha|\neq|\beta|$$
7. The generalized log-mean:
$$L_p:=L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right],\ \alpha\neq\beta,\ p\in\mathbb{R}\backslash\{-1,0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b,a) = M(b,a)$ in (2.4), (2.10) and (2.14), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f(a)+f(a+M(a,b))}{2}\int_a^{a+M(a,b)} w(x)\,dx - \int_a^{a+M(a,b)} f(x)\,w(x)\,dx\right| \leq \frac{M(a,b)}{4}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt. \tag{3.1}$$

$$\left|\frac{f(a)+f(a+M(a,b))}{2}\int_a^{a+M(a,b)} w(x)\,dx - \int_a^{a+M(a,b)} f(x)\,w(x)\,dx\right| \leq \frac{M(a,b)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}, \tag{3.2}$$

for $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ and

$$\left|\frac{f(a)+f(a+M(a,b))}{2}\int_a^{a+M(a,b)} w(x)\,dx - \int_a^{a+M(a,b)} f(x)\,w(x)\,dx\right| \leq \frac{M(a,b)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt. \tag{3.3}$$

for $q \geq 1$, where $U'(a,b,t) = a + \left(\frac{1+t}{2}\right)M(a,b)$, $L'(a,b,t) = a + \left(\frac{1-t}{2}\right)M(a,b)$. Letting $M = A$, G, H, P_r, I, L, L_p in (3.1), (3.2) and (3.3), we can get the required inequalities for different weight function $w(x)$, and the details are left to the interested reader.

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