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INEQUALITIES FOR POWER SERIES IN BANACH ALGEBRAS

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Abstract. Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$, a Banach algebra, with $\|x\|,\|y\|<R$ we show among others that

$$\left\|\frac{f(y)+f(x)}{2}-f\left(\frac{x+y}{2}\right)\right\|\leq\frac{1}{2}\|y-x\|\int_0^1 f_a'(\|(1-t)x+ty\|)\,dt$$

where $f_a(\lambda)=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n$.

Inequalities for the commutator such as

$$\|f(yx)-f(xy)\|\leq f_a'\left(M^2\right)\|yx-xy\|,$$

if $\|x\|,\|y\|\leq M<R^{1/2}$, as well as some inequalities for exponential and resolvent functions are also provided.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further:

$$\|ab\|\leq\|a\|\|b\|$$

for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a,b\in$Inv$\mathcal{B}$ then $ab\in$Inv$\mathcal{B}$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in$Inv$\mathcal{B}$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset$Inv$\mathcal{B}$;
(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map Inv$\mathcal{B}\ni a\longmapsto a^{-1}\in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{\lambda\in\mathbb{C}:\ \lambda-a\in\text{Inv}\mathcal{B}\};$$

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the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \mathrm{Inv}\mathcal{B}$,

$$R_a(\lambda) := (\lambda - a)^{-1}.$$

For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that

$$\sigma(a) \subset \{\lambda \in \mathbb{C} : \ |\lambda| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \leq \nu(a)\nu(b) \text{ and } \nu(a+b) \leq \nu(a) + \nu(b).$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have

$$a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi;$$

(v) We have

$$\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series*

$$f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \ \ (|\lambda| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\|<1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b=-\sum_{n=1}^{\infty}\frac{1}{n}(1-a)^{n},$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b)=a$.

In this paper we establish some upper bounds for the following quantities

$$\|f(xy)-f(yx)\|,$$

$$\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\|$$

and

$$\left\|\frac{f(x^{2})+f(y^{2})}{2}-f(xy)\right\|$$

that can naturally be associated with the analytic functions $f(\lambda):=\sum_{j=0}^{\infty}\alpha_{j}\lambda^{j}$ defined on the open disk $D(0,R)$ and the elements x and y of the unital Banach algebra $\mathcal{B}$.

Some applications for functions of interest such as the exponential map on $\mathcal{B}$ are provided as well.

2. Some Power Inequalities

The following simple result holds.

Lemma 1. *For any $x,y\in\mathcal{B}$ and $n\geq 1$ we have*

$$\|y^{n}-x^{n}\|\leq n\|y-x\|\int_{0}^{1}\|(1-t)x+ty\|^{n-1}dt. \tag{2.1}$$

Proof. We use the identity (see for instance [1, p. 254])

$$a^{n}-b^{n}=\sum_{j=0}^{n-1}a^{n-1-j}(a-b)b^{j} \tag{2.2}$$

that holds for any $a,b\in\mathcal{B}$ and $n\geq 1$.

For $x,y\in\mathcal{B}$ we consider the function $\varphi:[0,1]\to\mathcal{B}$ defined by $\varphi(t)=[(1-t)x+ty]^{n}$. For $t\in(0,1)$ and $\varepsilon\neq 0$ with $t+\varepsilon\in(0,1)$ we have from (2.2) that

$$\begin{aligned}\varphi(t+\varepsilon)-\varphi(t)&=[(1-t-\varepsilon)x+(t+\varepsilon)y]^{n}-[(1-t)x+ty]^{n}\\&=\varepsilon\sum_{j=0}^{n-1}[(1-t-\varepsilon)x+(t+\varepsilon)y]^{n-1-j}(y-x)[(1-t)x+ty]^{j}\end{aligned}$$

Dividing with $\varepsilon\neq 0$ and taking the limit over $\varepsilon\to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\begin{aligned}\varphi'(t)&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\varphi(t+\varepsilon)-\varphi(t)]\\&=\sum_{j=0}^{n-1}[(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^{j}.\end{aligned} \tag{2.3}$$

Integrating on $[0,1]$ we get from (2.3) that

$$\int_0^1 \varphi'(t)\,dt = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt$$

and since

$$\int_0^1 \varphi'(t)\,dt = \varphi(1)-\varphi(0) = y^n - x^n$$

then we get the following *equality of interest*

$$y^n - x^n = \sum_{j=0}^{n-1} \int_0^1 [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$.

Taking the norm and utilising the properties of Bochner integral for vector valued functions (see for instance [9, p. 21]) we have

$$\begin{aligned} \|y^n - x^n\| &\leq \sum_{j=0}^{n-1} \left\| \int_0^1 [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j\,dt \right\| \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)x+ty]^{n-1-j}(y-x)[(1-t)x+ty]^j \right\| dt \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)x+ty]^{n-1-j} \right\| \|y-x\| \left\| [(1-t)x+ty]^j \right\| dt \\ &\leq \sum_{j=0}^{n-1} \int_0^1 \|(1-t)x+ty\|^{n-1-j} \|y-x\| \|(1-t)x+ty\|^j\,dt \\ &= n\|y-x\| \int_0^1 \|(1-t)x+ty\|^{n-1}\,dt \end{aligned} \tag{2.4}$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$. □

Remark 1. *Utilising the Hermite-Hadamard inequality for convex functions*

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right],$$

(see for instance [4, p. 2]*) we have the sequence of inequalities*

$$\begin{aligned} \|y^n - x^n\| &\leq n\|y-x\| \int_0^1 \|(1-t)x+ty\|^{n-1}\,dt \\ &\leq \frac{1}{2}n\|y-x\| \left[\left\|\frac{x+y}{2}\right\|^{n-1} + \frac{\|x\|^{n-1}+\|y\|^{n-1}}{2}\right] \\ &\leq \frac{1}{2}n\|y-x\| \left[\|x\|^{n-1}+\|y\|^{n-1}\right] \\ &\leq n\|y-x\| \max\left\{\|x\|^{n-1}, \|y\|^{n-1}\right\}. \end{aligned} \tag{2.5}$$

For other Hermite-Hadamard type inequalities that may be utilised to obtain such upper bounds, see for instance [2], [3], [6], [7], [10], [11], [12], [13], [14], [15] *and* [16].

We also have

$$\begin{aligned}
\|y^n - x^n\| &\le n\|y-x\| \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\
&\le n\|y-x\| \int_0^1 ((1-t)\|x\| + t\|y\|)^{n-1} dt \\
&\le \frac{1}{2} n \|y-x\| \left[\left(\frac{\|x\| + \|y\|}{2} \right)^{n-1} + \frac{\|x\|^{n-1} + \|y\|^{n-1}}{2} \right] \\
&\le \frac{1}{2} n \|y-x\| \left[\|x\|^{n-1} + \|y\|^{n-1} \right] \\
&\le n\|y-x\| \max\left\{ \|x\|^{n-1}, \|y\|^{n-1} \right\}.
\end{aligned} \tag{2.6}$$

We observe that if $\|y\| \neq \|x\|$, *then by the change of variable* $s = (1-t)\|x\| + t\|y\|$ *we have*

$$\begin{aligned}
\int_0^1 ((1-t)\|x\| + t\|y\|)^{n-1} dt &= \frac{1}{\|y\| - \|x\|} \int_{\|x\|}^{\|y\|} s^{n-1} ds \\
&= \frac{1}{n} \cdot \frac{\|y\|^n - \|x\|^n}{\|y\| - \|x\|}.
\end{aligned}$$

If $\|y\| = \|x\|$, *then*

$$\int_0^1 ((1-t)\|x\| + t\|y\|)^{n-1} dt = \|x\|^{n-1}.$$

Utilising these observations we then get the following divided difference inequality for $x \neq y$

$$\begin{aligned}
\frac{\|y^n - x^n\|}{\|y-x\|} &\le n \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\
&\le \begin{cases} \frac{\|y\|^n - \|x\|^n}{\|y\| - \|x\|} & \text{if } \|y\| \neq \|x\|, \\ n\|x\|^{n-1} & \text{if } \|y\| = \|x\|. \end{cases}
\end{aligned} \tag{2.7}$$

Remark 2. *We observe that the quantity* $n\int_0^1 \|(1-t)x + ty\|^{n-1} dt$, *which might be difficult to compute in various examples of Banach algebras, has got the simpler bounds*

$$B_1(x,y) := \frac{1}{2} n \left[\left\| \frac{x+y}{2} \right\|^{n-1} + \frac{\|x\|^{n-1} + \|y\|^{n-1}}{2} \right]$$

and

$$B_2(x,y) := \begin{cases} \frac{\|y\|^n - \|x\|^n}{\|y\| - \|x\|} & \text{if } \|y\| \neq \|x\|, \\ n\|x\|^{n-1} & \text{if } \|y\| = \|x\|. \end{cases}$$

It is natural then to ask which of these bounds is better?

Let $m \ge 1$. *Then*

$$B_1(x,y) = \frac{1}{2} m \left[\left\| \frac{x+y}{2} \right\|^{m-1} + \frac{\|x\|^{m-1} + \|y\|^{m-1}}{2} \right]$$

and

$$B_2(x,y)=\begin{cases}\|y\|^{m-1}+\|y\|^{m-2}\|x\|+...+\|x\|^{m-1} & \text{if } \|y\|\neq\|x\|,\\ \\ m\|x\|^{m-1} & \text{if } \|y\|=\|x\|.\end{cases}$$

If we take $y=tx$ with $\|x\|=1$ and $|t|\neq 1$ then we get

$$B_1(t)=\frac{1}{2}m\left[\left|\frac{1+t}{2}\right|^{m-1}+\frac{1+|t|^{m-1}}{2}\right]$$

and

$$B_2(t)=|t|^{m-1}+...+|t|+1.$$

If we take $m=4$ and plot the difference

$$d(t):=2\left(\left|\frac{t+1}{2}\right|^3+\frac{1+|t|^3}{2}\right)-\left(|t|^3+|t|^2+|t|+1\right)$$

on the interval $[-8,8]$, then we can conclude that some time the first bound is better than the second, while other time the conclusion is the other way around. The details for the plot are nor presented here.

3. Inequalities for a Generalized Commutator

Now, by the help of power series $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n\geq 0$, then $f_a=f$.

The following result is valid.

Theorem 1. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|xy\|,\|yx\|<R$ we have*

$$\|f(xy)-f(yx)\|\leq\|xy-yx\|\int_0^1 f_a'(\|(1-t)xy+tyx\|)\,dt. \tag{3.1}$$

Proof. For any $m\geq 1$, by making use of the inequality (2.1) we have succesively

$$\begin{aligned}&\left\|\sum_{n=0}^{m}\alpha_n(xy)^n-\sum_{n=0}^{m}\alpha_n(yx)^n\right\| \\ &=\left\|\sum_{n=1}^{m}\alpha_n\left((xy)^n-(yx)^n\right)\right\| \\ &\leq\sum_{n=1}^{m}|\alpha_n|\,\|(xy)^n-(yx)^n\| \\ &\leq\|xy-yx\|\sum_{n=1}^{m}n|\alpha_n|\int_0^1\|(1-t)xy+tyx\|^{n-1}dt \\ &=\|xy-yx\|\int_0^1\left(\sum_{n=1}^{m}n|\alpha_n|\,\|(1-t)xy+tyx\|^{n-1}\right)dt.\end{aligned} \tag{3.2}$$

Moreover, since $\|xy\|, \|yx\| < R$, then the series $\sum_{n=0}^{\infty} \alpha_n (xy)^n$, $\sum_{n=0}^{\infty} \alpha_n (yx)^n$ and

$$\sum_{n=1}^{\infty} n |\alpha_n| \|(1-t)xy + tyx\|^{n-1}$$

are convergent and

$$\sum_{n=0}^{\infty} \alpha_n (xy)^n = f(xy), \sum_{n=0}^{\infty} \alpha_n (yx)^n = f(yx)$$

while

$$\sum_{n=1}^{\infty} n |\alpha_n| \|(1-t)xy + tyx\|^{n-1} = f_a'(\|(1-t)xy + tyx\|).$$

Therefore, by taking the limit over $m \to \infty$ in the inequality (3.2) we deduce the desired result (3.1). □

Remark 3. *We observe that f_a' is monotonic nondecreasing and convex on the interval $[0, R)$ and since the function $\psi(t) := \|(1-t)xy + tyx\|$ is convex on $[0,1]$ we have that $f_a' \circ \psi$ is also convex on $[0,1]$. Utilising the Hermite-Hadamard inequality for convex functions (see for instance [4, p. 2]) we have the sequence of inequalities*

$$\begin{aligned}
&\|f(xy) - f(yx)\| \\
&\le \|xy - yx\| \int_0^1 f_a'(\|(1-t)xy + tyx\|)\, dt \\
&\le \frac{1}{2} \|xy - yx\| \left[f_a'\left(\left\|\frac{xy + yx}{2}\right\|\right) + \frac{f_a'(\|xy\|) + f_a'(\|yx\|)}{2} \right] \\
&\le \frac{1}{2} \|xy - yx\| \left[f_a'(\|xy\|) + f_a'(\|yx\|) \right] \\
&\le \|xy - yx\| \max\{f_a'(\|xy\|), f_a'(\|yx\|)\} \\
&\le \|xy - yx\| f_a'(\|x\| \|y\|).
\end{aligned} \tag{3.3}$$

We also have

$$\begin{aligned}
&\|f(xy) - f(yx)\| \\
&\le \|xy - yx\| \int_0^1 f_a'(\|(1-t)xy + tyx\|)\, dt \\
&\le \|xy - yx\| \int_0^1 f_a'((1-t)\|xy\| + t\|yx\|)\, dt \\
&\le \frac{1}{2} \|xy - yx\| \left[f_a'\left(\frac{\|xy\| + \|yx\|}{2}\right) + \frac{f_a'(\|xy\|) + f_a'(\|yx\|)}{2} \right] \\
&\le \frac{1}{2} \|xy - yx\| \left[f_a'(\|xy\|) + f_a'(\|yx\|) \right] \\
&\le \|xy - yx\| \max\{f_a'(\|xy\|), f_a'(\|yx\|)\} \\
&\le \|xy - yx\| f_a'(\|x\| \|y\|).
\end{aligned} \tag{3.4}$$

We observe that if $\|yx\| \neq \|xy\|$, *then by the change of variable* $s = (1-t)\|x\| + t\|y\|$ *we have*

$$\begin{aligned}\int_0^1 f_a'((1-t)\|xy\| + t\|yx\|)\,dt &= \frac{1}{\|yx\| - \|xy\|}\int_{\|xy\|}^{\|yx\|} f_a'(s)\,ds \\ &= \frac{f_a(\|yx\|) - f_a(\|xy\|)}{\|yx\| - \|xy\|}.\end{aligned}$$

If $\|yx\| = \|xy\|$, *then*

$$\int_0^1 f_a'((1-t)\|xy\| + t\|yx\|)\,dt = f_a'(\|xy\|).$$

Utilising these observations we then get the following divided difference inequality for $xy \neq yx$

$$\begin{aligned}\frac{\|f(yx) - f(xy)\|}{\|yx - xy\|} &\le \int_0^1 f_a'(\|(1-t)xy + tyx\|)\,dt \\ &\le \begin{cases} \frac{f_a(\|yx\|) - f_a(\|xy\|)}{\|yx\| - \|xy\|} & \text{if } \|yx\| \neq \|xy\|, \\ f_a'(\|xy\|) & \text{if } \|yx\| = \|xy\|. \end{cases}\end{aligned} \tag{3.5}$$

If $\|x\|, \|y\| \le M < R^{1/2}$, *then from the inequalities (3.3) we get the simpler inequality*

$$\|f(yx) - f(xy)\| \le f_a'\left(M^2\right)\|yx - xy\|. \tag{3.6}$$

If we consider the exponential function $\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n$, then for any $x, y \in \mathcal{B}$ we have the inequalities

$$\begin{aligned}&\|\exp(yx) - \exp(xy)\| \\ &\le \|yx - xy\| \int_0^1 \exp(\|(1-t)xy + tyx\|)\,dt \\ &\le \|yx - xy\| \times \begin{cases} \frac{1}{2}\left[\exp\left(\left\|\frac{xy+yx}{2}\right\|\right) + \frac{\exp(\|xy\|) + \exp(\|yx\|)}{2}\right], \\ \frac{\exp(\|yx\|) - \exp(\|xy\|)}{\|yx\| - \|xy\|} \text{ if } \|yx\| \neq \|xy\|, \\ \exp(\|xy\|) \text{ if } \|yx\| = \|xy\|. \end{cases}\end{aligned} \tag{3.7}$$

Now, if we consider the functions $(1-\lambda)^{-1}=\sum_{n=0}^{\infty}\lambda^{n}$ and $(1+\lambda)^{-1}=\sum_{n=0}^{\infty}(-1)^{n}\lambda^{n}$, then for any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<1$ we have

$$\begin{aligned}
&\left\|(1\pm yx)^{-1}-(1\pm xy)^{-1}\right\| \\
&\leq\|yx-xy\|\int_{0}^{1}\left(1-\|(1-t)\,xy+tyx\|\right)^{-2}dt \\
&\leq\|yx-xy\| \\
&\times\begin{cases}\frac{1}{2}\left[\left(1-\left\|\frac{xy+yx}{2}\right\|\right)^{-2}+\frac{(1-\|xy\|)^{-2}+(1-\|yx\|)^{-2}}{2}\right], \\ (1-\|yx\|)^{-1}(1-\|xy\|)^{-1}\ \text{ if }\ \|yx\|\neq\|xy\|, \\ (1-\|xy\|)^{-2}\ \text{ if }\ \|yx\|=\|xy\|.\end{cases}
\end{aligned}\tag{3.8}$$

4. Bounds for the Jensen Difference

In this section we establish some bounds for the norm of the *Jensen difference*, namely, the quantity

$$\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\|,$$

where $x,y\in\mathcal{B}$ and f is a function defined on the Banach algebra $\mathcal{B}$.

Theorem 2. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_{n}\lambda^{n}$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. For any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|<R$ we have*

$$\left\|\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right\|\leq\frac{1}{2}\|y-x\|\int_{0}^{1}f_{a}'\left(\|(1-t)\,x+ty\|\right)dt.\tag{4.1}$$

Proof. For any $x,y\in\mathcal{B}$ and $n\geq1$ we have from (2.1) that

$$\begin{aligned}
\left\|y^{n}-\left(\frac{x+y}{2}\right)^{n}\right\|&\leq n\left\|y-\frac{x+y}{2}\right\|\int_{0}^{1}\left\|(1-t)\,\frac{x+y}{2}+ty\right\|^{n-1}dt \\
&=\frac{1}{2}n\|y-x\|\int_{0}^{1}\left\|(1-t)\,\frac{x+y}{2}+ty\right\|^{n-1}dt
\end{aligned}\tag{4.2}$$

and

$$\left\|x^{n}-\left(\frac{x+y}{2}\right)^{n}\right\|\leq\frac{1}{2}n\|y-x\|\int_{0}^{1}\left\|(1-t)\,\frac{x+y}{2}+tx\right\|^{n-1}dt.\tag{4.3}$$

We add (4.2) with (4.3), use the triangle inequality and divide by 2 to get

$$
\begin{aligned}
&\left\| \frac{y^n + x^n}{2} - \left(\frac{x+y}{2} \right)^n \right\| \\
&\leq \frac{1}{2} n \|y - x\| \\
&\times \frac{1}{2} \int_0^1 \left[\left\| (1-t) \frac{x+y}{2} + ty \right\|^{n-1} + \left\| (1-t) \frac{x+y}{2} + tx \right\|^{n-1} \right] dt \\
&= \frac{1}{2} n \|y - x\| \\
&\times \frac{1}{2} \int_0^1 \left[\left\| s \frac{x+y}{2} + (1-s) y \right\|^{n-1} + \left\| s \frac{x+y}{2} + (1-s) x \right\|^{n-1} \right] ds,
\end{aligned} \tag{4.4}
$$

where we used for the last equality the change of variable $s = 1 - t$.

Now, using the change of variable $s = 2\tau$ we have

$$
\frac{1}{2} \int_0^1 \left\| s \frac{x+y}{2} + (1-s) x \right\|^{n-1} ds = \int_0^{1/2} \|(1-\tau) x + \tau y\|^{n-1} d\tau
$$

and by the change of variable $s = 1 - v$ we have

$$
\frac{1}{2} \int_0^1 \left\| s \frac{x+y}{2} + (1-s) y \right\|^{n-1} ds = \frac{1}{2} \int_0^1 \left\| (1-v) \frac{x+y}{2} + vy \right\|^{n-1} dv.
$$

Moreover, if we make the change of variable $v = 2\tau - 1$ we also have

$$
\frac{1}{2} \int_0^1 \left\| (1-v) \frac{x+y}{2} + vy \right\|^{n-1} dv = \int_{1/2}^1 \|(1-\tau) x + \tau y\|^{n-1} d\tau.
$$

Therefore

$$
\begin{aligned}
&\frac{1}{2} \int_0^1 \left[\left\| s \frac{x+y}{2} + (1-s) y \right\|^{n-1} + \left\| s \frac{x+y}{2} + (1-s) x \right\|^{n-1} \right] ds \\
&= \int_0^{1/2} \|(1-\tau) x + \tau y\|^{n-1} d\tau + \int_{1/2}^1 \|(1-\tau) x + \tau y\|^{n-1} d\tau \\
&= \int_0^1 \|(1-\tau) x + \tau y\|^{n-1} d\tau.
\end{aligned}
$$

Utilising (4.4) we get the inequality

$$
\left\| \frac{y^n + x^n}{2} - \left(\frac{x+y}{2} \right)^n \right\| \leq \frac{1}{2} n \|y - x\| \int_0^1 \|(1-t) x + ty\|^{n-1} dt \tag{4.5}
$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$.

Now, for any $m \geq 1$, by making use of the inequality (4.5) we have

$$
\begin{aligned}
&\left\| \frac{1}{2} \left(\sum_{n=0}^{m} \alpha_n y^n + \sum_{n=0}^{m} \alpha_n x^n \right) - \sum_{n=0}^{m} \alpha_n \left(\frac{x+y}{2} \right)^n \right\| \\
&= \left\| \sum_{n=1}^{m} \alpha_n \left[\frac{y^n + x^n}{2} - \left(\frac{x+y}{2} \right)^n \right] \right\| \\
&\leq \sum_{n=1}^{m} |\alpha_n| \left\| \frac{y^n + x^n}{2} - \left(\frac{x+y}{2} \right)^n \right\| \\
&\leq \frac{1}{2} \|y - x\| \sum_{n=1}^{m} n |\alpha_n| \int_0^1 \|(1-t)x + ty\|^{n-1} dt \\
&= \frac{1}{2} \|y - x\| \int_0^1 \left(\sum_{n=1}^{m} n |\alpha_n| \|(1-t)x + ty\|^{n-1} \right) dt
\end{aligned} \tag{4.6}
$$

for any $x, y \in \mathcal{B}$ and $n \geq 1$.

Since all the series whose partial sums are involved in (4.6) are convergent, then by letting $m \to \infty$ in (4.6) and taking into account that

$$
\sum_{n=1}^{\infty} n |\alpha_n| \|(1-t)x + ty\|^{n-1} = f_a' (\|(1-t)x + ty\|), t \in [0,1],
$$

we get the desired inequality (4.1). $\square$

Remark 4. *We observe that f_a' is monotonic nondecreasing and convex on the interval $[0, R)$ and since the function $\psi(t) := \|(1-t)x + ty\|$ is convex on $[0,1]$ we have that $f_a' \circ \psi$ is also convex on $[0,1]$. Utilising the Hermite-Hadamard inequality for convex functions we have the sequence of inequalities:*

$$
\begin{aligned}
&\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2} \right) \right\| \\
&\leq \frac{1}{2} \|y - x\| \int_0^1 f_a' (\|(1-t)x + ty\|) dt \\
&\leq \frac{1}{4} \|y - x\| \left[f_a' \left(\left\| \frac{x+y}{2} \right\| \right) + \frac{f_a'(\|x\|) + f_a'(\|y\|)}{2} \right] \\
&\leq \frac{1}{4} \|y - x\| \left[f_a'(\|x\|) + f_a'(\|y\|) \right] \\
&\leq \frac{1}{2} \|y - x\| \max \left\{ f_a'(\|x\|), f_a'(\|y\|) \right\}.
\end{aligned} \tag{4.7}
$$

We also have

$$\left\| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| \tag{4.8}$$
$$\leq \frac{1}{2}\|y-x\| \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt$$
$$\leq \frac{1}{2}\|y-x\| \int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt$$
$$\leq \frac{1}{4}\|y-x\| \left[f_a'\left(\frac{\|x\|+\|y\|}{2}\right) + \frac{f_a'(\|x\|)+f_a'(\|y\|)}{2} \right]$$
$$\leq \frac{1}{4}\|y-x\| \left[f_a'(\|x\|)+f_a'(\|y\|) \right]$$
$$\leq \frac{1}{2}\|y-x\| \max\left\{ f_a'(\|x\|), f_a'(\|y\|) \right\}.$$

We observe that if $\|y\| \neq \|x\|$, *then by the change of variable* $s=(1-t)\|x\|+t\|y\|$ *we have*

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt = \frac{1}{\|y\|-\|x\|}\int_{\|x\|}^{\|y\|} f_a'(s)\,ds = \frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|}.$$

If $\|y\| = \|x\|$, *then*

$$\int_0^1 f_a'((1-t)\|x\|+t\|y\|)\,dt = f_a'(\|x\|).$$

Utilising these observations we then get the following divided difference inequality

$$\left\| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| \tag{4.9}$$
$$\leq \frac{1}{2}\|y-x\| \int_0^1 f_a'(\|(1-t)x+ty\|)\,dt$$
$$\leq \frac{1}{2}\|y-x\| \times \begin{cases} \frac{f_a(\|y\|)-f_a(\|x\|)}{\|y\|-\|x\|} & \text{if } \|y\| \neq \|x\|, \\ f_a'(\|x\|) & \text{if } \|y\| = \|x\|. \end{cases}$$

If $\|x\|, \|y\| \leq M < R$, *then from the inequalities (4.8) we have the simpler inequality*

$$\left\| \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{2} f_a'(M)\|y-x\|. \tag{4.10}$$

If we consider the exponential function $\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n$, then for any $x, y \in \mathcal{B}$ we have:

$$\begin{aligned}
(4.11)\qquad & \left\| \frac{\exp(x) + \exp(y)}{2} - \exp\left(\frac{x+y}{2}\right) \right\| \\
& \leq \frac{1}{2}\|y - x\| \int_0^1 \exp\left(\|(1-t)x + ty\|\right) dt \\
& \leq \frac{1}{2}\|y - x\| \times \begin{cases} \frac{1}{2}\left[\exp\left(\left\|\frac{x+y}{2}\right\|\right) + \frac{\exp(\|x\|)+\exp(\|y\|)}{2}\right], \\ \frac{\exp(\|y\|)-\exp(\|x\|)}{\|y\|-\|x\|} \text{ if } \|y\| \neq \|x\|, \\ \exp(\|x\|) \text{ if } \|y\| = \|x\|, \end{cases} \\
& \leq \frac{1}{2}\|y - x\| \exp(M),
\end{aligned}$$

where, for the last inequality we assume that $\|y\|, \|x\| \leq M$.

Now, if we consider the functions $(1-\lambda)^{-1}$ and $(1+\lambda)^{-1}$, then for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < 1$ we have:

$$\begin{aligned}
(4.12)\qquad & \left\| \frac{(1 \pm x)^{-1} + (1 \pm y)^{-1}}{2} - \left(1 \pm \frac{x+y}{2}\right)^{-1} \right\| \\
& \leq \frac{1}{2}\|y - x\| \int_0^1 \left(1 - \|(1-t)x + ty\|\right)^{-2} dt \\
& \leq \frac{1}{2}\|y - x\| \\
& \times \begin{cases} \frac{1}{2}\left[\left(1 - \left\|\frac{x+y}{2}\right\|\right)^{-2} + \frac{(1-\|x\|)^{-2}+(1-\|y\|)^{-2}}{2}\right], \\ (1-\|x\|)^{-1}(1-\|y\|)^{-1} \text{ if } \|y\| \neq \|x\|, \\ (1-\|x\|)^{-2} \text{ if } \|y\| = \|x\|, \end{cases} \\
& \leq \frac{1}{2}\|y - x\| (1-M)^{-2},
\end{aligned}$$

where, for the last inequality we assume that $\|y\|, \|x\| \leq M < 1$.

5. Some Inequalities for Commuting Elements

For two commuting elements $x, y \in \mathcal{B}$ it is of interest to estimate the distance between $\frac{1}{2}\left[f\left(x^2\right) + f\left(y^2\right)\right]$ and $f(xy)$, namely the quantity

$$\left\| \frac{f\left(x^2\right) + f\left(y^2\right)}{2} - f(xy) \right\|,$$

where f is a function defined on the Banach algebra $\mathcal{B}$.

We have the following result:

Theorem 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any*

$x, y \in \mathcal{B}$ with $xy = yx$ and $\left\|x^2\right\|, \left\|y^2\right\|, \|xy\| < R$ we have

$$\left\|\frac{f\left(x^2\right)+f\left(y^2\right)}{2}-f\left(xy\right)\right\| \tag{5.1}$$
$$\le \frac{1}{2}\|y-x\|^2\left[\int_0^1 f_a'\left(\|(1-t)x+ty\|^2\right)dt + \int_0^1 \|(1-t)x+ty\|^2 f_a''\left(\|(1-t)x+ty\|^2\right)dt\right].$$

Proof. We have from (2.1), for $n \ge 1$, that

$$\begin{aligned}\|y^n-x^n\|^2 &\le n^2\|y-x\|^2\left(\int_0^1 \|(1-t)x+ty\|^{n-1}dt\right)^2 \\ &\le n^2\|y-x\|^2\int_0^1\|(1-t)x+ty\|^{2(n-1)}dt\end{aligned} \tag{5.2}$$

for any $x, y \in \mathcal{B}$.

The second inequality follows from the Cauchy-Bunyakovsky-Schwarz integral inequality

$$\left(\int_0^1 f(s)\,ds\right)^2 \le \int_0^1 f^2(s)\,ds.$$

Since $x, y \in \mathcal{B}$ are commutative, then

$$(y^n-x^n)^2 = y^{2n}-y^nx^n-x^ny^n+x^{2n} = 2\left(\frac{y^{2n}+x^{2n}}{2}-(xy)^n\right)$$

which gives that

$$\left\|\frac{y^{2n}+x^{2n}}{2}-(xy)^n\right\| = \frac{1}{2}\left\|(y^n-x^n)^2\right\| \le \frac{1}{2}\|y^n-x^n\|^2 \tag{5.3}$$

for $n \ge 1$.

Therefore, from (5.2) and (5.3) we have

$$\left\|\frac{y^{2n}+x^{2n}}{2}-(xy)^n\right\| \le \frac{1}{2}n^2\|y-x\|^2\int_0^1\|(1-t)x+ty\|^{2(n-1)}dt \tag{5.4}$$

for $n \ge 1$ and for any commuting elements $x, y \in \mathcal{B}$.

Using the generalised triangle inequality and the inequality (5.4) we have

$$
\begin{aligned}
(5.5)\qquad & \left\| \frac{1}{2}\left[\sum_{n=0}^{m}\alpha_n y^{2n} + \sum_{n=0}^{m}\alpha_n x^{2n}\right] - \sum_{n=0}^{m}\alpha_n (xy)^n \right\| \\
&= \left\| \sum_{n=1}^{m}\alpha_n \left[\frac{y^{2n}+x^{2n}}{2} - (xy)^n\right] \right\| \\
&\le \sum_{n=1}^{m} |\alpha_n| \left\| \frac{y^{2n}+x^{2n}}{2} - (xy)^n \right\| \\
&\le \frac{1}{2}\|y-x\|^2 \sum_{n=1}^{m} |\alpha_n| n^2 \int_0^1 \|(1-t)x+ty\|^{2(n-1)}\,dt \\
&= \frac{1}{2}\|y-x\|^2 \int_0^1 \sum_{n=1}^{m} n^2 |\alpha_n| \|(1-t)x+ty\|^{2(n-1)}\,dt.
\end{aligned}
$$

Consider, for $u \neq 0$, the series

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^{n-1} = \frac{1}{u}\sum_{n=0}^{\infty} n^2 \alpha_n u^n.$$

If we denote $g(u) := \sum_{n=0}^{\infty}\alpha_n u^n$, then

$$u g'(u) = \sum_{n=0}^{\infty} n \alpha_n u^n$$

and

$$u\left(u g'(u)\right)' = \sum_{n=0}^{\infty} n^2 \alpha_n u^n.$$

However

$$u\left(u g'(u)\right)' = u g'(u) + u^2 g''(u)$$

and then

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^{n-1} = g'(u) + u g''(u)$$

for $u \neq 0$.

Utilising the above relations we can conclude that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^2 |\alpha_n| \|(1-t)x+ty\|^{2(n-1)} \\
&= \sum_{n=0}^{\infty} n^2 |\alpha_n| \|(1-t)x+ty\|^{2(n-1)} \\
&= f_a'\left(\|(1-t)x+ty\|^2\right) + \|(1-t)x+ty\|^2 f_a''\left(\|(1-t)x+ty\|^2\right)
\end{aligned}
$$

for almost any $t \in [0,1]$.

Since all the series whose partial sums are involved in (5.5) are convergent, then by letting $m \to \infty$ in (5.5) we get the desired inequality (5.1). $\square$

Remark 5. *If we use the notation*

$$D_a^{(2)}(f)(u) := f_a'(u) + u f_a''(u), u \in D(0,R),$$

then the inequality (5.1) can be written in a simpler form as

$$\left\| \frac{f(x^2) + f(y^2)}{2} - f(xy) \right\| \leq \frac{1}{2} \|y - x\|^2 \int_0^1 D_a^{(2)}(f)\left(\|(1-t)x + ty\|^2\right) dt, \tag{5.6}$$

where $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x^2\|, \|y^2\|, \|xy\| < R$.

Remark 6. *Utilising the Hermite-Hadamard inequality for convex functions, we have*

$$\begin{aligned} &\int_0^1 \|(1-t)x + ty\|^{2(n-1)} dt \\ &\leq \frac{1}{2}\left[\left\|\frac{x+y}{2}\right\|^{2(n-1)} + \frac{\|x\|^{2(n-1)} + \|y\|^{2(n-1)}}{2}\right] \\ &\leq \frac{\|x\|^{2(n-1)} + \|y\|^{2(n-1)}}{2} \leq \max\left\{\|x\|^{2(n-1)}, \|y\|^{2(n-1)}\right\} \end{aligned}$$

for any $n \geq 1$.

If we multiply this inequality with $n^2 |\alpha_n|$ and sum, then we get

$$\begin{aligned} &\frac{1}{2} \|y - x\|^2 \int_0^1 D_a^{(2)}(f)\left(\|(1-t)x + ty\|^2\right) dt \\ &\leq \frac{1}{4} \|y - x\|^2 \\ &\times \left[D_a^{(2)}(f)\left(\left\|\frac{x+y}{2}\right\|^2\right) + \frac{D_a^{(2)}(f)\left(\|x\|^2\right) + D_a^{(2)}(f)\left(\|y\|^2\right)}{2}\right] \\ &\leq \frac{1}{2} \|y - x\|^2 \left[\frac{D_a^{(2)}(f)\left(\|x\|^2\right) + D_a^{(2)}(f)\left(\|y\|^2\right)}{2}\right] \\ &\leq \frac{1}{2} \|y - x\|^2 \max\left\{D_a^{(2)}(f)\left(\|x\|^2\right), D_a^{(2)}(f)\left(\|y\|^2\right)\right\}, \end{aligned} \tag{5.7}$$

where $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x^2\|, \|y^2\|, \|xy\| < R$, which provides some simpler upper bounds for the quantity

$$\left\| \frac{f(x^2) + f(y^2)}{2} - f(xy) \right\|.$$

Moreover, if we assume that $\|x\|, \|y\| \leq M$ with $M^2 < R$, then

$$\begin{aligned} D_a^{(2)}(f)\left(\|x\|^2\right), D_a^{(2)}(f)\left(\|y\|^2\right) &\leq D_a^{(2)}(f)\left(M^2\right) \\ &= f_a'\left(M^2\right) + M^2 f_a''\left(M^2\right) \end{aligned}$$

and from (5.6) and (5.7) we get the simple inequality

$$\left\| \frac{f\left(x^2\right)+f\left(y^2\right)}{2} - f\left(xy\right) \right\| \le \frac{1}{2}\left\|y-x\right\|^2 \left[f_a'\left(M^2\right) + M^2 f_a''\left(M^2\right) \right], \tag{5.8}$$

for any $x, y \in \mathcal{B}$ *with* $xy = yx$ *and* $\|x\|, \|y\| \le M$ *with* $M^2 < R$.

If we consider the exponential function $\exp(\lambda)$, then for any $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| \le M$ we have the inequality

$$\left\| \frac{\exp\left(x^2\right)+\exp\left(y^2\right)}{2} - \exp\left(xy\right) \right\| \le \frac{1}{2}\left\|y-x\right\|^2 \left(1+M^2\right)\exp\left(M^2\right). \tag{5.9}$$

Now, if we consider the functions $(1-\lambda)^{-1}$ and $(1+\lambda)^{-1}$, then for any $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| < M < 1$, we have the inequalities

$$\left\| \frac{\left(1 \pm x^2\right)^{-1}+\left(1 \pm y^2\right)^{-1}}{2} - \left(1 \pm xy\right)^{-1} \right\| \le \frac{1}{2}\left\|y-x\right\|^2 \frac{1+M^2}{\left(1-M^2\right)^3}. \tag{5.10}$$

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