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ON SEVERAL INEQUALITIES DEDUCED USING A POWER SERIES APPROACH

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ABSTRACT. The aim of this paper is to study what would become several inequalities using the power series method. Also some applications will be presented.

1. INTRODUCTION

It is necessary to recall the inequality of J. Radon which was published in [8].

For every real numbers $p > 0$, $x_k \geq 0$, $a_k > 0$ for $1 \leq k \leq n$, we have the following inequality:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n})^{p+1}}{(\sum_{k=1}^{n} a_k)^p}, \quad p > 0.$$

Theorem 1. *([4]) For $a_k,\ x_k > 0,\ p \geq 1,\ k \in \{1, 2, ..., n\},\ n \in N$ and $n \geq 2$ the inequality takes place,*

$$(1) \qquad \sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p} + \max_{1 \leq i < j \leq n} \{\frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i + x_j)^{p+1}}{(a_i + a_j)^p}\}$$

with equality if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = ... = \frac{x_n}{a_n}$.

Also we will use the following two inequalities which were enunciated and proved in [2] and [3].

Theorem 2. *([2]) If $n \in \mathbb{N}^* - \{1\},\ a \in \mathbb{R}_+,\ b,\ c,\ d,\ x_k \in \mathbb{R}_+^*,\ X_n = \sum_{k=1}^{n} x_k$, $cX_n > d \max_{1 \leq k \leq n} x_k$ and $m \in [1, \infty),\ p \in \mathbb{R}_+^*$, then:*

$$(2) \qquad \sum_{k=1}^{n} \frac{(aX_n + bx_k)^m}{(cX_n - dx_k)^p} \geq \frac{(an + b)^m}{(cn - d)^p} n^{p-m+1} X_n^{m-p}.$$

Theorem 3. *([3]) If $n \in \mathbb{N}^* - \{1\},\ a, b, x_k \in \mathbb{R}_+^*,\ k \in \{1, ..., n\},\ X_n = \sum_{k=1}^{n} x_k$ and $m, t, u \in [1, \infty)$, such that $aX_n^t > b \max_{1 \leq k \leq n} x_k^t$, then:*

$$(3). \qquad \sum_{k=1}^{n} \frac{x_k^m}{(aX_n^t - bx_k^t)^u} \geq \frac{n^{-m+tu+1}}{(an^t - b)^u} X_n^{m-tu}$$

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It is also necessary to recall the extension of an inequality which is stronger than the Radon's inequality, and was given by [6] and also a consequence.

Theorem 4. *([6]) For every $n \geq 2,\ p > 0,\ a_k > 0,\ x_k \geq 0,\ 1 \leq k \leq n$, it holds:*

$$\frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + ... + \frac{x_n^{p+1}}{a_n^p} \geq \frac{(x_1 + x_2 + ... + x_n)^{p+1}}{(a1 + a_2 + ... + a_n)^p} +$$

$$+p \cdot \max_{1 \leq i < j \leq n} \frac{(x_i + x_j)^{p-1}(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)^p}.$$

Denoting $x_k = \lambda_k a_k,\ 1 \leq k \leq n$, we have the equivalent form:

$$a_1 \lambda_1^{p+1} + a_2 \lambda_2^{p+1} + ... + a_n \lambda_n^{p+1} \geq \frac{(a_1 \lambda_1 + a_2 \lambda_2 + ... + a_n \lambda_n)^{p+1}}{(a_1 + a_2 + ... + a_n)^p} +$$

$$+p \cdot \max_{1 \leq i < j \leq n} \frac{a_i a_j (a_i \lambda_i + a_j \lambda_j)^{p-1}(\lambda_i - \lambda_j)^2}{(a_i + a_j)^p}.$$

Corollary 1. *([6]) For every $n \geq 2,\ p > 0,\ x_k \geq 0,\ 1 \leq k \leq n$, with $s = x_1 + x_2 + ... + x_n$, the following extension of Nesbitt's inequality holds:*

$$\sum_{k=1}^{n} \frac{x_k}{(s - x + k)^p} \geq \frac{1}{s^{p-1}} \cdot \left(\frac{n}{n-1}\right)^p +$$

$$+p \cdot \max_{1 \leq i < j \leq n} \frac{x_i x_j (x_i + x_j)^{p-1}(x_i - x_j)^2}{(s - x_i)(s - x_j)[(x_i + x_j)s - (x_i^2 + x_j^2)]^p}.$$

We use below also the next result, which is given in [9].

Theorem 5. *([9]) For every $n \geq 2,\ p \geq 1,\ a_k \geq 0,\ b_k > 0,\ 1 \leq k \leq n$, the following inequalities hold:*

$$(2.5), \qquad 0 \leq \Delta_n^{[p]}(a;b) \leq p\left(\Delta_n^{[p]}(a;b) - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\Delta_n^{[p-1]}(a;b)\right)$$

and

$$0 \leq \Delta_n^{[p]}(a;b) \leq \frac{p}{4}(M - m)(M_{p-1} - m^{p-1})\left(\sum_{i=1}^n b_i\right),$$

where $m \leq \frac{a_i}{b_1} \leq M$, for $i = 1, ..., n$.

2. The results

In next result an inequality obtained using power series for inequality (2) from Theorem 2, see [2] is given.

Theorem 6. *If $n \in \mathbb{N}^* - \{1\},\ a \in \mathbb{R}_+,\ b, c, d, x_k \in \mathbb{R}_+^*,\ X_n = \sum_{k=1}^n x_k,\ cX_n > d \max_{1 \leq k \leq n} x_k,\ m \in [1, \infty),\ p \in \mathbb{R}_+^*$, and in addition $aX_n + bx_k < 1,\ (\forall)k \in \{1, 2, ..., n\}$ then:*

$$\sum_{k=1}^{n} \frac{aX_n + bx_k}{(cX_n - dx_k)^p(1 - aX_n - bx_k)} \geq \frac{n^{p+1}}{X_n^{p-1}(n - (an + b)X_n)} \cdot \frac{an + b}{(cn - d)^p}$$

Proof. Using inequality (2),

$$\sum_{k=1}^{n} \frac{(aX_n + bx_k)^m}{(cX_n - dx_k)^p} \geq \left(\frac{an+b}{n} X_n\right)^m \cdot \frac{n^{p+1}}{(X_n(cn-d))^p}$$

when $m \in \mathbb{N}^*$ and replacing m by i and summing then for $i \in \{1, 2, ..., m\}$ we obtain

$$\sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(aX_n + bx_k)^i}{(cX_n - dx_k)^p} \geq \sum_{i=1}^{m} \left(\frac{an+b}{n} X_n\right)^i \cdot \frac{n^{p+1}}{(X_n(cn-d))^p}.$$

Now taking into account the hypothesis, $0 < aX_n + bx_k < 1$, $(\forall) k \in \{1, 2, ..., n\}$ and $a \in \mathbb{R}_+$, $b, c, d, x_k \in \mathbb{R}_+^*$, $k \in \{1, 2, ..., n\}$ we can notice that,

$$0 \leq \frac{an+b}{n} X_n = aX_n + b\frac{X_n}{n} < aX_n + b \max_{1 \leq k \leq n} < 1.$$

Therefore when m tends to infinity we have

$$\sum_{k=1}^{n} \frac{1}{(cX_n - dx_k)^p} \cdot \frac{aX_n + bx_k}{1 - (aX_n + bx_k)} \geq \frac{n^{p+1}}{X_n^p (cn-d)^p} \cdot \frac{(an+b)X_n}{n - (an+b)X_n}.$$

or

$$\sum_{k=1}^{n} \frac{1}{(cX_n - dx_k)^p} \cdot \frac{aX_n + bx_k}{1 - (aX_n + bx_k)} \geq \frac{n^{p+1}}{X_n^{p-1}(n - (an+b)X_n)} \frac{an+b}{(cn-d)^p}.$$

∎

Now we give below a form of inequality (1) from Theorem 1, see [4], obtained using a power series approach, see [7] and [5].

Theorem 7. *For* a_k, $x_k > 0$, $\{1, 2, ..., n\}$, $n \in \mathbb{N}$, $n \geq 2$ *if* $x_k < a_k$, $k \in \{1, 2, ..., n\}$ *the inequality takes place,*

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i - x_i} \geq \frac{(\sum_{i=1}^{n} x_i)^2}{\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} x_i} + \max_{1 \leq i < j \leq n} \{\frac{x_i^2}{a_i - x_i} + \frac{x_j^2}{a_j - x_j} - \frac{(x_i + x_j)^2}{a_i + a_j - (x_i + x_j)}\}.$$

Proof. We use the same method like before. By inequality (1), we have,

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p} + \max_{1 \leq i < j \leq n} \{\frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i + x_j)^{p+1}}{(a_i + a_j)^p}\} \geq$$

$$\geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p} + \frac{x_i^{p+1}}{a_i^p} + \frac{x_j^{p+1}}{a_j^p} - \frac{(x_i + x_j)^{p+1}}{(a_i + a_j)^p},$$

$(\forall)\ 1 \leq i < j \leq n$.

Taking into account inequality (1) when $p \in \mathbb{N}^*$ is replaced by l and summing for $l \in \{1, 2, ..., p\}$, we obtain:

$$\sum_{l=1}^{p} \sum_{k=1}^{n} a_k \left(\frac{x_k}{a_k}\right)^{l+1} \geq \sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_k\right) \cdot \left(\frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} a_k}\right)^{l+1} +$$

$$+ a_i \sum_{l=1}^{p} \left(\frac{x_i}{a_i}\right)^{l+1} + a_j \sum_{l=1}^{p} \left(\frac{x_j}{a_j}\right)^{l+1} - (a_i + a_j) \sum_{l=1}^{p} \left(\frac{x_i + x_j}{a_i + a_j}\right)^{l+1},$$

$(\forall)\ 1 \leq i < j \leq n$.

When p tends to infinity, because $\frac{x_k}{a_k} < 1,\ k \in \{1, 2, ..., n\}$ and $\frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n a_k} < 1$ the series are convergent and we have,

$$\sum_{k=1}^{n} a_k \left(\frac{1}{1 - \frac{x_k}{a_k}} - 1 - \frac{x_k}{a_k} \right) \geq \left(\sum_{k=1}^{n} a_k \right) \left(\frac{1}{1 - \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n a_k}} - 1 - \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n a_k} \right) +$$

$$+a_i(\frac{1}{1-\frac{x_i}{a_i}} - 1 - \frac{x_i}{a_i}) + a_j(\frac{1}{1-\frac{x_j}{a_j}} - 1 - \frac{x_j}{a_j}) - (a_i + a_j)(\frac{1}{1-\frac{x_i+x_j}{a_i+a_j}} - 1 - \frac{x_i + x_j}{a_i + a_j}),$$

$(\forall)\ 1 \leq i < j \leq n.$

Therefore

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i - x_i} \geq \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n a_i - \sum_{i=1}^n x_i} + \max_{1 \leq i < j \leq n} \{ \frac{x_i^2}{a_i - x_i} + \frac{x_j^2}{a_j - x_j} - \frac{(x_i + x_j)^2}{a_i + a_j - (x_i + x_j)} \}.$$

∎

We can also see what will become the inequality from Theorem 2, see [6] by using power series method.

Theorem 8. *For* $a_k,\ x_k > 0,\ \{1, 2, ..., n\}, n \in \mathbb{N},\ n \geq 2$ *if* $x_k < a_k,\ k \in \{1, 2, ..., n\}$ *the inequality takes place,*

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i - x_i} \geq \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n a_i - \sum_{i=1}^n x_i} +$$

$$+ \max_{1 \leq i < j \leq n} \{ \frac{(a_i + a_j)(a_i x_j - a_j x_i)^2}{a_i a_j} \cdot \frac{1}{[a_i + a_j - (x_i + x_j)]^2} \}.$$

Proof. We use the inequality from Theorem 4 and we have

$$\frac{x_1^{p+1}}{a_1^p} + \frac{x_2^{p+1}}{a_2^p} + ... + \frac{x_n^{p+1}}{a_n^p} \geq \frac{(x_1 + x_2 + ... + x_n)^{p+1}}{(a1 + a_2 + ... + a_n)^p} +$$

$$+p \cdot \max_{1 \leq i < j \leq n} \frac{(x_i + x_j)^{p-1}(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)^p} \geq \frac{(x_1 + x_2 + ... + x_n)^{p+1}}{(a1 + a_2 + ... + a_n)^p} +$$

$$+p \frac{(x_i + x_j)^{p-1}(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)^p},$$

$(\forall)\ 1 \leq i < j \leq n.$

When $p \in \mathbb{N}^*$ is replaced by l and then summing for $l \in \{1, 2, ..., p\}$, we obtain:

$$\sum_{l=1}^{p} \sum_{i=1}^{n} a_i \frac{x_i^{l+1}}{a_i^{l+1}} - \sum_{l=1}^{p} (a_i + a_j) \frac{(x_1 + x_2 + ... + x_n)^{l+1}}{(a_1 + a_1 + ... + a_n)^{l+1}} \geq$$

$$\geq \sum_{l=1}^{p} l \cdot \frac{(x_i + x_j)^{l-1}(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)^l} = \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)} \sum_{l=1}^{p} l \cdot \left(\frac{x_i + x_j}{a_i + a_j} \right)^{l-1},$$

$(\forall)\ 1 \leq i < j \leq n.$

By hypothesis, $x_i < a_i,\ 1 \leq i < j \leq n$ we see that $\frac{x_i + x_j}{a_i + a_j} < 1,\ 1 \leq i < j \leq n$ and therefore when p tends to infinity we obtain,

$$\sum_{i=1}^{n} \frac{x_i^2}{a_i - x_i} - \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n a_i - \sum_{i=1}^n x_i} \geq \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)} \frac{1}{\left(1 - \frac{x_i + x_j}{a_i + a_j}\right)^2} =$$

$$= \frac{(a_i + a_j)(a_i x_j - a_j x_i)^2}{a_i a_j} \cdot \frac{1}{[a_i + a_j - (x_i + x_j)]^2},$$

$(\forall)\ 1 \leq i < j \leq n.$

This implies the required inequality, if we take maximum for all $1 \leq i < j \leq n,\ i, j \in \mathbb{N}$ in the right side of the last inequality.

The well-known equality,

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$$

where $0 < x < 1$, see [5] was also used before.

∎

In next result also the power series is used in order to see what will become the inequality form Corollary 3, see [6] under more restrictive conditions on the numbers $x_1, ..., x_n$.

Corollary 2. *For every $n \geq 2,\ x_k \geq 0,\ 1 \leq k \leq n$, with $s = x_1 + x_2 + ... + x_n$, and $s - 1 > \max_{1 \leq i < j \leq n,\ 1 \leq k \leq n}\{x_k, \frac{1}{n-1}, \frac{x_i^2 + x_j^2}{x_i + x_j}\}$ the following inequality takes place:*

$$\sum_{k=1}^{n} \frac{x_k}{s - 1 - x_k} \geq \frac{ns}{n(s-1) - s} +$$

$$+ \max_{1 \leq i < j \leq n} \frac{x_i x_j (x_i - x_j)^2}{(s - x_i)(s - x_j)} \cdot \frac{[(x_i + x_j)s - (x_i^2 + x_j^2)]}{[(x_i + x_j)(s-1) - (x_i^2 + x_j^2)]^2}.$$

Proof. Using the hypothesis we have, $s - x_k > 1,\ s - 1 > \frac{1}{n-1}$ and $x_i + x_j < (x_i + x_j)s - (x_i^2 + x_j^2)$ or $\frac{1}{s - x_k} < 1,\ \frac{n}{(n-1)s} < 1$ and $\frac{x_i + x_j}{(x_i + x_j)s - (x_i^2 + x_j^2)} < 1$ so if we change p in l, where $l \in \{1, 2, ..., p\}$ and $p \in \mathbb{N}^*$ and take the below sum, we obtain,

$$\sum_{l=1}^{p} \sum_{k=1}^{n} \frac{x_k}{(s - x_k)^l} \geq \sum_{l=1}^{p} \frac{1}{s^{l-1}} \cdot \left(\frac{n}{n-1}\right)^l +$$

$$+ \sum_{l=1}^{p} l \cdot \frac{x_i x_j (x_i + x_j)^{l-1} (x_i - x_j)^2}{(s - x_i)(s - x_j)[(x_i + x_j)s - (x_i^2 + x_j^2)]^l}$$

and when p tends to infinity we have,

$$\sum_{k=1}^{n} x_k \frac{\frac{1}{s - x_k}}{1 - \frac{1}{s - x_k}} \geq \frac{n}{n-1} \cdot \frac{s(n-1)}{s(n-1) - n} + \frac{x_i x_j (x_i - x_j)^2 [(x_i + x_j)s - (x_i^2 + x_j^2)]}{[(x_i + x_j)(s-1) - (x_i^2 + x_j^2)]^2},$$

$(\forall)\ 1 \leq i < j \leq n.$

Then using the same technique as in previous theorem we obtain the desired maximum and the required inequality.

∎

Considering Theorem 3 we can obtain below two different inequalities. First case is when m tends to infinity and the second inequality is obtained when u tends to infinity. Then we study the case when and m and u tends to infinity.

Theorem 9. *If* $n \in \mathbb{N}^* - \{1\}$, $a, b, x_k \in \mathbb{R}_+^*$, $k \in \{1, ..., n\}$, $X_n = \sum_{k=1}^n x_k$ *and* $t, u \in [1, \infty)$, *such that* $aX_n^t > b \max_{1 \leq k \leq n} x_k^t$, *and* $x_k < 1$, $k \in \{1, ..., n\}$ *then:*

$$\sum_{k=1}^n \frac{1}{\left(aX_n^t - bx_k^t\right)^u} \cdot \frac{x_k}{1 - x_k} \geq \frac{n^{tu+1}}{\left(an^t - b\right)^u X_n^{tu-1}} \cdot \frac{1}{n - X_n}.$$

Proof. Like in previous demonstration summing when m tends to infinity we have,

$$\sum_{k=1}^n \frac{1}{\left(aX_n^t - bx_k^t\right)^u} \left(\frac{1}{1 - x_k} - 1\right) \geq \frac{n^{tu+1}}{\left(an^t - b\right)^u X_n^{tu}} \left(\frac{1}{1 - \frac{X_n}{n}} - 1\right).$$

∎

Theorem 10. *If* $n \in \mathbb{N}^* - \{1\}$, $a, b, x_k \in \mathbb{R}_+^*$, $k \in \{1, ..., n\}$, $X_n = \sum_{k=1}^n x_k$ *and* $t, m \in [1, \infty)$, $t \in \mathbb{N}$ *such that* $aX_n^t > bx_k^t + 1$, $(\forall)$ $k \in \{1, ..., n\}$ *then:*

$$\sum_{k=1}^n \frac{x_k^m}{aX_n^t - bx_k^t - 1} \geq \frac{n^{t+1-m} X_n^{m-t}}{an^t - b - \left(\frac{n}{X_n}\right)^t}.$$

Proof. In inequality (3), we consider l instead of u, $l \in \{1, ..., u\}$ and u a natural number, $u \geq 1$ and summing we obtain:

$$\sum_{l=1}^u \sum_{k=1}^n \frac{x_k^m}{\left(aX_n^t - bx_k^t\right)^l} \geq \sum_{l=1}^u \frac{n^{-m+tl+1}}{\left(an^t - b\right)^l} X_n^{m-tl}.$$

Because $aX_n^t > bx_k^t + 1$, $(\forall)$ $k \in \{1, ..., n\}$ and $a, b, x_k \in \mathbb{R}_+^*$ results $naX_n^t \geq b \sum_{k=1}^n x_k^t + n$ or $aX_n^t \geq b \frac{\sum_{k=1}^n x_k^t}{n} + 1$. It is known that

$$\left(\frac{x_1 + ... + x_n}{n}\right)^t \leq \frac{1}{n}(x_1^t + ... + x_n^t)$$

if $t \in \mathbb{N} \bigcup \{0\}$, see [5] and therefore

$$aX_n^t - b\frac{X_n^t}{n^t} > 1$$

or

$$\frac{n^t}{X_n^t} \cdot \frac{1}{an^t - b} < 1.$$

If u tends to infinity we have

$$\sum_{k=1}^n x_k^m \left(\frac{1}{1 - \frac{1}{aX_n^t - bx_k^t}} - 1\right) \geq n^{1-m} X_n^m \left(\frac{1}{1 - \left(\frac{n}{X_n}\right)^t \frac{1}{an^t - b}} - 1\right).$$

∎

Consequence 1. *If* $n \in \mathbb{N}^* - \{1\}$, $a, b, x_k \in \mathbb{R}_+^*$, $k \in \{1, ..., n\}$, $X_n = \sum_{k=1}^n x_k$ *and* $t, m \in [1, \infty)$, $t \in \mathbb{N}$ *such that* $aX_n^t > bx_k^t + 1$, $(\forall)$ $k \in \{1, ..., n\}$ *and* $x_k < 1$, $k \in \{1, ..., n\}$ *then:*

$$\sum_{k=1}^n \frac{1}{aX_n^t - bx_k^t - 1} \cdot \frac{x_k}{1 - x_k} \geq \frac{n^{t+1}}{X_n^t(an^t - b) - n^t} \cdot \frac{X_n}{n - X_n}.$$

It would be also interesting to see what will become some applications of Theorem 1 and Theorem 2, see [3].

Now we see what will become the inequalities (2.5) and (2.6) from Theorem 2.3, see [9] if on numbers a_i, b_i M and m we have some restrictions.

Theorem 11. *For every $n \geq 2$, $a_k \geq 0$, $b_k > 0$, $1 \leq k \leq n$, the following inequalities hold:*

$$0 \leq \sum_{i=1}^{n} \frac{a_i b_i}{b_i - a_i} - \frac{\left(\sum_{i=1}^{n} b_i\right)\left(\sum_{i=1}^{n} a_i\right)}{\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i} \leq$$

$$\leq \sum_{i=1}^{n} \frac{a_i b_i^2}{(b_i - a_i)^2} - \frac{\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)^2}{\left(\sum_{i=1}^{n} b_i\right)^2 - \left(\sum_{i=1}^{n} a_i\right)^2} -$$

$$-\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}\left(\sum_{i=1}^{n} \frac{b_i^3}{(b_i - a_i)^2} - \frac{\left(\sum_{i=1}^{n} b_i\right)^3}{\left(\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i\right)^2}\right),$$

and

$$0 \leq \sum_{i=1}^{n} \frac{a_i b_i}{b_i - a_i} - \frac{\sum_{i=1}^{n} b_i \sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i} \leq \frac{1}{4}(M - m)\left(\frac{1}{(1-M)^2} - \frac{1}{(1-m)^2}\right)\sum_{i=1}^{n} b_i,$$

where $m \leq \frac{a_i}{b_i} \leq M < 1$, $(\forall)\ i \in \{1, ..., n\}$.

Proof. We deduce the inequality by the same technique as before, taking into account that the condition $m \leq \frac{a_i}{b_i} \leq M < 1$, $(\forall)\ i \in \{1, ..., n\}$ result $\frac{a_i}{b_i} < 1$ and $\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} < 1$, $(\forall)\ i \in \{1, ..., n\}$.
∎

We shall enunciate below the integral form of this last inequality using the same techniques as in [1].

Theorem 12. *Let $f(x) \geq 0$, $g(x) > 0$ and if $f, g : [a, b] \to \mathbb{R}_+$ be two integrable functions on $[a, b]$ with $m \leq \frac{f(x)}{g(x)} \leq M$, $(\forall)\ x \in [a, b]$ and $M < 1$ then*

$$0 \leq \int_a^b \frac{f(x)g(x)}{g(x) - f(x)} dx - \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx - \int_a^b f(x)dx} \leq$$

$$\leq \int_a^b \frac{f(x)g^2(x)dx}{g(x) - f(x)^2} dx - \frac{\int_a^b f(x)dx \left(\int_a^b g(x)dx\right)^2}{\left(\int_a^b g(x)dx\right)^2 - \left(\int_a^b f(x)dx\right)^2} -$$

$$-\frac{\int_a^b f(x)dx}{\int_a^b g(x)dx} \cdot \left(\int_a^b \frac{g^3(x)}{(g(x) - f(x))^2} dx - \frac{\left(\int_a^b g(x)dx\right)^3}{\left(\int_a^b g(x)dx - \int_a^b f(x)dx\right)^2}\right)$$

and

$$0 \leq \int_a^b \frac{f(x)g(x)}{g(x) - f(x)} dx - \frac{\left(\int_a^b g(x)dx\right)^2}{\int_a^b g(x)dx - \int_a^b f(x)dx} \leq$$

$$\leq \frac{1}{4}(M-m)\left(\frac{1}{(1-M)^2}-\frac{1}{(1-m)^2}\right)\int_a^b g(x)dx.$$

Proof. Let $n \in \mathbb{N}$ and $x_k = k + \frac{b-a}{n}$, $k \in \{0,1,...,n\}$. We use previous theorem, we put $f(x_k)$ instead of a_k and $g(x_k)$ instead of b_k , then we multiply by $\frac{b-a}{n}$ and obtain the corresponding Riemann sums of the functions $\frac{fg}{g-f}$, f, g, $\frac{fg^2}{(g-f)^2}$ and $\frac{g^3}{(g-f)^2}$ below in our inequality:

$$0 \leq \sigma\left(\frac{fg}{g-f},\Delta_n,x_k\right)-\frac{\sigma(f,\Delta_n,x_k)\,\sigma(g,\Delta_n,x_k)}{\sigma(g,\Delta_n,x_k)-\sigma(f,\Delta_n,x_k)} \leq$$

$$\leq \sigma\left(\frac{fg^2}{(g-f)^2},\Delta_n,x_k\right)-\frac{\sigma(f,\Delta_n,x_k)\left(\sigma(g,\Delta_n,x_k)\right)^2}{\left(\sigma(g,\Delta_n,x_k)\right)^2-\left(\sigma(f,\Delta_n,x_k)\right)^2}-$$

$$-\frac{\sigma(f,\Delta_n,x_k)}{\sigma(g,\Delta_n,x_k)}\left(\sigma\left(\frac{g^3}{(g-f)^2},\Delta_n,x_k\right)-\frac{\left(\sigma(g,\Delta_n,x_k)\right)^3}{\left(\sigma(g,\Delta_n,x_k)-\sigma(f,\Delta_n,x_k)\right)^2}\right)$$

and

$$0 \leq \sigma\left(\frac{fg}{g-f},\Delta_n,x_k\right)-\frac{\sigma(f,\Delta_n,x_k)\,\sigma(g,\Delta_n,x_k)}{\sigma(g,\Delta_n,x_k)-\sigma(f,\Delta_n,x_k)} \leq$$

$$\leq \frac{1}{4}(M-m)\left(\frac{1}{(1-M)^2}-\frac{1}{(1-m)^2}\right)\sigma(g,\Delta_n,x_k),$$

where $\Delta_n = (x_0,x_1,...,x_n)$ is the division, x_k are the intermediate points and $m \leq \frac{f(x_k)}{g(x_k)} \leq M < 1$, $(\forall)\ k \in \{1,...,n\}$. When n tends to infinity, we obtain the inequalities.

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