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ERROR BOUNDS FOR TRAPEZOID TYPE QUADRATURE RULES WITH APPLICATIONS FOR THE MEAN AND VARIANCE

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Abstract. In this paper, we establish some inequalities of trapezoid type to give tight bounds for the expectation and variance of a probability density function. The approach is also demonstrated for higher order moments.

Key words and Phrases: trapezoid rule, first moment, second moment, expectation, variance
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1. Introduction

The trapezoid rule is a method to approximate the integral $\int_a^b f(x)\,dx$, by approximating the area under the curve of $f(x)$ as a trapezoid:

$$\int_a^b f(x)\,dx \approx (b-a)\frac{f(a)+f(b)}{2}.$$

Some inequalities have been established to give bounds for the error of this approximation, and we summarised the result in the following proposition (cf. Cerone and Dragomir [5]).

Proposition 1.1. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a function and $u, v \in I$ with $u < v$. Consider the approximation of the integral of g on $[u, v]$ by the trapezoid rule, that is, find abound for the quantity:*

$$\left|\frac{g(u)+g(v)}{2}(v-u) - \int_u^v g(t)\,dt\right|. \tag{1}$$

The following bounds for (1) holds for any $u, v \in I$ with $u < v$:
a. If $g \in BV[u, v]$, then

$$\left|\frac{g(u)+g(v)}{2}(v-u) - \int_u^v g(t)\,dt\right| \le \frac{1}{2}|v-u|\left|\bigvee_u^v (g)\right|. \tag{2}$$

b. If g is Lipschitz continuous with Lipschitz constant L, then

$$\left|\frac{g(u)+g(v)}{2}(v-u) - \int_u^v g(t)\,dt\right| \le \frac{1}{4}L(v-u)^2 \tag{3}$$

c. If g'' exists and bounded, then

$$\left|\frac{g(u)+g(v)}{2}(v-u) - \int_u^v g(t)\,dt\right| \le \frac{1}{12}(v-u)^3\|g''\|_\infty. \tag{4}$$

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d. If the first derivative of g exists and is absolutely continuous on $[u,v]$, then

$$\left|\frac{g(u)+g(v)}{2}(v-u)-\int_u^v g(t)\,dt\right| \tag{5}$$
$$\leq \begin{cases} \frac{1}{4}\|g'\|_\infty (v-u)^2, & \text{if } g' \in L_\infty[u,v]; \\ \frac{\|g'\|_p (v-u)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}, & \text{if } g' \in L_p[u,v],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \frac{1}{2}\|g'\|_1 (v-u), & \text{if } g' \in L_1[u,v]. \end{cases}$$

We refer the reader to Cerone and Dragomir [5] for more details on the trapezoidal type inequalities.

One of the applications of integral inequalities is to obtain bounds for the expectation, variance and moments of continuous random variables defined over a finite interval [1]. In Barnett et. al. [1], it is noted that some Ostrowski type inequalities may be used to obtain these bounds (see, for example, Brnetić and Pečarić [2]). We refer the readers to the monograph by Barnett et al. [1], for an overview of these inequalities.

There are other inequalities which provides bounds for means and variances. Chernoff [7], for instance, proved that for any Gaussian random variable X and an absolutely function G, we have $\mathrm{Var}(G(X)) \leq E(G'(X))^2$. This inequality is then generalised with higher-order derivatives in Houdré and Kagan [9]. A characterisation of distributions (normal, gamma, negative binomial or Poisson) is given in [10] by means of a Chernoff type inequality. We refer to the papers by Cacaoullos [3], Cacoullos and Papadatos [4], Chang and Richards [6] and Dharmadhikari and Joag-Dev [8], for further inequalities involving variances.

In this paper, we aim to provide some inequalities of trapezoid type to give tight bounds for the expectation and variance of a probability density function f. In Section 2, we give approximations for the first and second moments of a function $f:[a,b]\to\mathbb{R}$ around the midpoint of the domain, i.e.,

$$\int_a^b \left(x-\frac{a+b}{2}\right) f(x)\,dx \quad \text{and} \quad \int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx.$$

We make use of the trapezoid type inequalities to obtain error bounds for the approximation. In Section 4, we apply the results to obtain bounds for the expectation and variance of a probability density function f. Remark 2.10 demonstrates the applicability of the approach for higher order moments.

2. Main results

Firstly, we note that inequality (4) also holds when we weaken the assumption, as presented in the next proposition.

Proposition 2.1. *Let $g: I \to \mathbb{R}$ be a function and and $u,v \in I$ with $u<v$. If g' is absolutely continuous and $g'' \in L_\infty[u,v]$, then,*

$$\left|\frac{g(u)+g(v)}{2}(v-u)-\int_u^v g(t)\,dt\right| \leq \frac{1}{12}(v-u)^3\|g''\|_\infty,$$

for all $u,v\in I$.

Proof. Since g'' exists almost everywhere, we have

$$\begin{aligned}\frac{1}{2}\int_u^v (t-u)(v-t)g''(t)\,dt &= \frac{1}{2}\left[(t-u)(v-t)g'(t)\Big|_u^v - \int_u^v (u+v-2t)g'(t)\,dt\right]\\ &= \frac{1}{2}\left[(2t-u-v)g(t)\Big|_u^v - 2\int_u^v g(t)\,dt\right]\\ &= \frac{g(u)+g(v)}{2}(v-u) - \int_u^v g(t)\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\frac{g(u)+g(v)}{2}(v-u) - \int_u^v g(t)\,dt\right| &\le \frac{1}{2}\int_u^v (t-u)(v-t)|g''(t)|\,dt\\ &\le \frac{1}{2}\|g''\|_\infty \int_u^v (t-u)(v-t)\,dt\\ &= \frac{1}{12}\|g''\|_\infty (v-u)^3,\end{aligned}$$

as desired. □

2.1. **Error bounds for the first moment approximation.** Utilising (1) we have the following approximation for the first moment of a function f.

Lemma 2.2. *Let $f:[a,b]\to\mathbb{R}$ be an integrable function. We have the following approximation for the first moment of f:*

$$\int_a^b \left(x-\frac{a+b}{2}\right) f(x)\,dx \approx \frac{b-a}{3}\left(\int_{\frac{a+b}{2}}^b f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx\right). \tag{6}$$

Proof. Setting $f\equiv g$, $u=\frac{a+b}{2}$ and $v=x$ in (1), we have

$$\left|\frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^x f(t)\,dt\right|.$$

Integrating the above on $[a,b]$, we have:

$$\left|\int_a^b \frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right)\,dx - \int_a^b\left(\int_{\frac{a+b}{2}}^x f(t)\,dt\right)\,dx\right|. \tag{7}$$

Now,

$$\int_a^b \frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right)\,dx = \frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)f(x)\,dx.$$

We also have

$$\begin{aligned}&\int_a^b\left(\int_{\frac{a+b}{2}}^x f(t)\,dt\right)\,dx\\ =\ &\left(x-\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^x f(t)\,dt\Big|_a^b - \int_a^b\left(x-\frac{a+b}{2}\right)f(x)\,dx\\ =\ &\frac{b-a}{2}\int_{\frac{a+b}{2}}^b f(t)dt - \frac{b-a}{2}\int_a^{\frac{a+b}{2}} f(t)dt - \int_a^b\left(x-\frac{a+b}{2}\right)f(x)\,dx.\end{aligned}$$

Thus, (7) becomes

$$\left|\frac{3}{2}\int_a^b\left(x-\frac{a+b}{2}\right)f(x)\,dx - \frac{b-a}{2}\left(\int_{\frac{a+b}{2}}^b f(t)dt - \int_a^{\frac{a+b}{2}} f(t)dt\right)\right|.$$

Multiplying the above by $\frac{2}{3}$ completes the proof. □

Let f be an integrable real-valued function defined on $[a,b]$, and set

$$T_1(f) := \int_a^b \left(x - \frac{a+b}{2}\right) f(x)\,dx - \frac{b-a}{3}\left(\int_{\frac{a+b}{2}}^b f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx\right).$$

In the next theorem, we give bounds for $|T_1|$, i.e. the error bounds for the approximation in Lemma 2.2, for different classes of functions.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function.*

a. If f of bounded variation on $[a,b]$, then,

$$|T_1(f)| \le \begin{cases} \dfrac{(b-a)^2}{12}\left[\dfrac{1}{2}\bigvee_a^b(f) + \dfrac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(f) - \bigvee_{\frac{a+b}{2}}^b(f)\right|\right] \\ \dfrac{b-a}{6}\displaystyle\int_a^b \left|\bigvee_{\frac{a+b}{2}}^x(f)\right| dx \end{cases} \le \frac{1}{6}(b-a)^2 \bigvee_a^b(f). \tag{8}$$

b. If f is L-Lipschitz, then

$$|T_1(f)| \le \frac{1}{72} L(b-a)^3. \tag{9}$$

c. If f' is absolutely continuous and $f'' \in L_\infty[a,b]$, then

$$|T_1(f)| \le \frac{1}{576}\|f''\|_\infty (b-a)^4. \tag{10}$$

d. If f is differentiable and f' is absolutely continuous, then

$$|T_1(f)| \le \begin{cases} \dfrac{1}{72}\|f'\|_\infty (b-a)^3, & \text{if } f' \in L_\infty[a,b]; \\ \dfrac{q\|f'\|_p (b-a)^{2+\frac{1}{q}}}{3(2q+1)(q+1)^{\frac{1}{q}} 2^{1+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \dfrac{1}{12}\|f'\|_1 (b-a)^2, & \text{if } f' \in L_1[a,b]. \end{cases} \tag{11}$$

The proof of Theorem 2.3 is presented in Section 3. In the next propositions, we present the sharpness of the constants for some of the inequalities in Theorem 2.3.

Proposition 2.4. *The constant $\frac{1}{12}$ in the first case of (8) is best possible.*

Proof. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$f(x) = \begin{cases} 1, & a \le x < \frac{a+b}{2}; \\ 0, & x = \frac{a+b}{2}; \\ -1, & \frac{a+b}{2} < x \le b. \end{cases}$$

We have

$$\bigvee_a^{\frac{a+b}{2}}(f) = 1, \quad \bigvee_{\frac{a+b}{2}}^b(f) = 1, \text{ and } \bigvee_a^b(f) = 2.$$

Let us assume that (8) holds for constants $A > 0$ instead $\frac{1}{12}$ i.e.

$$\begin{aligned} &\left|\int_a^b \left(x - \frac{a+b}{2}\right) f(x)\,dx - \frac{b-a}{3}\left(\int_{\frac{a+b}{2}}^b f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx\right)\right| \\ &\le A(b-a)^2\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(f) - \bigvee_{\frac{a+b}{2}}^b(f)\right|\right]. \end{aligned} \tag{12}$$

With the above choice of f, we observe the terms on the left hand side of (12) in the following:

$$\int_a^b \left(x - \frac{a+b}{2}\right) f(x)\,dx = -\frac{1}{4}(b-a)^2;$$

$$\int_{\frac{a+b}{2}}^b f(x)\,dx = -\frac{(b-a)}{2}, \text{ and } \int_a^{\frac{a+b}{2}} f(x)\,dx = \frac{b-a}{2}.$$

We observe the term on the right hand side of (12) in the following:

$$\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(f) - \bigvee_{\frac{a+b}{2}}^b(f)\right| = 1.$$

Thus, (12) becomes:

$$\frac{1}{12}(b-a)^2 \le A(b-a)^2,$$

which asserts that $A \ge \frac{1}{12}$. □

Proposition 2.5. *The constant* $\frac{1}{576}$ *in* (10) *is best possible.*

Proof. Let us assume that (10) holds for constants $B > 0$ instead $\frac{1}{576}$ i.e.

$$\begin{aligned}(13)\qquad &\left|\int_a^b \left(x - \frac{a+b}{2}\right) f(x)\,dx - \frac{b-a}{3}\left(\int_{\frac{a+b}{2}}^b f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx\right)\right| \\ \le\ & B\|f''\|_\infty (b-a)^4.\end{aligned}$$

Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$f(x) = \begin{cases} \left(x - \frac{a+b}{2}\right)^2, & a \le x \le \frac{a+b}{2}; \\ -\left(x - \frac{a+b}{2}\right)^2, & \frac{a+b}{2} < x \le b. \end{cases}$$

We note that f'' exists almost everywhere and $\|f''\|_\infty = 2$. With this choice of f, the left-hand side of (13) becomes

$$\begin{aligned} &\left|\int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right)^3 dx - \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right)^3 dx \right. \\ &\left. -\frac{b-a}{3}\left(\int_{\frac{a+b}{2}}^b -\left(x - \frac{a+b}{2}\right)^2 dx - \int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right)^2 dx\right)\right| \\ =\ & \left|-\frac{1}{64}(b-a)^4 - \frac{1}{64}(b-a)^4 + \frac{1}{72}(b-a)^4 + \frac{1}{72}(b-a)^4\right| \\ =\ & \frac{1}{288}(b-a)^4. \end{aligned}$$

The right-hand side of (13) becomes

$$B\|f''\|_\infty (b-a)^4 = 2B(b-a)^4.$$

Thus, (13) becomes

$$\frac{1}{288}(b-a)^4 \le 2B(b-a)^4$$

which asserts that $B \ge \frac{1}{576}$. □

2.2. **Error bounds for the second moment approximation.** Utilising (1) we have the following approximation for the second moment of a function f.

Lemma 2.6. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. We have the following approximation for f:*

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx \approx \frac{1}{8}(b-a)^2 \int_a^b f(t)\,dt - \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3. \tag{14}$$

Proof. Set $f \equiv g$, $u = \frac{a+b}{2}$ and $v = x$ in (1) and let $F(x) = \int_a^x f(t)\,dt$ to obtain:

$$\left| F(x) - F\left(\frac{a+b}{2}\right) - \frac{f(x) + f(\frac{a+b}{2})}{2}\left(x - \frac{a+b}{2}\right)\right|. \tag{15}$$

If we multiply (15) with $|x - \frac{a+b}{2}|$, we get

(16)
$$\left|\left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right) - \frac{1}{2}\left[f(x) + f\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right)^2\right|.$$

Integrate (16) on $[a,b]$, we have:

$$\begin{aligned} &\left|\int_a^b \left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right) dx \right. \\ &\left. -\frac{1}{2}\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx - \frac{1}{2} f\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx\right|. \end{aligned} \tag{17}$$

Now, observe that

$$\begin{aligned} & \int_a^b \left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right) dx \\ = & \int_a^b F(x)\left(x - \frac{a+b}{2}\right) dx \\ = & \frac{1}{2}\int_a^b F(x) d\left(\left(x - \frac{a+b}{2}\right)^2\right) \\ = & \frac{1}{2}\left[F(x)\left(x - \frac{a+b}{2}\right)^2\Big|_a^b - \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx\right] \\ = & \frac{1}{2}\left[(F(b) - F(a))\frac{(b-a)^2}{4} - \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx\right] \\ = & \frac{1}{8}(b-a)^2\int_a^b f(t)\,dt - \frac{1}{2}\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx; \end{aligned}$$

and

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx = \frac{1}{3}\left(x - \frac{a+b}{2}\right)^3\Big|_a^b = \frac{1}{12}(b-a)^3.$$

Then, (17) becomes

$$\left|\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx - \frac{1}{8}(b-a)^2\int_a^b f(t)\,dt + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3\right|$$

as desired. □

Let $f : [a,b] \to \mathbb{R}$ be an integrable function and set

$$T_2(f) := \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx - \frac{1}{8}(b-a)^2 \int_a^b f(t)\,dt + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3.$$

In the next theorem, we have bounds for $|T_2|$, i.e. the error bounds for the approximation in Lemma 2.6, for different classes of functions.

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function.*
a. If f is of bounded variation, then

$$|T_2(f)| \le \frac{1}{48} \bigvee_a^b (f)(b-a)^3. \tag{18}$$

b. If f is L-Lipschitz function, then

$$|T_2(f)| \le \frac{1}{128} L(b-a)^4. \tag{19}$$

c. If f' is absolutely continuous and $f'' \in L_\infty[a,b]$, then

$$|T_2(f)| \le \frac{1}{960} \|f''\|_\infty (b-a)^5. \tag{20}$$

d. If f' exists and is absolutely continuous, then

$$|T_2(f)| \le \begin{cases} \dfrac{1}{128}\|f'\|_\infty (b-a)^4, & \text{if } f' \in L_\infty[a,b]; \\ \dfrac{q\|f'\|_p (b-a)^{3+\frac{1}{q}}}{(3q+1)(q+1)^{\frac{1}{q}} 2^{3+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{24}\|f'\|_1 (b-a)^3 & \text{if } f' \in L_1[a,b]. \end{cases} \tag{21}$$

The proof of Theorem 2.7 is presented in Section 3. In the next propositions, we present the sharpness of the constants for some of the inequalities in Theorem 2.7.

Proposition 2.8. *The constant $\frac{1}{48}$ in (18) is best possible.*

Proof. We now prove the sharpness of the constant $\frac{1}{48}$. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$f(x) = \begin{cases} 0, & x = \frac{a+b}{2}; \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, $\bigvee_a^b (f) = 2$. Let us assume that (18) holds for a constant $C > 0$ instead of $\frac{1}{48}$, i.e.

$$\left| \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx - \frac{1}{8}(b-a)^2 \int_a^b f(t)\,dt + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 \right| \le C \bigvee_a^b (f)(b-a)^3. \tag{22}$$

With the above choice of f, we observe the terms on the left hand side of (22) in the following:

$$\frac{1}{8}(b-a)^2 \int_a^b f(t)\,dt = \frac{1}{8}(b-a)^3;$$

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx = \frac{1}{12}(b-a)^3; \text{ and } \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 = 0.$$

Thus, (22) becomes:

$$\frac{1}{24}(b-a)^3 \le 2C(b-a)^3$$

which asserts that $C \geq \frac{1}{48}$, hence the constant $\frac{1}{48}$ is best possible. □

Proposition 2.9. *The constant* $\frac{1}{960}$ *in* (20) *is best possible.*

Proof. Assume that (20) holds for a constant K instead of $\frac{1}{960}$, i.e.,

$$\left|\int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx - \frac{1}{8}(b-a)^2\int_a^b f(t)\,dt + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3\right| \leq K\|f''\|_\infty (b-a)^5. \tag{23}$$

Let $f:[a,b]\to\mathbb{R}$ defined by $f(x)=\frac{1}{2}(x-\frac{a+b}{2})^2$. So, $f''(x)=1$ for all $x\in[a,b]$ and thus $\|f''\|_\infty = 1$. Therefore, (23) becomes

$$\frac{1}{960}(b-a)^5 \leq K(b-a)^5$$

which yields $K \geq \frac{1}{960}$. □

Remark 2.10. Utilising a similar technique to that of Lemma 2.6, we are able to obtain the higher order moments can be derived from (1). Set $f\equiv g$, $u=\frac{a+b}{2}$ and $v=x$ in (1), let $F(x)=\int_a^x f(t)\,dt$, multiply with $|x-\frac{a+b}{2}|^n$ $(n\geq 1)$ and integrate with respect to x on $[a,b]$

$$\int_a^b \left(x-\frac{a+b}{2}\right)^{n+1} f(x)\,dx \approx \frac{2(n+1)}{n+3}\left[\frac{1}{n+1}\left(\frac{b-a}{2}\right)^{n+1}\int_a^b f(t)\,dt - \int_a^{\frac{a+b}{2}} f(x)\,dx \int_a^b \left(x-\frac{a+b}{2}\right)^n dx - \frac{1}{2} f\left(\frac{a+b}{2}\right)\int_a^b\left(x-\frac{a+b}{2}\right)^{n+1} dx\right].$$

The integral $\int_a^b \left(x-\frac{a+b}{2}\right)^k dx$ vanishes when k is odd; and when k is even,

$$\int_a^b \left(x-\frac{a+b}{2}\right)^k dx = \frac{(b-a)^{k+1}}{2^k(k+1)}.$$

3. Proof of the main theorems

3.1. **Proof of Theorem 2.3.** Let f be a function of bounded variation on $[a,b]$. Setting $f\equiv g$, $u=x$ and $v=\frac{a+b}{2}$ in (2), we have

$$\left|\frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^x f(t)\,dt\right| \leq \frac{1}{2}\left|x-\frac{a+b}{2}\right|\left|\bigvee_{\frac{a+b}{2}}^x (f)\right|.$$

Integrating the above on $[a,b]$, we have

$$\begin{aligned}
&\left|\int_a^b \frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right)dx - \int_a^b\left(\int_{\frac{a+b}{2}}^x f(t)\,dt\right)dx\right| \\
\leq\ & \int_a^b \left|\frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right) - \int_{\frac{a+b}{2}}^x f(t)\,dt\right| dx \\
\leq\ & \frac{1}{2}\int_a^b \left|x-\frac{a+b}{2}\right|\left|\bigvee_{\frac{a+b}{2}}^x (f)\right| dx.
\end{aligned} \tag{24}$$

Following the proof of Lemma 2.2, the first term of (24) is $\frac{3}{2}|T_1(f)|$. Furthermore, we have

$$\int_a^b \left|x-\frac{a+b}{2}\right|\left|\bigvee_{\frac{a+b}{2}}^{x}(f)\right|dx \le \begin{cases} \max\limits_{x\in[a,b]}\left|\bigvee_{\frac{a+b}{2}}^{x}(f)\right|\int_a^b\left|x-\frac{a+b}{2}\right|dx \\ \max\limits_{x\in[a,b]}\left|x-\frac{a+b}{2}\right|\int_a^b\left|\bigvee_{\frac{a+b}{2}}^{x}(f)\right|dx \end{cases}$$

$$\le \begin{cases} \max\left\{\bigvee_a^{\frac{a+b}{2}}(f), \bigvee_{\frac{a+b}{2}}^{b}(f)\right\}\dfrac{(b-a)^2}{4} \\ \dfrac{b-a}{2}\int_a^b\left|\bigvee_{\frac{a+b}{2}}^{x}(f)\right|dx \end{cases}$$

$$\le \begin{cases} \dfrac{(b-a)^2}{4}\left[\dfrac{1}{2}\bigvee_a^b(f)+\dfrac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^{b}(f)\right|\right] \\ \dfrac{b-a}{2}\int_a^b\left|\bigvee_{\frac{a+b}{2}}^{x}(f)\right|dx \end{cases} =: I.$$

Thus, (24) becomes

$$\frac{3}{2}|T_1(f)| \le \frac{1}{2}I \le \frac{1}{4}\bigvee_a^b(f).$$

Multiplying the above by $\frac{2}{3}$ gives us (8).

Let f be L-Lipschitz. We apply similar steps as above and utilise (3) to obtain

$$\frac{3}{2}|T_1(f)| \le \int_a^b \frac{1}{4}L\left|x-\frac{a+b}{2}\right|^2 dx = \frac{1}{4}L\left(\frac{1}{3}\left(x-\frac{a+b}{2}\right)^3\Big|_a^b\right) = \frac{1}{48}L(b-a)^3.$$

Multiply the above with $\frac{2}{3}$ gives us (9).

Let f' is absolutely continuous and $f'' \in L_\infty[a,b]$. We apply similar steps as above and utilise Proposition 2.1 to obtain

$$\begin{aligned} \frac{3}{2}|T_1(f)| &\le \int_a^b \frac{1}{12}\|f''\|_\infty\left|x-\frac{a+b}{2}\right|^3 dx \\ &= \frac{1}{12}\|f''\|_\infty\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^3 dx + \int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)^3 dx\right] \\ &= \frac{1}{48}\|f''\|_\infty\left[-\left(\frac{a+b}{2}-x\right)^4\Big|_a^{\frac{a+b}{2}} + \left(x-\frac{a+b}{2}\right)^4\Big|_{\frac{a+b}{2}}^b\right] \\ &= \frac{1}{384}\|f''\|_\infty(b-a)^4. \end{aligned}$$

Multiply the above with $\frac{2}{3}$ gives us (10).

Let f' exists and absolutely continuous. We apply similar steps as above and utilise (5). We have

$$\int_a^b \frac{1}{4}\|f'\|_\infty\left(x-\frac{a+b}{2}\right)^2 dx = \frac{1}{48}\|f'\|_\infty(b-a)^3.$$

The second case of the right hand side of (5) becomes

$$\begin{aligned}
&\frac{1}{2(q+1)^{\frac{1}{q}}}\|f'\|_p \int_a^b \left|x-\frac{a+b}{2}\right|^{1+\frac{1}{q}} dx \\
=\ & \frac{1}{2(q+1)^{\frac{1}{q}}}\|f'\|_p \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}} dx + \int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)^{1+\frac{1}{q}} dx\right] \\
=\ & \frac{q}{(2q+1)(q+1)^{\frac{1}{q}}2^{2+\frac{1}{q}}}\|f'\|_p (b-a)^{2+\frac{1}{q}}.
\end{aligned}$$

The third case of the right hand side of (5) becomes

$$\begin{aligned}
&\int_a^b \frac{1}{2}\|f'\|_1 \left|x-\frac{a+b}{2}\right| dx \\
=\ & \frac{1}{2}\|f'\|_1 \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right) dx + \int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right) dx\right] = \frac{1}{8}\|f'\|_1 (b-a)^2.
\end{aligned}$$

Thus we have

$$\frac{3}{2}|T_1(f)| \le \begin{cases} \frac{1}{48}\|f'\|_\infty (b-a)^3, & \text{if } f' \in L_\infty[a,b]; \\ \frac{q}{(2q+1)(q+1)^{\frac{1}{q}}2^{2+\frac{1}{q}}}\|f'\|_p (b-a)^{2+\frac{1}{q}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{8}\|f'\|_1 (b-a)^2, & \text{if } f' \in L_1[a,b]. \end{cases}$$

Multiply the above with $\frac{2}{3}$ gives us (11).

3.2. **Proof of Theorem 2.7.** Let f be of bounded variation. Let $F(x) = \int_a^x f(t)\,dt$, we have the following by (2):

(25)
$$\left|F(x) - F\left(\frac{a+b}{2}\right) - \frac{f(x)+f(\frac{a+b}{2})}{2}\left(x-\frac{a+b}{2}\right)\right| \le \frac{1}{2}\left|x-\frac{a+b}{2}\right| \left|\bigvee_x^{\frac{a+b}{2}}(f)\right|.$$

If we multiply (25) with $|x-\frac{a+b}{2}|$, we get

(26)
$$\begin{aligned}
&\left|\left[F(x) - F\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right) - \frac{1}{2}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]\left(x-\frac{a+b}{2}\right)^2\right| \\
\le\ & \frac{1}{2}\left|x-\frac{a+b}{2}\right|^2 \left|\bigvee_x^{\frac{a+b}{2}}(f)\right|.
\end{aligned}$$

Integrate (26) on $[a,b]$ and follow the proof of Lemma 2.6 we have:

$$\begin{aligned}
|T_2(f)| &\le \frac{1}{2}\int_a^b \left|x-\frac{a+b}{2}\right|^2 \left|\bigvee_x^{\frac{a+b}{2}}(f)\right| dx \\
&= \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} \left(x-\frac{a+b}{2}\right)^2 \bigvee_x^{\frac{a+b}{2}}(f)\,dx + \int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)^2 \bigvee_{\frac{a+b}{2}}^x(f)\,dx\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\bigvee_{a}^{\frac{a+b}{2}}(f)\int_{a}^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)^{2}dx+\frac{1}{2}\bigvee_{\frac{a+b}{2}}^{b}(f)\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{2}dx\\
&= \frac{1}{2}\bigvee_{a}^{b}(f)\int_{a}^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)^{2}dx\\
&= \frac{1}{2}\bigvee_{a}^{b}(f)\frac{1}{3}\left(x-\frac{a+b}{2}\right)^{2}\Bigg|_{a}^{\frac{a+b}{2}}=\frac{1}{48}\bigvee_{a}^{b}(f)(b-a)^{3}.
\end{aligned}
$$

Let f be L-Lipschitz. We apply similar steps as above and utilise (3) to obtain:

$$
\begin{aligned}
|T_2(f)| &\leq \frac{1}{4}L\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{3}dx\\
&= \frac{1}{4}L\left[\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{3}dx+\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{3}dx\right]\\
&= \frac{1}{16}L\left[-\left(\frac{a+b}{2}-x\right)^{4}\Bigg|_{a}^{\frac{a+b}{2}}+\left(x-\frac{a+b}{2}\right)^{4}\Bigg|_{\frac{a+b}{2}}^{b}\right]=\frac{1}{128}L(b-a)^{4}.
\end{aligned}
$$

Let f be a function such that f' is absolutely continuous and $f''\in L_\infty[a,b]$. We apply similar steps as above and utilise Proposition 2.1 to obtain:

$$
\begin{aligned}
|T_2(f)|\leq\frac{1}{12}\|f''\|_\infty\int_{a}^{b}\left(x-\frac{a+b}{2}\right)^{4}dx &= \frac{1}{60}\|f''\|_\infty\left(x-\frac{a+b}{2}\right)^{5}\Bigg|_{a}^{b}\\
&= \frac{1}{960}\|f''\|_\infty(b-a)^{5}.
\end{aligned}
$$

Let f be a function such that f' exists and absolutely continuous. We apply similar steps as above and utilise (5) to obtain:

$$
\frac{1}{4}\|f'\|_\infty\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{3}dx = \frac{1}{128}\|f'\|_\infty(b-a)^{4}.
$$

The second case in the right hand side of (5) becomes

$$
\begin{aligned}
&\frac{1}{2(q+1)^{\frac{1}{q}}}\|f'\|_p\int_{a}^{b}\left|x-\frac{a+b}{2}\right|^{2+\frac{1}{q}}dx\\
&= \frac{1}{2(q+1)^{\frac{1}{q}}}\|f'\|_p\left[\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2+\frac{1}{q}}dx+\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)^{2+\frac{1}{q}}dx\right]\\
&= \frac{q}{(3q+1)(q+1)^{\frac{1}{q}}2^{3+\frac{1}{q}}}\|f'\|_p(b-a)^{3+\frac{1}{q}}.
\end{aligned}
$$

The last case in the right hand side of (5) becomes

$$
\frac{1}{2}\|f'\|_1\int_{a}^{b}\left(x-\frac{a+b}{2}\right)^{2}dx = \frac{1}{24}\|f'\|_1(b-a)^{3}.
$$

This completes the proof.

4. Applications to mean and variance

In this section we provide some applications of Theorems 2.3 and 2.7 to obtain bounds for expectation and variance of a probability density function.

4.1. **Applications to expectations.** Let f be a probability density function on $[a,b]$. Let $E_{[a,b]}(f) := \int_a^b xf(x)\,dx$. Thus, T_1 becomes

$$\begin{aligned}
&\int_a^b \left(x-\frac{a+b}{2}\right)f(x)\,dx - \frac{b-a}{3}\left(\int_{\frac{a+b}{2}}^b f(x)dx - \int_a^{\frac{a+b}{2}} f(x)dx\right)\\
=\ & E_{[a,b]}(f) - \frac{a+b}{2}\int_a^b f(x)\,dx - \frac{b-a}{3}\int_{\frac{a+b}{2}}^b f(x)dx + \frac{b-a}{3}\int_a^{\frac{a+b}{2}} f(x)dx\\
=\ & E_{[a,b]}(f) - \int_{\frac{a+b}{2}}^b f(x)dx\left[\frac{a+b}{2}+\frac{b-a}{3}\right] - \int_a^{\frac{a+b}{2}} f(x)dx\left[\frac{a+b}{2}-\frac{b-a}{3}\right]\\
=\ & E_{[a,b]}(f) - \frac{a+5b}{6}\int_{\frac{a+b}{2}}^b f(x)dx - \frac{5a+b}{6}\int_a^{\frac{a+b}{2}} f(x)dx =: T_E(f).
\end{aligned}$$

Then, we have the following results for $f : [a,b] \to \mathbb{R}$:

1. If f is of bounded variation, then

$$|T_E(f)| \le \frac{1}{6}(b-a)^2 \bigvee_a^b (f). \tag{27}$$

2. If f is L-Lipschitz function, then

$$|T_E(f)| \le \frac{1}{72}L(b-a)^3. \tag{28}$$

3. If f' is absolutely continuous and $f'' \in L_\infty[a,b]$, then

$$|T_E(f)| \le \frac{1}{576}\|f''\|_\infty (b-a)^4. \tag{29}$$

4. If f is differentiable and f' is absolutely continuous, then

$$|T_E(f)| \le \begin{cases} \frac{1}{72}\|f'\|_\infty (b-a)^3, & \text{if } f' \in L_\infty[a,b];\\ \frac{q\|f'\|_p (b-a)^{2+\frac{1}{q}}}{3(2q+1)(q+1)^{\frac{1}{q}}2^{1+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{12}\|f'\|_1 (b-a)^2, & \text{if } f' \in L_1[a,b]. \end{cases} \tag{30}$$

Consider

$$\delta:\ a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and set

$$\begin{aligned}
T_E^\delta(f) \ :=\ & E_{[a,b]}(f)\\
& -\sum_{i=0}^{n-1}\left(\frac{x_i+5x_{i+1}}{6}\int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} f(x)dx + \frac{5x_i+x_{i+1}}{6}\int_{x_i}^{\frac{x_i+x_{i+1}}{2}} f(x)dx\right).
\end{aligned}$$

Write (27) for $[x_i, x_{i+1}]$, $i \in \{0,\dots,n-1\}$ and then use the generalised triangle inequality, we get:

$$\begin{aligned}
|T_E^\delta(f)| \ &\le\ \frac{1}{6}\sum_{i=0}^{n-1} h_i^2 \bigvee_{x_i}^{x_{i+1}}(f)\\
&\le\ \frac{1}{6}\max_{i\in\{0,\dots,n-1\}} h_i^2 \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f) = \frac{1}{2}\Delta^2(\delta)\bigvee_a^b (f)
\end{aligned}$$

where $\Delta(\delta) := \max_{i\in\{0,\dots,n-1\}} h_i$ and $h_i = x_{i+1} - x_i$, assuming f is of bounded variation.

Now, f is L-Lipschitz function, (28) becomes

$$|T_E^\delta(f)| \leq \frac{1}{72} L \sum_{i=0}^{n-1} h_i^3 \leq \frac{1}{72} L \Delta^3(\delta).$$

Now, f is twice differentiable and f'' is bounded, (29) becomes

$$|T_E^\delta(f)| \leq \frac{1}{576} \|f''\|_\infty \sum_{i=0}^{n-1} h_i^4 \leq \frac{1}{576} \Delta^4(\delta) \|f''\|_\infty.$$

Now, f is differentiable and f'' is absolutely continuous, (30) becomes

$$\begin{aligned} |T_E^\delta(f)| &\leq \begin{cases} \frac{1}{72}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^3, & \text{if } f' \in L_\infty[a,b]; \\ \frac{q\|f'\|_p \sum_{i=0}^{n-1} h_i^{2+\frac{1}{q}}}{3(2q+1)(q+1)^{\frac{1}{q}} 2^{1+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{12}\|f'\|_1 \sum_{i=0}^{n-1} h_i^2, & \text{if } f' \in L_1[a,b]; \end{cases} \\ &\leq \begin{cases} \frac{1}{72}\|f'\|_\infty \Delta^3(\delta), & \text{if } f' \in L_\infty[a,b]; \\ \frac{q\|f'\|_p \Delta^{2+\frac{1}{q}}(\delta)}{3(2q+1)(q+1)^{\frac{1}{q}} 2^{1+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{12}\|f'\|_1 \Delta^2(\delta), & \text{if } f' \in L_1[a,b]. \end{cases} \end{aligned}$$

Remark 4.1. We note that $T_E(f)$ can be simplified as follows:

$$T_E(f) = E_{[a,b]}(f) - \frac{a+5b}{6} + \frac{2(b-a)}{3} \int_a^{\frac{a+b}{2}} f(x)dx;$$

or

$$T_E(f) = E_{[a,b]}(f) - \frac{5a+b}{6} - \frac{2(b-a)}{3} \int_{\frac{a+b}{2}}^b f(x)dx;$$

in which the partition over $[a,b]$ can be halved.

4.2. **Applications to variance.** Let f be a probability density function on $[a,b]$. Let $\mathrm{Var}_{[a,b]}(f) := \int_a^b x^2 f(x)\,dx - [E_{[a,b]}(f)]^2$. Thus,

$$\begin{aligned} &\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx \\ =\ & \int_a^b x^2 f(x)\,dx - (a+b)\int_a^b x f(x)\,dx + \left(\frac{a+b}{2}\right)^2 \int_a^b f(x)\,dx \\ =\ & \mathrm{Var}_{[a,b]}(f) + [E_{[a,b]}(f)]^2 - (a+b)E_{[a,b]}(f) + \left(\frac{a+b}{2}\right)^2. \end{aligned}$$

Therefore, T_2 becomes

$$\begin{aligned}
&\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x)\,dx - \frac{1}{8}(b-a)^2 \int_a^b f(x)\,dx + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 \\
= \quad & \mathrm{Var}_{[a,b]}(f) + [E_{[a,b]}(f)]^2 - (a+b)E_{[a,b]}(f) + \left(\frac{a+b}{2}\right)^2 \\
& -\frac{1}{8}(b-a)^2 + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 \\
= \quad & \mathrm{Var}_{[a,b]}(f) + \left[E_{[a,b]}(f) - \frac{a+b}{2}\right]^2 - \frac{1}{8}(b-a)^2 + \frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 =: T_V(f).
\end{aligned}$$

Then, we have the following results:
1. If f is of bounded variation, then

$$|T_V(f)| \le \frac{1}{48} \bigvee_a^b (f)(b-a)^3. \tag{31}$$

2. If f is L-Lipschitz function, then

$$|T_V(f)| \le \frac{1}{128} L(b-a)^4. \tag{32}$$

3. If f' is absolutely continuous and $f'' \in L_\infty[a,b]$, then

$$|T_V(f)| \le \frac{1}{960} \|f''\|_\infty (b-a)^5. \tag{33}$$

4. If f' exists and is absolutely continuous, then

$$|T_V(f)| \le \begin{cases} \frac{1}{128}\|f'\|_\infty (b-a)^4, & \text{if } f' \in L_\infty[a,b]; \\ \frac{q\|f'\|_p (b-a)^{3+\frac{1}{q}}}{(3q+1)(q+1)^{\frac{1}{q}} 2^{3+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{24}\|f'\|_1 (b-a)^3, & \text{if } f' \in L_1[a,b]. \end{cases} \tag{34}$$

Consider

$$\delta : \ a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and set

$$\begin{aligned}
T_V^\delta(f) \quad := \quad & \mathrm{Var}_{[a,b]}(f) + \sum_{i=1}^{n-1} \Bigg[\left(E_{[x_i,x_{i+1}]}(f) - \frac{x_i + x_{i+1}}{2}\right)^2 \\
& -\frac{1}{8}(x_{i+1}-x_i)^2 + \frac{1}{24} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1}-x_i)^3 \Bigg].
\end{aligned}$$

Write (31) for $[x_i, x_{i+1}]$, $i \in \{0, \ldots, n-1\}$ and then use the generalised triangle inequality, we get:

$$\begin{aligned}
|T_V^\delta(f)| \quad \le \quad & \frac{1}{48} \sum_{i=0}^{n-1} h_i^3 \bigvee_{x_i}^{x_{i+1}} (f). \\
\le \quad & \frac{1}{48} \max_{i \in \{0,\ldots,n-1\}} h_i^3 \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f) = \frac{1}{2}\Delta^3(\delta) \bigvee_a^b (f)
\end{aligned}$$

where $\Delta(\delta) := \max_{i \in \{0,\ldots,n-1\}} h_i$ and $h_i = x_{i+1} - x_i$, assuming f is of bounded variation.

Now, f is L-Lipschitz function, (32) becomes

$$|T_V^\delta(f)| \quad \le \quad \frac{1}{128} L \sum_{i=1}^{n-1} h_i^4 \le \frac{1}{128} L \Delta^4(\delta).$$

Now, f is twice differentiable and f'' is bounded, (33) becomes

$$|T_V^\delta(f)| \leq \frac{1}{960}\|f''\|_\infty \sum_{i=1}^{n-1} h_i^5 \leq \frac{1}{960}\|f''\|_\infty \Delta^5(\delta).$$

Now, f is differentiable and f'' is absolutely continuous, (34) becomes

$$\begin{aligned} |T_V^\delta(f)| &\leq \begin{cases} \dfrac{1}{128}\|f'\|_\infty \displaystyle\sum_{i=1}^{n-1} h_i^4, & \text{if } f' \in L_\infty[a,b]; \\ \dfrac{q\|f'\|_p \displaystyle\sum_{i=1}^{n-1} h_i^{3+\frac{1}{q}}}{(3q+1)(q+1)^{\frac{1}{q}} 2^{3+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{24}\|f'\|_1 \displaystyle\sum_{i=1}^{n-1} h_i^3, & \text{if } f' \in L_1[a,b]; \end{cases} \\ &\leq \begin{cases} \dfrac{1}{128}\|f'\|_\infty \Delta^4(\delta), & \text{if } f' \in L_\infty[a,b]; \\ \dfrac{q\|f'\|_p \Delta^{3+\frac{1}{q}}(\delta)}{(3q+1)(q+1)^{\frac{1}{q}} 2^{3+\frac{1}{q}}}, & \text{if } f' \in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{24}\|f'\|_1 \Delta^3(\delta), & \text{if } f' \in L_1[a,b]. \end{cases} \end{aligned}$$

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