Received 06/09/13

ON SEVERAL INEQUALITIES OBTAINED FROM GENERALIZATIONS OF YOUNG'S INEQUALITY BY A POWER SERIES APPROACH

LOREDANA CIURDARIU

ABSTRACT. The aim of this paper is to give different inequalities which result from various generalizations of Young inequalities in some specific cases for a and b using the power series method used by C. Mortici in his paper, [10]. Also several integral forms for some inequalities deduced by power series method starting from certain inequalities which generalize Radon's inequality will be presented.

1. INTRODUCTION

We will recall the inequality of J. Radon which was published in [11].

For every real numbers $p > 0$, $x_k \geq 0$, $a_k > 0$ for $1 \leq k \leq n$, we have the following inequality:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}, \quad p > 0.$$

In [12], the authors consider two n-tuples $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_1, ..., b_n)$ where $ab = (a_1 b_1, a_2 b_2, ..., a_n b_n)$ and $a^m = (a_1^m, a_2^m, ..., a_n^m)$, for any real number m. Then $a > 0$ and $b > 0$ if $a_i > 0$ and $b_i > 0$ for every $1 < i < n$. We consider the expression:

$$\Delta_n^{[p]}(a; b) := \sum_{i=1}^{n} \frac{a_i^p}{b_i^{p-1}} - \frac{(\sum_{i=1}^{n} a_i)^p}{(\sum_{i=1}^{n} b_i)^{p-1}}, \tag{1.1}$$

for real number $p > 1$ and for n-tuples $a \geq 0$ and $b \geq 0$.

Then the well-known Radon's inequality can be written as:

$$\Delta_n^{[p]}(a; b) \geq 0. \tag{1.2}$$

The scalar Heinz inequality says that if $a, b \geq 0$ and $0 \leq \nu \leq 1$ then,

$$a^\nu b^{1-\nu} + a^{1-\nu} b^\nu \leq a + b$$

and the scalar Young inequality says that under the same hypothesis for a and b we have

$$a^\nu b^{1-\nu} \leq \nu a + (1 - \nu) b$$

with equality if and only if $a = b$.

Date: August 18, 2013.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. power series.

Theorem 1. *([12]) For $n \geq 2,\ p \geq 1$, we have the following inequalities:*

$$\Delta_n^{[p]}(a;b) \geq \max_{1\leq i<j\leq n}\left[\frac{a_i^p}{b_i^{p-1}}+\frac{a_j^p}{b_j^{p-1}}-\frac{(a_i+a_j)^p}{(b_i+b_j)^{p-1}}\right], \tag{2.16}$$

and

$$0 \leq \Delta_n^{[p]}(a;b) \leq \left[M^p+m^p-\frac{(M+m)^p}{2^{p-1}}\right]\left(\sum_{i=1}^n b_i\right),$$

where $m \leq \frac{a_i}{b_i} \leq M,\ a_i \geq,\ b_i > 0,\ 1 \leq i \leq n$.

Theorem 2. *([12]) There is the inequality:*
(2.19)

$$0 \leq \Delta_n^{[p]}(a;b) \leq \frac{[(M+m)\sum_{i=1}^n b_i-\sum_{i=1}^n a_i]^p}{(\sum_{i=1}^n b_i)^{p-1}}-\frac{(M+m)^p}{2^{p-1}}\left(\sum_{i=1}^n b_i\right)+\left(\sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}}\right),$$

where $m \leq \frac{a_i}{b_i} \leq M,\ a_i \geq 0,\ b_i > 0,\ 1 \leq i \leq n,\ p \geq 1,\ n \geq 2$.

Theorem 3. *([12]) If $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are n-tuples then we have the inequality:*

$$\frac{p(p-1)m^{p-2}}{2\sum_{i=1}^n b_i}\sum_{1\leq i<j\leq n}\frac{(a_ib_j-a_jb_i)^2}{b_ib_j} \leq \tag{2.13}$$

$$\leq \Delta_n^{[p]}(a;b) \leq \frac{p(p-1)M^{p-2}}{2\sum_{i=1}^n b_i}\sum_{1\leq i<j\leq n}\frac{(a_ib_j-a_jb_i)^2}{b_ib_j},$$

where $m \leq \frac{a_i}{b_i} \leq M,\ p > 1,\ a_i \geq 0,\ b_i > 0$, for $i = 1, ..., n$.

Theorem 4. *([2]) If $a, b \geq 0$ and $0 \leq \nu \leq 1$ then*

$$(\nu a+(1-\nu)b)^2 \leq (a^\nu b^{1-\nu})^2+s_0^2(a-b)^2, \tag{2.2}$$

where $s_0 = \max\{\nu, 1-\nu\}$.

The following result is a reverse of an inequality obtained by Kittaneh and Manasrah, see [7] or [2], who obtained a refinement of Heinz inequality.

Theorem 5. *([2]) If $a, b \geq 0$ and $0 \leq \nu \leq 1$, then*

$$(a+b)^2 \leq (a^\nu b^{1-\nu}+a^{1-\nu}b^\nu)^2+2s_0(a-b)^2, \tag{2.3}$$

where $s_0 = \max\{\nu, 1-\nu\}$.

The famous arithmetic-geometric mean inequality, called Young inequality was refined by F. Kittaneh and Y. Manasrah in [7]:

$$(1-\nu)a+\nu b \geq a^{1-\nu}b^\nu+r(\sqrt{a}-\sqrt{b})^2,$$

where $r = \min\{\nu, 1-\nu\}$ and the conditions for a, b and ν are as in Young inequality.

In [8], N. Minculete has given a refinement of the Kittaneh-Manasrah inequality in some special cases as an application:

Proposition 1. *For* $0 < a, b \leq 1$ *and* $\lambda \in (0,1)$ *we have:*

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right),$$

where $r = \min\{\lambda, 1-\lambda\}$*,* $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ *and* $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

Lemma 1. *(*[2]*) For* $\nu \in [0,1]$*, and* $a, b \geq 0$ *we have:*

$$\nu a^2 + (1-\nu)b^2 \leq (a^{\nu}b^{1-\nu})^2 + s_0(a-b)^2$$

, where $s_0 = \max\{\nu, 1-\nu\}$.

We also need to use in this paper the second inequality, difference-type reverse inequality from below

Corollary 1. *(*[6]*) For* $a, b > 0$ *and* $\lambda \in [0,1]$ *the following inequalities hold:*
(i) (Ratio-type reverse inequality)

$$a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + b\lambda \leq a^{1-\lambda}b^{\lambda}\exp\{\frac{\lambda(1-\lambda)(a-b)^2}{d_1^2}\},$$

where $d_1 = \min\{a, b\}$.
(ii) (Difference-type reverse inequality)

$$a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + b\lambda \leq a^{1-\lambda}b^{\lambda} + \lambda(1-\lambda)\{\log\left(\frac{a}{b}\right)\}^2 d_2,$$

where $d_2 = \max\{a, b\}$.

2. Several inequalities obtained from generalizations of Young's inequality by a power series approach

The following two results present some inequalities using a power series approach starting from inequalities from Theorem 1 and 2, see also [12].

Theorem 6. *For* $n \geq 2$ *the following inequalities hold:*

$$\sum_{i=1}^{n}\frac{a_ib_i}{b_i-a_i} - \frac{\sum_{i=1}^{n}a_i\cdot\sum_{i=1}^{n}b_i}{\sum_{i=1}^{n}(b_i-a_i)} \geq \max_{1\leq i<j\leq n}\{\frac{a_ib_i}{b_i-a_i} + \frac{a_jb_j}{b_j-a_j} - \frac{(a_i+a_j)(b_i+b_j)}{b_i+b_j-(a_i+a_j)}\}$$

if $a_i < b_i$*,* $(\forall)\ i = \overline{1,n}$*, and*

$$0 \leq \sum_{i=1}^{n}\frac{a_ib_i}{b_i-a_i} - \frac{\sum_{i=1}^{n}a_i\cdot\sum_{i=1}^{n}b_i}{\sum_{i=1}^{n}(b_i-a_i)} \leq \left(\frac{M}{1-M} + \frac{m}{1-m} - \frac{2(M+m)}{2-(M+m)}\right)\sum_{i=1}^{n}b_i,$$

if $m \leq \frac{a_i}{b_i} \leq M < 1$*,* $(\forall)\ i = \overline{1,n}$.

Proof. We use the power series method, see[10] and [9], as in [3] or [4] for inequality from Theorem 1, see [12]. ∎

Theorem 7. *For $n \geq 2$ we have,*

$$0 \leq \sum_{i=1}^{n} \frac{a_i b_i}{b_i - a_i} - \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i}{\sum_{i=1}^{n}(b_i - a_i)} \leq$$

$$\leq \sum_{i=1}^{n} b_i \frac{(M+m)\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i}{\sum_{i=1}^{n}(b_i + a_i) - (M+m)\sum_{i=1}^{n} b_i} - 2\frac{M+m}{2-(M+m)} \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \frac{a_i b_i}{b_i - a_i},$$

if $m \leq \frac{a_i}{b_i} \leq M < 1,\ (\forall)\ i = \overline{1,n}$.

Proof. With the same method as in [10] as in [3] or [4] for inequality from Theorem 2, see [12] by calculus we find the required inequality. ∎

Now the integral forms for inequalities from last two theorems will be given in the next theorem:

Theorem 8. *Let $f(x) \geq 0,\ g(x) > 0$ and if $f, g : [a,b] \to \mathbb{R}_+$ be two integrable functions on $[a,b]$ with $m \leq \frac{f(x)}{g(x)} \leq M,\ (\forall)\ x \in [a,b]$ and $M < 1$ then we have*

$$0 \leq \int_a^b \frac{f(x)g(x)}{g(x)-f(x)} dx - \frac{\int_a^b f(x)dx \int_a^b g(x)dx}{\int_a^b [f(x)-g(x)]dx} \leq$$

$$\leq \left(\frac{M}{1-M} + \frac{m}{1-m} - \frac{2(M+m)}{2-M+m} \right) \int_a^b g(x)dx$$

and

$$0 \leq \int_a^b \frac{f(x)g(x)}{g(x)-f(x)} dx - \frac{\int_a^b f(x)dx \int_a^b g(x)dx}{\int_a^b [f(x)-g(x)]dx} \leq$$

$$\leq \int_a^b g(x)dx \frac{(M+m)\int_a^b g(x)dx - \int_a^b f(x)dx}{\int_a^b [f(x)+g(x)]dx - (M+m)\int_a^b g(x)dx} -$$

$$-2\frac{M+m}{2-(M+m)} \int_a^b g(x)dx + \int_a^b \frac{f(x)g(x)}{f(x)-g(x)} dx.$$

Proof. We will use the definition of Riemann integral and the same techniques as in [1], [3]. ∎

A power series variant of inequality from Theorem 4 is presented below, under more particular conditions on a and b.

Theorem 9. *If $0 < a < 1,\ 0 < b < 1$ and $0 \leq \nu \leq 1$ then*

$$\nu^2 \frac{1}{1-a^2} + (1-\nu)^2 \frac{1}{1-b^2} + 2\nu(1-\nu)\frac{1}{1-ab} \leq$$

$$\leq \frac{1}{1-a^{2\nu}b^{2(1-\nu)}} + s_0^2 \cdot \left(\frac{1}{1-a^2} + \frac{1}{1-b^2} - 2\frac{1}{1-ab} \right),$$

where $s_0 = \max\{\nu, 1-\nu\}$.

Under previous conditions we also have:

$$\nu^2 \frac{a^2}{(1-a^2)^2} + (1-\nu)^2 \frac{b^2}{(1-b^2)^2} + 2\nu(1-\nu)\frac{ab}{(1-ab)^2} \leq$$

$$\leq \frac{a^{2\nu}b^{2(1-\nu)}}{(1-a^{2\nu}b^{2(1-\nu)})^2} + s_0^2 \left(\frac{a^2}{(1-a^2)^2} + \frac{b^2}{(1-b^2)^2} - \frac{2ab}{(1-ab)^2} \right).$$

Proof. We will use the power series method(see [10]), inequality (2.2) from Theorem 2.1(Theorem 4), see [2] with a^l instead of a and b^l instead of b and then summing when $l = \overline{1,n}$ we will obtain:

$$\sum_{l=0}^{n}[\nu a^l + (1-\nu)b^l]^2 \le \sum_{l=0}^{n}[(a^{l\nu}b^{l(1-\nu)})^2 + s_0^2(a^l - b^l)^2],$$

or

$$\sum_{l=0}^{n}[\nu^2 a^{2l} + (1-\nu)^2 b^{2l} + 2\nu(1-\nu)a^l b^l] \le \sum_{l=0}^{n}[a^{2l\nu}b^{2l(1-\nu)} + s_0^2(a^{2l} + b^{2l} - 2a^l b^l)].$$

Taking into account the hypothesis, $0 < a < 1$ and $0 < b < 1$ when n tends to infinity, and using the well-known identity,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},\ x \in (0,1),$$

we obtain the desired inequality.

Using the same inequality (2.2) and summing when $l = \overline{1,n}$ we have,

$$\sum_{l=1}^{n} l[\nu^2 a^{2l} + (1-\nu)^2 b^{2l} + 2\nu(1-\nu)a^l b^l] \le \sum_{l=1}^{n} l[a^{2l\nu}b^{2l(1-\nu)} + s_)^2(a^{2l} + b^{2l} - 2a^l b^l)].$$

Now applying the well-known identity,

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2},\ x \in (0,1)$$

in our case we obtain the desired inequality.

∎

The next result is an inequality obtained by the power series method applied to a generalization of Heinz inequality.

Theorem 10. *If* $0 \le a < 1$, $0 \le b < 1$ *and* $0 \le \nu \le 1$ *then*

$$\frac{1}{1-a^2} + \frac{1}{1-b^2} \le \frac{1}{1-a^{2\nu}b^{2(1-\nu)}} + \frac{1}{1-a^{2(1-\nu)}b^{2\nu}} + 2s_0\left(\frac{1}{1-a^2} + \frac{1}{1-b^2} - \frac{2}{1-ab}\right),$$

and

$$\frac{4}{1-\left(\frac{a+b}{2}\right)^2} \le \frac{1}{1-a^{2\nu}b^{2(1-\nu)}} + \frac{1}{1-a^{2(1-\nu)}b^{2\nu}} + 2s_0\left(\frac{1}{1-a^2} + \frac{1}{1-b^2}\right) + \frac{2(1-s_0)}{1-ab},$$

where $s_0 = \max\{\nu, 1-\nu\}$.

Moreover, under previous conditions, the following inequality holds:

$$\frac{a^2}{(1-a^2)^2} + \frac{b^2}{(1-b^2)^2} \le \frac{a^{2\nu}b^{2(1-\nu)}}{(1-a^{2\nu}b^{2(1-\nu)})^2} + \frac{a^{2(1-\nu)}b^{2\nu}}{(1-a^{2(1-\nu)}b^{2\nu})^2} +$$

$$+2s_0\left(\frac{a^2}{(1-a^2)^2} + \frac{b^2}{(1-b^2)^2} - 2\cdot\frac{ab}{(1-ab)^2}\right).$$

Proof. Taking a^l instead of a and b^l instead of b in inequality from Theorem 2.2, see [2] and then summing when $l=\overline{1,n}$, we have,

$$\sum_{l=0}^{n}(a^l+b^l)^2 \le \sum_{l=0}^{n}(a^{l\nu}b^{l(1-\nu)}+a^{l(1-\nu)}b^{l\nu})^2+\sum_{l=0}^{n}2s_0(a^l-b^l)^2.$$

Now for the first inequality by computation we obtain,

$$\sum_{l=0}^{n}(a^{2l}+b^{2l}) \le \sum_{l=0}^{n}(a^{2l\nu}b^{2l(1-\nu)}+a^{2l(1-\nu)}b^{2l\nu})+2s_0\sum_{l=0}^{n}(a^{2l}+b^{2l}-2a^lb^l)$$

and for the second one, using the generalized means inequality, we deduce that

$$4\sum_{l=0}^{n}\left(\frac{a+b}{2}\right)^{2l} \le \sum_{l=0}^{n}(a^{2l\nu}b^{2l(1-\nu)}+a^{2l(1-\nu)}b^{2l\nu}+2a^lb^l)+2s_0\sum_{l=0}^{n}(a^{2l}+b^{2l}-2a^lb^l),$$

and when n tends to infinity, we find the first and the second inequality.

The last inequality will be deduced by

$$\sum_{l=1}^{n}l(a^{2l}+b^{2l}+2a^lb^l) \le \sum_{l=1}^{n}l(a^{2l\nu}b^{2l(1-\nu)}+a^{2l(1-\nu)}b^{2l\nu}+2a^lb^l)+2s_0\sum_{l=1}^{n}l()a^{2l}+b^{2l}-2a^lb^l,$$

taking into account the identity,

$$\sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2},\ x\in(0,1).$$

∎

Theorem 11. *For $0<a,b<1$ and $\lambda\in(0,1)$ the following inequality holds:*

$$r\left(\frac{1}{1-a}+\frac{1}{1-b}-\frac{2}{1-a^{\frac12}b^{\frac12}}\right)+A(\lambda)\frac{ab(1+ab)}{(1-ab)^3}\log^2\left(\frac{a}{b}\right)\le$$

$$\le\lambda\frac{1}{1-a}+(1-\lambda)\frac{1}{1-b}-\frac{1}{1-a^{\lambda}b^{1-\lambda}}\le$$

$$\le(1-r)\left(\frac{1}{1-a}+\frac{1}{1-b}-\frac{2}{1-a^{\frac12}b^{\frac12}}\right)+B(\lambda)\frac{ab(1+ab)}{(1-ab)^3}\log^2\left(\frac{a}{b}\right),$$

where $r=\min\{\lambda,1-\lambda\}$, $A(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$.

Moreover, under previous conditions, we have:

$$r\left(\frac{a}{(1-a)^2}+\frac{b}{(1-b)^2}-\frac{2a^{\frac12}b^{\frac12}}{(1-a^{\frac12}b^{\frac12})^2}\right)+A(\lambda)\frac{ab(a^2b^2+4ab+1)}{(1-ab)^4}\log^2\left(\frac{a}{b}\right)\le$$

$$\le\lambda\frac{a}{(1-a)^2}+(1-\lambda)\frac{b}{(1-b)^2}-\frac{a^{\lambda}b^{1-\lambda}}{(1-a^{\lambda}b^{1-\lambda})^2}\le$$

$$\le(1-r)\left(\frac{a}{(1-a)^2}+\frac{b}{(1-b)^2}-\frac{2a^{\frac12}b^{\frac12}}{(1-a^{\frac12}b^{\frac12})^2}\right)+B(\lambda)\frac{ab(a^2b^2+4ab+1)}{(1-ab)^4}\log^2\left(\frac{a}{b}\right),$$

Proof. Like before, we put a^l instead of a and b^l instead b in inequality from Proposition 1, see [8] and obtain,

$$r(\sqrt{a^l}-\sqrt{b^l})^2+\log^2\left(\frac{a}{b}\right)A(\lambda)l^2a^lb^l\leq$$
$$\leq \lambda a^l+(1-\lambda)b^l-a^{l\lambda}b^{l(1-\lambda)}\leq$$
$$\leq (1-r)(\sqrt{a^l}-\sqrt{b^l})^2+\log^2\left(\frac{a}{b}\right)B(\lambda)l^2a^lb^l.$$

When $l=\overline{1,n}$ we have,

$$r\sum_{l=0}^{n}(\sqrt{a^l}-\sqrt{b^l})^2+\log^2\left(\frac{a}{b}\right)A(\lambda)\sum_{l=0}^{n}l^2a^lb^l\leq$$
$$\leq \lambda\sum_{l=0}^{n}a^l+(1-\lambda)\sum_{l=0}^{n}b^l-\sum_{l=0}^{n}a^{l\lambda}b^{l(1-\lambda)}\leq$$
$$\leq (1-r)\sum_{l=0}^{n}(\sqrt{a^l}-\sqrt{b^l})^2+\log^2\left(\frac{a}{b}\right)B(\lambda)\sum_{l=0}^{n}l^2a^lb^l.$$

Therefore if n tends to infinity, we have

$$r\left(\frac{1}{1-a}+\frac{1}{1-b}-\frac{2}{1-a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)+A(\lambda)S(ab)\log^2\left(\frac{a}{b}\right)\leq$$
$$\leq \lambda\frac{1}{1-a}+(1-\lambda)\frac{1}{1-b}-\frac{1}{1-a^{\lambda}b^{1-\lambda}}\leq$$
$$\leq (1-r)\left(\frac{1}{1-a}+\frac{1}{1-b}-\frac{2}{1-a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)+B(\lambda)S(ab)\log^2\left(\frac{a}{b}\right),$$

because the fractions which appear are the sums of some convergent geometric series and $S(x)=\sum_{n=0}^{\infty}n^2x^n$ has the sum $\frac{x(1+x)}{(1-x)^3}$, for $x\in(0,1)$.

For the second inequality we proceed like before, we multiply by l the same inequality from Proposition 1, but we consider the sum from $l=1$ to n and then when n tends to infinity, we have

$$r\left(\frac{a}{(1-a)^2}+\frac{b}{(1-b)^2}-\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}}{(1-a^{\frac{1}{2}}b^{\frac{1}{2}})^2}\right)+A(\lambda)G(ab)\log^2\left(\frac{a}{b}\right)\leq$$
$$\leq \lambda\frac{a}{(1-a)^2}+(1-\lambda)\frac{b}{(1-b)^2}-\frac{a^{\lambda}b^{1-\lambda}}{(1-a^{\lambda}b^{1-\lambda})^2}\leq$$
$$\leq (1-r)\left(\frac{a}{(1-a)^2}+\frac{b}{(1-b)^2}-\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}}{(1-a^{\frac{1}{2}}b^{\frac{1}{2}})^2}\right)+B(\lambda)G(ab)\log^2\left(\frac{a}{b}\right),$$

where $G(x)=\sum_{n=1}^{\infty}n^3x^n$ for $x\in(0,1)$ and has the sum $x\left(\frac{x(1+x)}{(1-x)^3}\right)'$, i.e. $\frac{x(x^2+4x+1)}{(1-x)^4}$.

In order to compute the sums of two uniform convergent series on $(0,1)$ let us notice that $\frac{S(x)}{x}=\sum_{n=0}^{\infty}n^2x^{n-1}$ and if we denote $\int\frac{S(x)}{x}dx$ by $A(x)$ then $A(x)=\sum_{n=0}^{\infty}nx^n$. Therefore $\frac{A(x)}{x}=\sum_{n=1}^{\infty}nx^{n-1}$ and then $\int\frac{A(x)}{x}dx=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$ for $x\in(0,1)$. This result let us to find $A(x)$ by derivation and then $S(x)$.

For the second sum, we can also see that $\frac{G(x)}{x}=\sum_{n=1}^{\infty}n^3x^{n-1}$ and then $\int\frac{G(x)}{x}=\sum_{n=1}^{\infty}n^2x^n=\frac{x(1+x)}{(1-x)^3}$, for $x\in(0,1)$.

∎

As a particular case of previous theorem, we have a power series variant of inequality from the Proposition 1.1, see [5].

Consequence 1. *If* $0 < a, b < 1$ *and* $r = \min\{\nu, 1-\nu\}$ *then we have,*

$$(1-\nu)\frac{1}{1-a} + \nu\frac{1}{1-b} \geq \frac{1}{1-a^{1-\nu}b^{\nu}} + r\left(\frac{1}{1-a} + \frac{1}{1-b} - 2\frac{1}{1-a^{\frac{1}{2}}b^{\frac{1}{2}}}\right).$$

In the above theorem if we take $\nu = \frac{1}{p}$, a^p instead of a and b^q instead of b then $1-\nu = \frac{1}{q}$ and the following result, Theorem 1, [10], holds:

Consequence 2. *If* $0 < a, b < 1$ *and* $p, q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have:*

$$\frac{q}{1-a^p} + \frac{p}{1-b^q} \geq \frac{pq}{1-ab}$$

and

$$\frac{a^p}{p(1-a^p)^1} + \frac{b^p}{q(1-b^q)^2} \geq \frac{ab}{(1-ab)^2}.$$

Another variant of inequality from Lemma 2.1, see [2], deduced by power series method, will be presented also below.

Consequence 3. *For* $\nu \in (0,1)$ *and* $a, b \in (0,1)$ *the following inequality holds:*

$$\nu\frac{1}{1-a^2} + (1-\nu)\frac{1}{1-b^2} \leq \frac{1}{1-a^{2\nu}b^{2(1-\nu)}} + s_0\left(\frac{1}{1-a^2} + \frac{1}{1-b^2} - 2\frac{1}{1-ab}\right),$$

where $s_0 = \max\{\nu, 1-\nu\}$.

A power series variant of Corollary 2.2 (ii), see [6], is given below:

Proposition 2. *For* $a, b \in (0,1)$ *and* $\lambda \in [0,1]$ *the following inequality takes place:*

$$(1-\lambda)\frac{1}{1-a} + \lambda\frac{1}{1-b} \leq \frac{1}{1-a^{1-\lambda}b^{\lambda}} + \lambda(1-\lambda)\{\log\left(\frac{a}{b}\right)\}^2\frac{d_2(1+d_2)}{(1-d_2)^3},$$

where $d_2 = \max\{a, b\}$.

Proof. We use the same technique and the fact that the sum of the series, $\sum_{n=0}^{\infty} n^2x^n$ is $\frac{x(1+x)}{(1-x)^3}$ when $x \in (0,1)$. ∎

References

[1] D. M. Batinetu-Giurgiu, D. Marghidanu and O. T. Pop, A new generalization of Radon's inequalities and applications, *Creative Math. Inform.*, **20** (2011), No. 1, 62-73.

[2] Chuanjiang He and Limin Zou, Some inequalities involving unitarily invariant norms, *Mathematical Inequalities and Applications*, 12, 4(2012), 767-776.

[3] L. Ciurdariu, *On several inequalities deduced using a power series approach*, submitted.

[4] L. Ciurdariu, *Integral forms for some inequalities*, RGMIA Res. Rep. Coll., 2013, 7 pp.

[5] S. Furuichi, On refined Young inequalities and reverse inequalities, *Journal of Mathematical Inequalities*, Vol. 5, 1 (2011), 21-31.

[6] S. Furuichi, N. Minculete, Alternative reverse inequalities for Young's inequality, *Journal of Mathematical Inequalities*, Vol. 5, Nr. 4 (2011), 595-600.

[7] F. Kittaneh, Y. Manasrah, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.* 361 (2010), 262-269.

[8] N. Minculete, A refinement of the Kittaneh-Manasrah inequalities, *Creat. Math. Inform.* 20 (2011), No. 2, 157-162.

[9] J. Moonja, Inequalities via power series and Cauchy-Schwarz inequality, *J. Koreean Soc. Math. Educ. Ser. B: Pure Appl. Math.*, Vol. 19, Number 3 (August 2012), 305-313.

[10] C. Mortici, A Power Series Approach to Some Inequalities, *The American Mathematical Monthly*, Vol. 119, No. 2(February 2012), pp. 147-151.

[11] J. Radon, Uber die absolut additiven Mengenfunktionen, Wiener Sitzungsber **122** (1913), 1295-1438.

[12] A. Ratiu, N. Minculete, Several refinements and counterparts of Radon's inequality, submitted.

Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara