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HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR DIFFERENTIABLE m-PREINVEX AND (α, m)-PREINVEX FUNCTIONS

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Abstract. In this paper, the notion of m-preinvex and (α, m)-preinvex functions is introduced and then several inequalities of Hermite-Hadamard type for differentiable m-preinvex and (α, m)-preinvex functions are established. The obtained inequalities for m-convex and (α, m)-convex functions, are then extended to functions of several variables.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for every x, $y \in I$ and $t \in [0, 1]$.

The following celebrated double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds for convex functions and is well-known in literature as the Hermite-Hadamard inequality. Both of the inequalities in (1.1) hold in reversed direction if f is concave.

The inequality (1.1) has been a subject of extensive research since its discovery and a number of papers have been written providing noteworthy extensions, generalizations and refinements see for example [6], [7], [25], [26] and [33].

The classical convexity that is stated above was generalized as m-convexity by G. Toader in [30] as follows:

Definition 1. *The function $[0, b^*]$, $b^* > 0$, is said to be m-convex , where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b^*]$ and $t \in 0, 1]$. We say that f is m-concave if $-f$ is m-convex.

Obviously, for $m = 1$ the Definition 1 recaptures the concept of standard convex functions on $[0, b^*]$.

The notion of m-convexity has been further generalized in [14] as it is stated in the following definition:

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Definition 2. *The function* $[0,b^*]$, $b^*>0$, *is said to be* (α,m)*-convex , where* $(\alpha,m)\in[0,1]^2$, *if we have*

$$f\left(tx+m\left(1-t\right)y\right)\le t^{\alpha}f\left(x\right)+m\left(1-t^{\alpha}\right)f\left(y\right)$$

for all $x,y\in[0,b^*]$ *and* $t\in 0,1]$ *.*

It can easily be seen that for $\alpha=1$, the class of m-convex functions are derived from the above definition and for $\alpha=m=1$ a class of convex functions are derivived.

For several results concerning Hermite-Hadamard type inequalities for m-convex and (α,m)-convex functions we refer the interested reader to [8] and [9].

More recently, a number of mathematicians have attempted to generalize the concept of classical convexity. For example in [10], Hason gave the notion of invexity as significant generalization of classical convexity. Ben-Israel and Mond [4] introduced the concept of preinvex functions, which is a special case of invex functions. Let us first recall the definition of preinvexity and some related results.

Let K be a subset in $\mathbb{R}^n$ and let $f:K\to\mathbb{R}$ and $\eta:K\times K\to\mathbb{R}^n$ be continuous functions. Let $x\in K$, then the set K is said to be invex at x with respect to $\eta\left(\cdot,\cdot\right)$, if

$$x+t\eta(y,x)\in K,\forall x,y\in K,t\in[0,1].$$

K is said to be an invex set with respect to η if K is invex at each $x\in K$. The invex set K is also called a η-connected set.

Definition 3. [24] *The function* f *on the invex set* K *is said to be preinvex with respect to* η, *if*

$$f(u+t\eta(v,u))\le\left(1-t\right)f(u)+tf(v),\forall u,v\in K,t\in[0,1].$$

The function f *is said to be preconcave if and only if* $-f$ *is preinvex.*

It is to be noted that every convex function is preinvex with respect to the map $\eta\left(x,y\right)=x-y$ but the converse is not true see for instance [23].

In a recent paper, Noor [17] obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1. [17] *Let* $f:[a,a+\eta(b,a)]\to(0,\infty)$ *be a preinvex function on the interval of the real numbers* K° *(the interior of* K*) and* a, $b\in K^{\circ}$ *with* $a<a+\eta(b,a)$. *Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\le\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f\left(x\right)dx\le\frac{f\left(a\right)+f\left(b\right)}{2}. \tag{1.2}$$

Barani, Ghazanfari and Dragomir in [3], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 2. [3] *Let* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$. *Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. If* $\left|f^{'}\right|$ *is preinvex on* K, *for* *every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.3}$$

Theorem 3. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.4}$$

For several new results on inequalities for preinvex functions, we refer the interested reader to [3] and [27] and the references therein.

In the present paper we first give the concept of m-preinvex and (α, m)-preinvex functions, which generalize the concept of preinvex functions, and then we will present new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are m-preinvex and (α, m)-preinvex. Our results generalize those results presented in very recent paper [3] concerning Hermite-Hadamard type inequalities for preinvex functions. We also present extensions to sveral variables of some inequalities for m-convex and (α, m)-convex functions which are special cases of our established results.

2. Main Results

To establish our main results we first give the following essential definitions and Lemmas:

Definition 4. *The function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$, is said to be m-preinvex with respect to η if*

$$f(u + t\eta(v,u)) \leq (1-t) f(u) + mtf\left(\frac{v}{m}\right)$$

holds for all $u, v \in K$, $t \in [0,1]$ and $m \in (0,1]$. The function f is said to be m-preconcave if and only if $-f$ is m-preinvex.

Definition 5. *The function f on the invex set $K \subseteq [0, b^*]$, $b^* > 0$, is said to be (α, m)-preinvex with respect to η if*

$$f(u + t\eta(v,u)) \leq (1-t^{\alpha}) f(u) + mt^{\alpha} f\left(\frac{v}{m}\right)$$

holds for all $u, v \in K$, $t \in [0,1]$ and $(\alpha, m) \in (0,1] \times (0,1]$. The function f is said to be (α, m)-preconcave if and only if $-f$ is (α, m)-preinvex.

Remark 1. *If in definition 4, $m = 1$, then one obtain the usual definition of preinvexity. If $\alpha = m = 1$, then definition 5 recaptures the usual definition of the the preinvex functions. It is to be noted that every m-preinvex function and (α, m)-preinvex functions are m-convex and (α, m)-convex with respect to $\eta(v,u) = v - u$ respectively.*

Lemma 1. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\ b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L([a, a+\eta(b,a)])$, then the following equality holds:*

$$-\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx = \frac{\eta(b,a)}{2}\int_0^1 (1-2t)\, f^{'}(a+t\eta(b,a))\,dt. \tag{2.1}$$

Now we establish results for functions whose derivatives in absolute values raise to some certain power are m-preinvex and (α, m)-preinvex.

Theorem 4. *Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\ b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L([a, a+\eta(b,a)])$. If $\left|f^{'}\right|$ is m-preinvex on K, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{8}\left[\left|f^{'}(a)\right| + m\left|f^{'}\left(\frac{b}{m}\right)\right|\right]. \tag{2.2}$$

Proof. From lemma 1, we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f^{'}(a+t\eta(b,a))\right|dt. \tag{2.3}$$

Since $\left|f^{'}\right|$ is m-preinvex on K, for every $a,\ b \in K$ and $t \in [0,1]$, $m \in (0,1]$, we have

$$\left|f^{'}(a+t\eta(b,a))\right| \le (1-t)\left|f^{'}(a)\right| + mt\left|f^{'}\left(\frac{b}{m}\right)\right|. \tag{2.4}$$

Hence we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{2}\left[\left|f^{'}(a)\right|\int_0^1 |1-2t|(1-t)\,dt + m\left|f^{'}\left(\frac{b}{m}\right)\right|\int_0^1 |1-2t|\,tdt\right]. \tag{2.5}$$

Since

$$\int_0^1 |1-2t|(1-t)\,dt = \int_0^1 |1-2t|\,tdt$$
$$= \int_0^{\frac{1}{2}} (1-2t)(1-t)\,dt - \int_{\frac{1}{2}}^1 (1-2t)(1-t)\,dt = \frac{1}{4}.$$

We get the desired inequality from (2.5). This completes the proof of theorem 4. □

Corollary 1. *If $\eta(b,a) = b-a$ in theorem 4, then (2.2) reduces to the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|\right]. \tag{2.6}$$

Theorem 5. *Let $K \subseteq [0,b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\ b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $\left|f'\right|^q$ is m-preinvex on K for $q > 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By lemma 1 and using the well known Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.8}$$

Since $\left|f'\right|^q$ is m-preinvex on K, for every $a,\ b \in [a,b]$ with $a < a+\eta(b,a)$ and $m \in (0,1]$, we have

$$\left|f'(a+t\eta(b,a))\right|^q \le (1-t)\left|f'(a)\right|^q + mt\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Hence

$$\int_0^1 \left|f'(a+t\eta(b,a))\right|^q dt \le \int_0^1 \left[(1-t)\left|f'(a)\right|^q + mt\left|f'\left(\frac{b}{m}\right)\right|^q\right] dt$$
$$= \frac{1}{2}\left|f'(a)\right|^q + \frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Moreover, by using basic calculus we have

$$\int_0^1 |1-2t|^p\,dt = \int_0^{\frac{1}{2}} (1-2t)^p\,dt + \int_{\frac{1}{2}}^1 (2t-1)^p\,dt = \frac{1}{p+1}.$$

A usage of the last two inequalities in (2.8) gives the desired result. This completes the proof of theorem 5. □

Corollary 2. *If we take $\eta(b,a) = b-a$ in theorem 5, then (2.7) becomes the following inequality:*

(2.9)
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}$$

A similar result may be stated as follows:

Theorem 6. *Let $K \subseteq [0,b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\ b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $\left|f'\right|^q$ is m-preinvex on K for $q \ge 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{4}\left[\frac{\left|f'(a)\right|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.10}$$

Proof. For $q = 1$, the proof is the same as that of theorem 4. Suppose now that $q > 1$. Using lemma 1 and the well-known power-mean integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.11}$$

Applying the m-preinvex convexity of $\left|f^{'}\right|^{q}$ on K in the second integral on the right side of (2.11), we have

$$\begin{aligned}(2.12)\quad & \int_0^1 |1-2t| \left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|^{q} dt \\ & \leq \int_0^1 |1-2t| \left[(1-t)\left|f^{'}\left(a\right)\right|^{q} + mt\left|f^{'}\left(\frac{b}{m}\right)\right|^{q}\right] dt \\ & = \left|f^{'}\left(a\right)\right|^{q} \int_0^1 |1-2t|\left(1-t\right) dt + m\left|f^{'}\left(\frac{b}{m}\right)\right|^{q} \int_0^1 t\left|1-2t\right| dt \\ & = \frac{1}{4}\left|f^{'}\left(a\right)\right|^{q} + \frac{m}{4}\left|f^{'}\left(\frac{b}{m}\right)\right|^{q}.\end{aligned}$$

Utilizing inequality (2.12) in (2.11), we get the inequality (2.10). This completes the proof of the theorem. □

Corollary 3. *Suppose $\eta\left(b,a\right) = b-a$, then one has the following inequality:*

$$(2.13)\quad \left|\frac{f\left(a\right)+f\left(b\right)}{2} - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f^{'}\left(a\right)\right|^{q} + m\left|f^{'}\left(\frac{b}{m}\right)\right|^{q}}{2}\right]^{\frac{1}{q}}.$$

Remark 2. *For $q=1$, (2.13) reduces to the inequality proved in theorem 4. If $q=\frac{p}{p-1}$ $(p>1)$, we have $4^{p} > p+1$ for $p>1$ and accordingly*

$$\frac{1}{4} < \frac{1}{2\left(p+1\right)^{\frac{1}{p}}}.$$

This reveals that the inequality (2.10) is better than the one given by (2.7) in theorem 5.

Now we give our results for (α, m)-preinvex functions.

Theorem 7. *Let $K \subseteq [0, b^{*}]$, $b^{*} > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $a < a + \eta\left(b,a\right)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L\left(\left[a, a+\eta\left(b,a\right)\right]\right)$. If $\left|f^{'}\right|$ is (α, m)-preinvex on K, then we have the following inequality:*

$$\begin{aligned}(2.14)\quad & \left|\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2} - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx\right| \\ & \leq \frac{\eta\left(b,a\right)}{2}\left[\nu_2\left|f^{'}\left(a\right)\right| + m\nu_1\left|f^{'}\left(\frac{b}{m}\right)\right|\right],\end{aligned}$$

where $\nu_1 = \frac{1+\alpha\cdot 2^{\alpha}}{2^{\alpha}(1+\alpha)(2+\alpha)}$ and $\nu_2 = \frac{1}{2} - \nu_1$.

Proof. From lemma 1, we have

$$\begin{aligned}(2.15)\quad & \left|\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2} - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx\right| \\ & \leq \frac{\eta\left(b,a\right)}{2}\int_0^1 |1-2t|\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right| dt.\end{aligned}$$

Since $\left|f^{'}\right|$ is (α,m)-preinvex on K, we have for every $t\in[0,1]$ that

$$\begin{aligned}(2.16)\quad &\int_0^1 |1-2t|\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|dt\\ &\leq \left|f^{'}\left(a\right)\right|\int_0^1 |1-2t|\left(1-t^{\alpha}\right)dt+m\left|f^{'}\left(\frac{b}{m}\right)\right|\int_0^1 t^{\alpha}\left|1-2t\right|dt\\ &=\left(\frac{1}{2}-\nu_1\right)\left|f^{'}\left(a\right)\right|+m\nu_1\left|f^{'}\left(\frac{b}{m}\right)\right|,\end{aligned}$$

where

$$\int_0^1 |1-2t|\,t^{\alpha}dt=\frac{1+\alpha\cdot 2^{\alpha}}{2^{\alpha}\left(1+\alpha\right)\left(2+\alpha\right)}=\nu_1$$

and

$$\int_0^1 |1-2t|\left(1-t^{\alpha}\right)dt=\frac{1}{2}-\frac{1+\alpha\cdot 2^{\alpha}}{2^{\alpha}\left(1+\alpha\right)\left(2+\alpha\right)}=\frac{1}{2}-\nu_1.$$

Utilizing (2.15) in (2.14), we get the required inequality and hence the proof of the theorem is completed. □

Corollary 4. *If $\eta(b,a)=b-a$ in theorem 7, the we have the inequality:*

$$(2.17)\quad \left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|\leq\frac{b-a}{2}\left[\nu_2\left|f^{'}\left(a\right)\right|+m\nu_1\left|f^{'}\left(\frac{b}{m}\right)\right|\right],$$

where $\nu_1=\frac{1+\alpha\cdot 2^{\alpha}}{2^{\alpha}(1+\alpha)(2+\alpha)}$ and $\nu_2=\frac{1}{2}-\nu_1$.

Theorem 8. *Let $K\subseteq[0,b^*]$, $b^*>0$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and a, $b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f^{'}\in L([a,a+\eta(b,a)])$. If $\left|f^{'}\right|^q$ is (α,m)-preinvex on K, $q>1$, then we have the following inequality:*

$$\begin{aligned}(2.18)\quad &\left|\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2}-\frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)}f\left(x\right)dx\right|\\ &\leq\frac{\eta\left(b,a\right)}{2\left(p+1\right)^{\frac{1}{p}}}\left[\frac{\alpha\left|f^{'}\left(a\right)\right|^q+m\left|f^{'}\left(\frac{b}{m}\right)\right|^q}{1+\alpha}\right]^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using lemma 1 and the Hölder's integral inequality, we have

$$\begin{aligned}(2.19)\quad &\left|\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2}-\frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)}f\left(x\right)dx\right|\\ &\leq\frac{\eta\left(b,a\right)}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

By the (α,m)-preinvexity of $\left|f^{'}\right|^q$, we have for every $t\in[0,1]$

$$\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|^q\leq\left(1-t^{\alpha}\right)\left|f^{'}\left(a\right)\right|^q+mt^{\alpha}\left|f^{'}\left(\frac{b}{m}\right)\right|^q$$

for $(\alpha, m) \in (0,1] \times (0,1]$. Hence

$$\int_0^1 \left|f'(a+t\eta(b,a))\right|^q dt \leq \left|f'(a)\right|^q \int_0^1 (1-t^\alpha)\,dt + m\left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1 t^\alpha dt = \frac{\alpha}{1+\alpha}\left|f'(a)\right|^q + \frac{m}{1+\alpha}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

An application of the above inequality in (2.19) and the fact

$$\int_0^1 |1-2t|^p\,dt = \frac{1}{p+1}$$

gives the desired inequality. □

Corollary 5. *If in theorem 8, we take $\eta(b,a) = b-a$, we get the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{\alpha\left|f'(a)\right|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{1+\alpha}\right]^{\frac{1}{q}}, \tag{2.20}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 9. *Let $K \subseteq [0,b^*]$, $b^*>0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $\left|f'\right|^q$ is (α, m)-preinvex on K, $q \geq 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\nu_2\left|f'(a)\right|^q + m\nu_1\left|f'(b)\right|^q\right]^{\frac{1}{q}}, \tag{2.21}$$

where $\nu_2 = \frac{1}{2} - \nu_1$ and $\nu_1 = \frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$.

Proof. For $q=1$, the proof is similar to that of theorem 7. Suppose that $q>1$. Using lemma 1, we have that the following inequality holds:

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.22}$$

By the (α, m)-preinvexity of $\left|f'\right|^q$ on K, for every $t \in [0,1]$ and $(\alpha, m) \in (0,1] \times (0,1]$ we have

$$\int_0^1 |1-2t| \left|f'(a+t\eta(b,a))\right|^q dt \leq \int_0^1 |1-2t| \left[(1-t)^\alpha \left|f'(a)\right|^q + mt^\alpha \left|f'(b)\right|^q\right] dt = \left|f'(a)\right|^q \int_0^1 |1-2t|(1-t)^\alpha dt + m\left|f'(b)\right|^q \int_0^1 |1-2t| t^\alpha dt = \nu_2 \left|f'(a)\right|^q + m\nu_1 \left|f'(b)\right|^q. \tag{2.23}$$

Using (2.23) in (2.22), we get the required inequality (2.21). This completes the proof of the theorem. □

Corollary 6. *Suppose $\eta(b,a) = b-a$ in theorem 9, then one has the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\nu_2 \left|f'(a)\right|^q + m\nu_1 \left|f'(b)\right|^q\right]^{\frac{1}{q}}, \tag{2.24}$$

where $\nu_2 = \frac{1}{2} - \nu_1$ and $\nu_1 = \frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$.

Remark 3. *If we take $m=1$ in theorem 4 an theorem 5 or if we take $\alpha = m = 1$ in theorem 7 and theorem 8 we get those results proved in theorem 2 and theorem 3 respectively. This shows that our results are more general than those proved in* [3].

Remark 4. *If we take $m=1$ in theorem 4 and theorem 5 or if we take $\alpha = m = 1$ in theorem 7 and theorem 8 with $\eta(b,a) = b-a$, we get those results proved in* [6] *and* [25].

3. An Extension to Functions of Several Variables

In this section we will extend Corollary 1 and corollary 4 to functions of several variables defined on an invex subset of $\mathbb{R}^n$. To this end, we need the following property of invex functions.

Condition C [34]**:** Let $K \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}^n$. For any $x, y \in K$ and any $t \in [0,1]$,

$$\eta(y, y+t\eta(x,y)) = -t\eta(x,y)$$

and

$$\eta(x, y+t\eta(x,y)) = (1-t)\eta(x,y).$$

It is to be noted from **Condition C** that for every $x, y \in K$ and every $t_1, t_2 \in [0,1]$, we have

$$\eta(y+t_2\eta(x,y), y+t_1\eta(x,y)) = (t_2-t_1)\eta(x,y). \tag{3.1}$$

Proposition 1. *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ is a function. Suppose that f satisfies* ***Condition C*** *on K. Then* *for every $x, y \in K$ the function f is m-preinvex with respect to η on η-path P_{xv}, $v = x + \eta(x, y)$, if and only if the function $\varphi : [0, 1] \to \mathbb{R}$ defined by*

$$\varphi(t) := f(x + t\eta(y, x))$$

is m-convex on $[0, 1]$, $m \in (0, 1]$.

Proof. Suppose that φ is m-convex on$[0, 1]$ and $z_1 := x + t_1\eta(y, x) \in P_{xv}$ and $z_2 := x + t_2\eta(y, x) \in P_{xv}$. Fix $\lambda \in [0; 1]$. Since f satisfies **Condition C**, by (3.1) we have

$$\begin{aligned} f(z_1 + \lambda\eta(z_2, z_1)) &= f(x + ((1-\lambda)t_1 + \lambda t_2))\eta(y, x) \\ &= \varphi((1-\lambda)t_1 + \lambda t_2) \\ &\le (1-\lambda)\varphi(t_1) + m\lambda\varphi\left(\frac{t_2}{m}\right) \\ &= (1-\lambda)f(z_1) + m\lambda f\left(\frac{z_2}{m}\right). \end{aligned}$$

Conversely, let $x, y \in K$ and the function f be m-preinvex with respect to η on η-path P_{xv}. Suppose that $t_1, t_2 \in [0, 1]$. Then for every $\lambda \in [0, 1]$, $m \in (0, 1]$ and using (3.1), we have

$$\begin{aligned} \varphi((1-\lambda)t_1 + \lambda t_2) &= f(x + ((1-\lambda)t_1 + \lambda t_2)\eta(y, x)) \\ &= f(x + t_1\eta(y, x) + \lambda(t_2 - t_1)\eta(y, x)) \\ &= f(x + t_1\eta(y, x) + \lambda\eta(x + t_2\eta(x, y), x + t_1\eta(x, y))) \\ &\le (1-\lambda)f(x + t_1\eta(y, x)) + m\lambda f\left(\frac{x + t_2\eta(x, y)}{m}\right) \\ &= (1-\lambda)\varphi(t_1) + m\lambda\varphi\left(\frac{t_2}{m}\right). \end{aligned}$$

Hence φ is m-preinvex function on $[0, 1]$. □

Proposition 2. *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ is a function. Suppose that η satisfies **Condition C** on K. Then for every $x, y \in K$ the function f is (α, m)-preinvex with respect to η on η-path P_{xv}, $v = x + \eta(x, y)$, if and only if the function $\varphi : [0, 1] \to \mathbb{R}$ defined by*

$$\varphi(t) := f(x + t\eta(y, x))$$

is (α, m)-convex on $[0, 1]$, $(\alpha, m) \in (0, 1] \times (0, 1]$.

Proof. The proof is similar to that of the proof of proposition 1, therefore we omit the details. □

Theorem 10. *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}^+$ is a function. Suppose that η satisfies Condition C on K. Suppose that for every $x, y \in K$ the function f is m-preinvex with respect to η on η-path P_{xv}, $m \in (0, 1]$. Then for every $a, b \in (0, 1)$ with $a < b$ the following inequality* *holds:*

$$\left|\frac{1}{2}\left[\int_0^a f(x+s\eta(y,x))ds+\int_0^b f(x+s\eta(y,x))ds\right]\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left(\int_0^s f(x+t\eta(y,x))dt\right)ds\right|$$
$$\le\frac{b-a}{8}\left[f(x+a\eta(y,x))+mf\left(x+\frac{b}{m}\eta(y,x)\right)\right]. \tag{3.2}$$

Proof. Let $x,y\in K$ and $a,b\in(0,1)$ with $a<b$. Since $f:K\to\mathbb{R}^+$ is m-preinvex with respect to η on η-path P_{xv}, $m\in(0,1]$, by proposition 1 the function $\varphi:[0,1]\to\mathbb{R}^+$ defined by

$$\varphi(t):=f(x+t\eta(y,x))$$

is m-convex on $[0,1]$. Now we define function $\phi:[0,1]\to\mathbb{R}^+$ as

$$\phi(t):=\int_0^t\varphi(s)ds=\int_0^t f(x+s\eta(y,x))ds.$$

It is clear that for every $t\in(0,1)$ we have

$$\phi^{'}(t)=\varphi(t)=f(x+t\eta(y,x))\ge 0,$$

hence $\left|\phi^{'}(t)\right|=\phi^{'}(t)$. Applying corollary 1 to the function ϕ, we get

$$\left|\frac{\phi(a)+\phi(b)}{2}-\frac{1}{b-a}\int_a^b\phi(s)\,ds\right|\le\frac{b-a}{8}\left[\left|\phi^{'}(a)\right|+m\left|\phi^{'}\left(\frac{b}{m}\right)\right|\right], \tag{3.3}$$

we deduce from (3.3) that (3.2) holds. This completes the proof of the theorem. □

Theorem 11. *Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$ and $f:K\to\mathbb{R}^+$ is a function. Suppose that η satisfies Condition C on K. Suppose that for every $x,y\in K$ the function f is (α,m)-preinvex with respect to η on η-path P_{xv}, $(\alpha,m)\in(0,1]$. Then for every $a,b\in(0,1)$ with $a<b$ the following inequality holds:*

$$\left|\frac{1}{2}\left[\int_0^a f(x+s\eta(y,x))ds+\int_0^b f(x+s\eta(y,x))ds\right]\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left(\int_0^s f(x+t\eta(y,x))dt\right)ds\right|$$
$$\le\frac{b-a}{8}\left[\nu_2 f(x+a\eta(y,x))+m\nu_1 f\left(x+\frac{b}{m}\eta(y,x)\right)\right], \tag{3.4}$$

where $\nu_1=\frac{1+\alpha\cdot 2^\alpha}{2^\alpha(1+\alpha)(2+\alpha)}$ and $\nu_2=\frac{1}{2}-\nu_1$.

Proof. The proof of is similar to that of theorem 10 using corollary 4 so we omit the details to the readers. □

Remark 5. *Let $\varphi(t):[0,1]\to\mathbb{R}^+$ be a function and q be a positive real number. Then φ is m-convex or (α,m)-convex function if and only if $\varphi(t)^q:[0,1]\to\mathbb{R}^+$ is m-convex or (α,m)-convex respectively. Hence similar results can be stated as* *those of proposition 1 and proposition 2 by using corollary 2, corollary 3, corollary 5 and corollary 6 and we omit the details for the interested reader.*

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