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# HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR $n$-TIMES DIFFERENTIABLE $m$-PREINVEX FUNCTIONS

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ABSTRACT. In this paper we establish inequalities of Hermite-Hadamard type for functions whose $n$th derivatives in absolute value are $m$-preinvex functions. The established results generalize several recent results proved for functions whose derivatives in absolute value are $m$-convex functions.

## 1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for every $x$, $y \in I$ and $t \in [0, 1]$.

The following celebrated double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds for convex functions and is known as the Hermite-Hadamard inequality. Both of the inequalities in (1.1) hold in reversed direction if $f$ is concave.

The inequalities Hermite-Hadamard inequalities (1.1) have been a source of inspiration for many mathematicians and hence its various refinements and its variant forms have been obtained in the literature by many researchers (see [6, 7, 11, 12, 28, 29] and [37]) and the references therein.

The classical convexity that is stated above was generalized as $m$-convexity by G. Toader in [33] as follows:

**Definition 1.** [33] *The function* $[0, b^*]$, $b^* > 0$, *is said to be* $m$*-convex , where* $m \in [0, 1]$, *if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

*for all* $x, y \in [0, b^*]$ *and* $t \in 0, 1]$ . *We say that* $f$ *is* $m$*-concave if* $-f$ *is* $m$*-convex.*

Obviously, for $m = 1$ the Definition 1 recaptures the concept of standard convex functions on $[0, b^*]$.

The notion of $m$-convexity has been further generalized in [17] as it is stated in the following definition:

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**Definition 2.** [17] *The function* $[0, b^*]$*,* $b^* > 0$*, is said to be* $(\alpha, m)$*-convex , where* $(\alpha, m) \in [0,1]^2$*, if we have*

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha) f(y)$$

*for all* $x, y \in [0, b^*]$ *and* $t \in 0, 1]$ *.*

It can easily be seen that for $\alpha = 1$, the class of $m$-convex functions are derived from the above definition and for $\alpha = m = 1$ a class of convex functions are derived.

For several results concerning Hermite-Hadamard type inequalities for $m$-convex and $(\alpha, m)$-convex functions we refer the interested reader to [5, 8, 9, 23, 24, 25, 26, 31, 32, 34] and [36].

More recently, a number of mathematicians have attempted to generalize the concept of classical convexity. For example in [10], Hason gave the notion of invexity as significant generalization of classical convexity. Ben-Israel and Mond [4] introduced the concept of preinvex functions, which is a special case of invex functions.

Let us first restate the definition of preinvexity as follows:

**Definition 3.** [35] *Let* $K$ *be a subset in* $\mathbb{R}^n$ *and let* $f : K \to \mathbb{R}$ *and* $\eta : K \times K \to \mathbb{R}^n$ *be continuous functions. Let* $x \in K$*, then the set* $K$ *is said to be invex at* $x$ *with respect to* $\eta(\cdot, \cdot)$*, if*

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

$K$ *is said to be an invex set with respect to* $\eta$ *if* $K$ *is invex at each* $x \in K$*. The invex set* $K$ *is also called a* $\eta$*-connected set.*

**Definition 4.** [35] *The function* $f$ *on the invex set* $K$ *is said to be preinvex with respect to* $\eta$*, if*

$$f(u + t\eta(v, u)) \leq (1-t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1].$$

*The function* $f$ *is said to be preconcave if and only if* $-f$ *is preinvex.*

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [35].

For several new results on Hermite-Hadamard type inequalities for preinvex functions, we refer the interested reader to [2, 3, 14, 15, 21, 22] and [30], and the references therein.

In the present paper, we first give the concept of $m$-preinvex and $(\alpha, m)$-preinvex functions in Section 2, which generalize the concept of preinvex functions and then we will present new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are $m$-preinvex. It can be viewed that our results generalize those results presented in recent paper [15] and some of the results given in [11] concerning Hermite-Hadamard type inequalities for functions whose $n$th derivatives in absolute value are $m$-convex functions and convex functions respectively. It can also be observed that some the the results from [15] have also been extended.

## 2. Main Results

To establish our main results we first give the following essential definitions and a Lemma:

**Definition 5.** *Let $K \subseteq [0, b^*]^n \subseteq [0,\infty)^n$, $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to [0,\infty)^n$ is said to be $m$-preinvex with respect to $\eta$ on $K$ if*

$$f(u + t\eta(v,u)) \leq (1-t) f(u) + mtf\left(\frac{v}{m}\right)$$

*holds for all $u, v \in K$, $t \in [0,1]$ and $m \in (0,1]$. The function $f$ is said to be $m$-preconcave if and only if $-f$ is $m$-preinvex.*

**Definition 6.** *Let $K \subseteq [0, b^*]^n \subseteq [0,\infty)^n$, $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to [0,\infty)^n$ is said to be $(\alpha, m)$-preinvex with respect to $\eta$ if*

$$f(u + t\eta(v,u)) \leq (1-t^\alpha) f(u) + mt^\alpha f\left(\frac{v}{m}\right)$$

*holds for all $u, v \in K$, $t \in [0,1]$ and $(\alpha, m) \in (0,1] \times (0,1]$. The function $f$ is said to be $(\alpha, m)$-preconcave if and only if $-f$ is $(\alpha, m)$-preinvex.*

**Remark 1.** *If in Definition 5, $m = 1$, then one obtain the usual definition of preinvexity. If $\alpha = m = 1$, then Definition 6 recaptures the usual definition of the the preinvex functions. It is to be noted that every $m$-preinvex function and $(\alpha, m)$-preinvex functions are $m$-convex and $(\alpha, m)$-convex with respect to $\eta(v,u) = v - u$ respectively.*

**Lemma 1.** [15] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a$, $b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ for $n \in \mathbb{N}$, $n \geq 1$ and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, we have the following equality:*

$$\begin{aligned} &-\frac{f(a) + f(a + \eta(b,a))}{2} + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \\ &+ \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a)) \\ &= \frac{(-1)^{n-1}(\eta(b,a))^n}{2n!} \int_0^1 t^{n-1}(n-2t) f^{(n)}(a + t\eta(b,a))dt, \end{aligned} \tag{2.1}$$

*where the sum above takes 0 when $n = 1$ and $n = 2$.*

Now we establish results for functions whose derivatives in absolute values raise to some certain power are $m$-preinvex.

**Theorem 1.** *Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a$, $b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$ is $m$-preinvex on $K$ for $n \in \mathbb{N}$, $n \geq 2$, $q \in [1,\infty)$, we have*

*the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\le\frac{(\eta(b,a))^n(n-1)^{1-\frac{1}{q}}}{2(n+1)!}\left[\frac{n|f^n(a)|^q+m(n^2-2)\left|f^n\left(\frac{b}{m}\right)\right|^q}{n+2}\right]^{\frac{1}{q}}. \tag{2.2}$$

*Proof.* From Lemma 1 and the Hölder integral inequality, we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\le\frac{(\eta(b,a))^n}{2n!}\left(\int_0^1t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.3}$$

Since $\left|f'\right|^q$ is $m$-preinvex on $K$, $q\ge1$, for every $a$, $b\in K$, $t\in[0,1]$ and $m\in(0,1]$, we have

$$|f^n(a+t\eta(b,a))|^q\le(1-t)|f^n(a)|^q+mt\left|f^n\left(\frac{b}{m}\right)\right|^q. \tag{2.4}$$

Hence we have

$$\begin{aligned}\int_0^1t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt&\le\int_0^1t^{n-1}(n-2t)\left[(1-t)|f^n(a)|^q+mt\left|f^n\left(\frac{b}{m}\right)\right|^q\right]dt\\&=|f^n(a)|^q\int_0^1(1-t)t^{n-1}(n-2t)\,dt+m\left|f^n\left(\frac{b}{m}\right)\right|^q\int_0^1t^n(n-2t)\,dt\\&=\frac{n|f^n(a)|^q}{(n+1)(n+2)}+\frac{m(n^2-2)\left|f^n\left(\frac{b}{m}\right)\right|^q}{(n+1)(n+2)}\end{aligned} \tag{2.5}$$

By using (2.5) and the fact

$$\int_0^1t^{n-1}(n-2t)\,dt=\frac{n-1}{n+1},$$

we get the desired inequality from (2.3). This completes the proof of theorem 1. □

**Corollary 1.** *Under the assumptions of Theorem 1, If $q=1$, we have*

$$\begin{aligned}(2.6)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq\frac{(\eta(b,a))^n}{2(n+1)!}\left[\frac{n|f^n(a)|+m\left(n^2-2\right)\left|f^n\left(\frac{b}{m}\right)\right|}{n+2}\right].\end{aligned}$$

*If $n=2$, we obtain the following result:*

$$\begin{aligned}(2.7)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\leq\frac{(\eta(b,a))^2\left[\left|f''(a)\right|+m\left|f''\left(\frac{b}{m}\right)\right|\right]}{24}.\end{aligned}$$

**Corollary 2.** *Under the assumptions of Theorem 1, if $m=1$, we have*

$$\begin{aligned}(2.8)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq\frac{(\eta(b,a))^n(n-1)^{1-\frac{1}{q}}}{2(n+1)!}\left[\frac{n|f^n(a)|^q+\left(n^2-2\right)|f^n(b)|^q}{n+2}\right]^{\frac{1}{q}}.\end{aligned}$$

**Corollary 3.** *Under the assumptions of Theorem 1, if $n=2$, we have*

$$\begin{aligned}(2.9)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\leq\frac{(\eta(b,a))^2}{12}\left[\frac{|f^n(a)|^q+m\left|f^n\left(\frac{b}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

**Theorem 2.** *Let $K\subseteq[0,b^*]$, $b^*>0$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a$, $b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 2$. If $\left|f^{(n)}\right|^q$ is $m$-preinvex on $K$ for $n\in\mathbb{N}$, $n\geq 2$, $q\in(1,\infty)$, we have*

*the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx -\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \le \frac{(\eta(b,a))^n}{2n!}\left(\frac{q-1}{nq-1}\right)^{1-\frac{1}{q}}\left\{\left[\frac{n^{q+1}(2q-n+4)+(n-2)^{q+2}}{4(q+1)(q+2)}\right]|f^n(a)|^q +m\left[\frac{n^{q+2}-(n-2)^{q+2}(2q+n+2)}{4(q+1)(q+2)}\right]\left|f^n\left(\frac{b}{m}\right)\right|^q\right\}^{\frac{1}{q}}. \tag{2.10}$$

*Proof.* From Lemma 1 and the Hölder integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx -\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \le \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{\frac{q(n-1)}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(n-2t)^q\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.11}$$

By the $m$-preinvexity of $\left|f^{(n)}\right|^q$ on $K$ for $n\in\mathbb{N}$, $n\ge 2$, $q\in(1,\infty)$, we have

$$\begin{aligned}\int_0^1(n-2t)^q\left|f^{(n)}(a+t\eta(b,a))\right|^q dt &\le |f^n(a)|^q\int_0^1(n-2t)^q(1-t)\,dt+m\left|f^n\left(\frac{b}{m}\right)\right|^q\int_0^1 t(n-2t)^q\,dt\\ &=\left[\frac{n^{q+1}(2q-n+4)+(n-2)^{q+2}}{4(q+1)(q+2)}\right]|f^n(a)|^q\\ &\quad+m\left[\frac{n^{q+1}-(n-2)^{q+2}(2q+n+2)}{4(q+1)(q+2)}\right]\left|f^n\left(\frac{b}{m}\right)\right|^q\end{aligned} \tag{2.12}$$

By (2.12) and

$$\int_0^1 t^{\frac{q(n-1)}{q-1}}dt=\frac{q-1}{nq-1},$$

we get the required inequality from (2.11). This completes the proof of the Theorem. □

**Corollary 4.** *Under the assumptions of Theorem 2, if $m=1$, we have*

$$\begin{aligned}(2.13)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq\frac{(\eta(b,a))^n}{2n!}\left(\frac{q-1}{nq-1}\right)^{1-\frac{1}{q}}\left\{\left[\frac{n^{q+1}(2q-n+4)+(n-2)^{q+2}}{4(q+1)(q+2)}\right]|f^n(a)|^q\right.\\ &\left.+\left[\frac{n^{q+2}-(n-2)^{q+2}(2q+n+2)}{4(q+1)(q+2)}\right]|f^n(b)|^q\right\}^{\frac{1}{q}}.\end{aligned}$$

**Corollary 5.** *If assumptions of the Theorem 2 are satisfied and $n=2$, we have the following inequality:*

$$\begin{aligned}(2.14)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\leq\frac{(\eta(b,a))^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\frac{(q+1)\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{4(q+1)(q+2)}\right].\end{aligned}$$

A similar result may be stated as follows:

**Theorem 3.** *Let $K\subseteq[0,b^*]$, $b^*>0$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 2$. If $\left|f^{(n)}\right|^q$ is m-preinvex on $K$ for $n\in\mathbb{N}$, $n\geq2$, $q\in(1,\infty)$, we have the following inequality:*

$$\begin{aligned}(2.15)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq\frac{(\eta(b,a))^n}{2n!}\left[\frac{(q-1)\left(n^{\frac{2q-1}{q-1}}-(n-2)^{\frac{2q-1}{q-1}}\right)}{2(2q-1)}\right]^{1-\frac{1}{q}}\left[\frac{|f^n(a)|^q+m(nq-q+1)\left|f^n\left(\frac{b}{m}\right)\right|^q}{(nq-q+1)(nq-q+2)}\right]^{\frac{1}{q}}.\end{aligned}$$

*Proof.* From Lemma 1 and the Hölder integral inequality, we have

$$\begin{aligned}(2.16)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq\frac{(\eta(b,a))^n}{2n!}\left(\int_0^1(n-2t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{(n-1)q}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

By the $m$-preinvexity of $\left|f^{(n)}\right|^q$ on $K$ for $n \in \mathbb{N}$, $n \geq 2$, $q \in (1, \infty)$, we have

$$\int_0^1 t^{(n-1)q} \left|f^{(n)}(a + t\eta(b,a))\right|^q dt \leq |f^n(a)|^q \int_0^1 t^{(n-1)q}(1-t) + m\left|f^n\left(\frac{b}{m}\right)\right|^q \int_0^1 t^{(n-1)q+1} dt = \frac{|f^n(a)|^q + m(nq - q + 1)\left|f^n\left(\frac{b}{m}\right)\right|^q}{(nq - q + 1)(nq - q + 2)}. \tag{2.17}$$

Applying (2.17) and

$$\int_0^1 (n - 2t)^{\frac{q}{q-1}} dt = \frac{(q-1)\left(n^{\frac{2q-1}{q-1}} - (n-2)^{\frac{2q-1}{q-1}}\right)}{2(2q-1)}$$

in (2.16), we get the required inequality. This completes the proof of the Theorem. □

**Corollary 6.** *Under the assumptions of Theorem 3, if $m = 1$, we get the following inequality:*

$$\left|\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a))\right| \leq \frac{(\eta(b,a))^n}{2n!}\left[\frac{(q-1)\left(n^{\frac{2q-1}{q-1}} - (n-2)^{\frac{2q-1}{q-1}}\right)}{2(2q-1)}\right]^{1-\frac{1}{q}} \left[\frac{|f^n(a)|^q + (nq - q + 1)|f^n(b)|^q}{(nq - q + 1)(nq - q + 2)}\right]^{\frac{1}{q}}. \tag{2.18}$$

**Corollary 7.** *Under the assumptions of Theorem 3, if $n = 2$, we get the following inequality:*

$$\left|\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{2}\left[\frac{q-1}{2q-1}\right]^{1-\frac{1}{q}} \left[\frac{\left|f''(a)\right|^q + m(q+1)\left|f''\left(\frac{b}{m}\right)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}. \tag{2.19}$$

**Theorem 4.** *Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$ is $m$-preinvex on $K$ for $n \in \mathbb{N}$, $n \geq 2$, $q \in (1, \infty)$, we have*

*the following inequality:*

$$\begin{aligned}(2.20)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq \frac{(\eta(b,a))^n(n-1)}{2n!}\left[\frac{(q-1)(nq-2)}{(nq-1)(nq+q-2)}\right]^{1-\frac{1}{q}}\left[\frac{(3n-2)|f^n(a)|^q+m(3n-4)\left|f^n\left(\frac{b}{m}\right)\right|^q}{6}\right]^{\frac{1}{q}}.\end{aligned}$$

*Proof.* From Lemma 1 and the Hölder integral inequality, we have

$$\begin{aligned}(2.21)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1(n-2t)\,t^{\frac{q(n-1)}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

By the $m$-preinvexity of $\left|f^{(n)}\right|^q$ on $K$ for $n\in\mathbb{N}$, $n\geq 2$, $q\in(1,\infty)$, we have

$$\begin{aligned}(2.22)\quad &\int_0^1 t^{(n-1)q}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\\ &\leq |f^n(a)|^q\int_0^1(n-2t)(1-t)+m\left|f^n\left(\frac{b}{m}\right)\right|^q\int_0^1(n-2t)\,tdt\\ &=\frac{(3n-2)|f^n(a)|^q+m(3n-4)\left|f^n\left(\frac{b}{m}\right)\right|^q}{6}.\end{aligned}$$

Using (2.22) and

$$\int_0^1(n-2t)^{\frac{q}{q-1}}\,dt=\frac{(q-1)(nq-2)(n-1)}{(nq-1)(nq+q-2)}$$

in (2.21), we get the required inequality. This completes the proof of the Theorem. □

**Corollary 8.** *Under the assumptions of Theorem 4, if $m=1$, we have*

$$\begin{aligned}(2.23)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\leq \frac{(\eta(b,a))^n(n-1)}{2n!}\left[\frac{(q-1)(nq-2)}{(nq-1)(nq+q-2)}\right]^{1-\frac{1}{q}}\left[\frac{(3n-2)|f^n(a)|^q+(3n-4)|f^n(b)|^q}{6}\right]^{\frac{1}{q}}.\end{aligned}$$

**Corollary 9.** *Under the assumptions of Theorem 4, if $n = 2$, we obtain the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{4}\left[\frac{2(q-1)^2}{(2q-1)(3q-2)}\right]^{1-\frac{1}{q}}\left[\frac{2\left|f''(a)\right|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{3}\right]^{\frac{1}{q}}. \tag{2.24}$$

**Theorem 5.** *Let $K \subseteq [0, b^*]$, $b^* > 0$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a$, $b \in K$ with $a < a + \eta(b, a)$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on $K$ and $f^{(n)}$ is integrable on $[a, a + \eta(b, a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$ is $m$-preinvex on $K$ for $n \in \mathbb{N}$, $n \geq 2$, $q \in (1, \infty)$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \leq \frac{(\eta(b,a))^n}{2n!}\left\{\frac{(q-1)\left[(q-1)n^{\frac{3q-2}{q-1}}-(n(q-1)-2(3q-2))(n-2)^{\frac{2q-1}{q-1}}\right]}{4(2q-1)(3q-2)}\right\}^{1-\frac{1}{q}} \times\left[\frac{|f^n(a)|^q+m(nq-2q+2)\left|f^n\left(\frac{b}{m}\right)\right|^q}{(nq-2q+2)(nq-2q+3)}\right]^{\frac{1}{q}}. \tag{2.25}$$

*Proof.* From Lemma 1 and the Hölder integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx - \sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t(n-2t)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{nq-2q+1}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.26}$$

By the $m$-preinvexity of $\left|f^{(n)}\right|^q$ on $K$ for $n \in \mathbb{N}$, $n \geq 2$, $q \in (1, \infty)$, we have

$$\int_0^1 t^{(n-1)q}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt \leq |f^n(a)|^q\int_0^1 t^{nq-2q+1}(1-t)+m\left|f^n\left(\frac{b}{m}\right)\right|^q\int_0^1 t^{nq-2q+2}dt = \frac{|f^n(a)|^q+m(nq-2q+2)\left|f^n\left(\frac{b}{m}\right)\right|^q}{(nq-2q+2)(nq-2q+3)}. \tag{2.27}$$

Utilizing (2.27) and

$$\int_0^1 t(n-2t)^{\frac{q}{q-1}}dt = \frac{(q-1)\left[n^{\frac{3q-2}{q-1}}(q-1)-(n-2)^{\frac{2q-1}{q-1}}(n(q-1)-2(3q-2))\right]}{4(2q-1)(3q-2)}$$

in (2.26), we get the required inequality. This completes the proof of the Theorem. □

**Corollary 10.** *If in theorem 5, we take $m=1$, we get the following inequality:*

$$\begin{aligned}(2.28)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.\\ &\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\ &\le \frac{(\eta(b,a))^n}{2n!}\left\{\frac{(q-1)\left[(q-1)n^{\frac{3q-2}{q-1}}-(n(q-1)-2(3q-2))(n-2)^{\frac{2q-1}{q-1}}\right]}{4(2q-1)(3q-2)}\right\}^{1-\frac{1}{q}}\\ &\quad\times\left[\frac{|f^n(a)|^q+(nq-2q+2)|f^n(b)|^q}{(nq-2q+2)(nq-2q+3)}\right]^{\frac{1}{q}}.\end{aligned}$$

**Corollary 11.** *Suppose the assumptions of Theorem 5 are fulfilled and $n=2$, we get the following inequality:*

$$\begin{aligned}(2.29)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\le \frac{(\eta(b,a))^2}{2}\left\{\frac{(q-1)^2}{4(2q-1)(3q-2)}\right\}^{1-\frac{1}{q}}\left[\frac{\left|f''(a)\right|^q+2m\left|f''\left(\frac{b}{m}\right)\right|^q}{6}\right]^{\frac{1}{q}}.\end{aligned}$$

**Remark 2.** *If we take $m=1$ in Theorem 1 and its related Corollaries, we get* [15, Theorem 2.4] *and the related Corollaries of* [15, Theorem 2.4].

**Remark 3.** *If we take $\eta(b,a)=b-a$ in all the results presented above, we get those results proved in* [34].

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