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Ostrowski's Type Inequalities for Complex Functions Defined on Unit Circle with Applications for Unitary Operators in Hilbert Spaces

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ABSTRACT. Some Ostrowski's type inequalities for the Riemann-Stieltjes integral $\int_a^b f(e^{it})\,du(t)$ of continuous complex valued integrands $f:\mathcal{C}(0,1)\to\mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of integrators $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

1. Introduction

The problem of approximating the *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ by the quantity $f(x)[u(b)-u(a)]$, which is a natural generalization of the Ostrowski problem analyzed in 1937 (see [**17**]), was apparently first considered in the literature by the author in 2000 (see [**9**]) where he obtained the following result:

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{1.1}$$
$$\leq H\left[(x-a)^r\bigvee_a^x(f)+(b-x)^r\bigvee_x^b(f)\right]$$
$$\leq H\times\begin{cases}[(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right];\\[2ex] [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}}\\ \qquad\qquad\text{if}\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f);\end{cases}$$

for each $x\in[a,b]$, provided f is of *bounded variation* on $[a,b]$, $\bigvee_a^b(f)$ is its *total variation* on $[a,b]$, while $u:[a,b]\to\mathbb{R}$ is $r-H$-*Hölder continuous*, i.e., we recall that:

$$|u(x)-u(y)|\leq H|x-y|^r\quad\text{for each } x,y\in[a,b]. \tag{1.2}$$

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The dual case, i.e., when the *integrand* f is $q-K-$Hölder continuous and the *integrator* u is of bounded variation was obtained by the author in 2001 and can be stated as [10]

$$\left|\left[u\left(b\right)-u\left(a\right)\right]f\left(x\right)-\int_{a}^{b}f\left(t\right)du\left(t\right)\right| \leq K\left[\frac{1}{2}\left(b-a\right)+\left|x-\frac{a+b}{2}\right|\right]^{q}\bigvee_{a}^{b}\left(u\right) \tag{1.3}$$

for each $x\in\left[a,b\right]$.

The above inequalities provide, as important consequences, the following *midpoint inequalities*:

$$\left|\left[u\left(b\right)-u\left(a\right)\right]f\left(\frac{a+b}{2}\right)-\int_{a}^{b}f\left(t\right)du\left(t\right)\right|\leq\begin{cases}\frac{1}{2^{r}}\left(b-a\right)^{r}H\bigvee_{a}^{b}\left(f\right)\\ \\ \frac{1}{2^{q}}\left(b-a\right)^{q}K\bigvee_{a}^{b}\left(u\right),\end{cases} \tag{1.4}$$

which can be numerically implemented and provide a quadrature rule for approximating the Riemann-Stieltjes integral $\int_{a}^{b}f\left(t\right)du\left(t\right)$.

For other inequalities for the Riemann-Stieltjes integral, see [1]-[5], [6]-[12] and the edited book [15].

Let U be a *selfadjoint operator* on the complex Hilbert space $\left(H,\left\langle .,.\right\rangle\right)$ with the spectrum $Sp\left(U\right)$ included in the interval $\left[m,M\right]$ for some real numbers $m<M$ and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its *spectral family.* Then for any continuous function $f:\left[m,M\right]\rightarrow\mathbb{R}$, it is well known that we have the following *spectral representation* in terms of the Riemann-Stieltjes integral:

$$\left\langle f\left(U\right)x,y\right\rangle=\int_{m-0}^{M}f\left(\lambda\right)d\left(\left\langle E_{\lambda}x,y\right\rangle\right), \tag{1.5}$$

for any $x,y\in H$. The function $g_{x,y}\left(\lambda\right):=\left\langle E_{\lambda}x,y\right\rangle$ is of *bounded variation* on the interval $\left[m,M\right]$ and

$$g_{x,y}\left(m-0\right)=0\text{ and }g_{x,y}\left(M\right)=\left\langle x,y\right\rangle$$

for any $x,y\in H$. It is also well known that $g_{x}\left(\lambda\right):=\left\langle E_{\lambda}x,x\right\rangle$ is *monotonic nondecreasing* and *right continuous* on $\left[m,M\right]$.

On utilizing the spectral representation (1.5) and the Ostrowski's type inequality (1.3) we obtained the following result for continuous functions of selfadjoint operators (see [14, p. 35]):

Theorem 1 ([11]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right)\subseteq\left[m,M\right]$ for some real numbers $m<M$ and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. If $f:\left[m,M\right]\rightarrow\mathbb{R}$ is $r-H$-Hölder continuous on $\left[m,M\right]$, then we have the inequality*

$$\begin{aligned}&\left|f\left(s\right)\left\langle x,y\right\rangle-\left\langle f\left(A\right)x,y\right\rangle\right|\\&\leq H\bigvee_{m-0}^{M}\left(\left\langle E_{\left(\cdot\right)}x,y\right\rangle\right)\left[\frac{1}{2}\left(M-m\right)+\left|s-\frac{m+M}{2}\right|\right]^{r}\\&\leq H\left\Vert x\right\Vert\left\Vert y\right\Vert\left[\frac{1}{2}\left(M-m\right)+\left|s-\frac{m+M}{2}\right|\right]^{r}\end{aligned} \tag{1.6}$$

for any $x, y \in H$ *and* $s \in [m, M]$.

The following result for functions of bounded variation also holds (see [**14**, p. 36]):

THEOREM 2 ([**12**]). *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$ *and let* $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f : [m, M] \to \mathbb{R}$ *is a continuous function of bounded variation on* $[m, M]$, *then we have the inequality*

$$\begin{aligned} (1.7) \qquad & |f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \\ & \leq \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ & + \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\ & \leq \|x\| \|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \leq \|x\| \|y\| \bigvee_m^M (f) \end{aligned}$$

for any $x, y \in H$ *and for any* $s \in [m, M]$, *where* 1_H *is the identity operator on* H.

For various recent inequalities for functions of selfadjoint operators on Hilbert spaces see the books [**13**] and [**14**].

Motivated by the above results, we investigate in the current paper the magnitude of the difference

$$f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) \text{ with } s \in [a, b] \subseteq [0, 2\pi]$$

for continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of functions $u : [a, b] \subseteq [0, 2\pi] \to \mathbb{C}$ of bounded variation. Natural applications for functions of unitary operators in Hilbert spaces are provided as well.

2. Scalar Ostrowski's Type Inequalities

THEOREM 3. *Assume that* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *satisfies the following Hölder's type condition*

$$(2.1) \qquad |f(z) - f(w)| \leq H |z - w|^r$$

for any $w, z \in \mathcal{C}(0,1)$, *where* $H > 0$ *and* $r \in (0, 1]$ *are given.*

If $[a, b] \subseteq [0, 2\pi]$ *and the function* $u : [a, b] \to \mathbb{C}$ *is of bounded variation on* $[a, b]$, *then*

$$(2.2) \qquad \left| f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) \right| \leq 2^r H \max_{t \in [a,b]} \left| \sin\left(\frac{s - t}{2}\right) \right|^r \bigvee_a^b (u)$$

for any $s \in [a, b]$.

PROOF. Observe that

$$(2.3) \qquad f\left(e^{is}\right)[u(b) - u(a)] - \int_a^b f\left(e^{it}\right) du(t) = \int_a^b \left[f\left(e^{is}\right) - f\left(e^{it}\right)\right] du(t)$$

for any $s \in [a, b]$.

It is known that if $p:[c,d]\to\mathbb{C}$ is a continuous function and $v:[c,d]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)\,dv(t)\right|\le\max_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.4}$$

Applying the property (2.4) to the identity (2.3) and utilizing the Hölder's type condition (2.1) we have successively

$$\begin{aligned}&\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \\ &=\max_{t\in[a,b]}\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|\bigvee_a^b(u)\le H\max_{t\in[a,b]}\left|e^{is}-e^{it}\right|^r\bigvee_a^b(u).\end{aligned} \tag{2.5}$$

Since

$$\begin{aligned}\left|e^{is}-e^{it}\right|^2&=\left|e^{is}\right|^2-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2\\&=2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)\end{aligned}$$

for any $t,s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^r=2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.6}$$

for any $t,s\in\mathbb{R}$.

Now, by (2.5) and (2.6) we deduce the desired result (2.2). □

Remark 1. *If $a=0$ and $b=2\pi$, then for any $s\in[0,2\pi]$ there exists a unique $t\in[0,2\pi]$ such that $\frac{1}{2}|t-s|=\frac{\pi}{2}$, therefore $\max_{t\in[0,2\pi]}\left|\sin\left(\frac{s-t}{2}\right)\right|=1$ for all $s\in[0,2\pi]$ and we deduce from (2.2) the following inequality of interest*

$$\left|f\left(e^{is}\right)[u(2\pi)-u(0)]-\int_0^{2\pi}f\left(e^{it}\right)du(t)\right|\le 2^rH\bigvee_0^{2\pi}(u) \tag{2.7}$$

that holds for each $s\in[0,2\pi]$.

Remark 2. *If $[a,b]\subset[0,2\pi]$ and $0<b-a\le\pi$ then for all $t,s\in[a,b]$ we have $\frac{1}{2}|t-s|\le\frac{1}{2}(b-a)\le\frac{\pi}{2}$. Since the function $\sin$ is increasing on $\left[0,\frac{\pi}{2}\right]$, then we have successively that*

$$\begin{aligned}\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|&=\sin\left(\max_{t\in[a,b]}\frac{1}{2}|t-s|\right)\\&=\sin\left(\frac{1}{2}\max\{b-s,s-a\}\right)\\&=\sin\left(\frac{1}{4}(b-a)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right)\end{aligned} \tag{2.8}$$

for any $s\in[a,b]$.

Therefore, under the assumptions of Theorem 3 and if $[a,b] \subset [0,2\pi]$ *with* $0 < b-a \leq \pi$, *then*

$$\left| f\left(e^{is}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du(t) \right| \leq 2^r H \sin^r \left[\frac{1}{4}(b-a) + \frac{1}{2}\left|s - \frac{a+b}{2}\right|\right] \bigvee_a^b (u) \leq 2^r H \sin^r \left[\frac{1}{2}(b-a)\right] \bigvee_a^b (u) \tag{2.9}$$

for all $s \in [a,b]$.

In particular, the best inequality we can get from (2.9) is incorporated in

$$\left| f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du(t) \right| \leq 2^r H \sin^r \left[\frac{1}{4}(b-a)\right] \bigvee_a^b (u). \tag{2.10}$$

The case when $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the Lipschitz condition $|f(z) - f(w)| \leq L|z-w|$ for any $w, z \in \mathcal{C}(0,1)$, where $L > 0$ is given, is of interest due to various examples one can consider. Also in this case we can show that the corresponding version of the inequality (2.11) is sharp.

Corollary 1. *Assume that* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the circle* $\mathcal{C}(0,1)$. *If* $[a,b] \subset [0,2\pi]$ *with* $0 < b-a \leq \pi$ *and the function* $u : [a,b] \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, *then we have*

$$\left| f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)] - \int_a^b f\left(e^{it}\right) du(t) \right| \leq 2L \sin\left[\frac{1}{4}(b-a)\right] \bigvee_a^b (u). \tag{2.11}$$

The constant 2 *cannot be replaced by a smaller quantity.*

Proof. We need to prove only the sharpness of the constant 2.

If we consider the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z$, then obviously f is Lipschitzian with the constant $L = 1$. Also, consider in (2.11) $a = 0$ and $b = \pi$ to get

$$\left| i[u(\pi) - u(0)] - \int_0^\pi e^{it} du(t) \right| \leq \sqrt{2} \bigvee_0^\pi (u). \tag{2.12}$$

Utilising the *integration by parts formula* for the Riemann-Stieltjes integral we have

$$\begin{aligned} \int_0^\pi e^{it} du(t) &= e^{it} u(t)\big|_0^\pi - i \int_0^\pi e^{it} u(t)\, dt \\ &= -u(\pi) - u(0) - i \int_0^\pi e^{it} u(t)\, dt \end{aligned}$$

and replacing into the inequality (2.12) we deduce

$$\left| i[u(\pi) - u(0)] + u(\pi) + u(0) + i \int_0^\pi e^{it} u(t)\, dt \right| \leq \sqrt{2} \bigvee_0^\pi (u)$$

which is equivalent with

$$\left|(i-1)\,u(\pi)+(i+1)\,u(0)-\int_0^{\pi}e^{it}u(t)\,dt\right|\leq\sqrt{2}\bigvee_0^{\pi}(u) \tag{2.13}$$

that holds for any functions of bounded variation $u:[0,\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that there exists a constant $C>0$ such that

$$\left|(i-1)\,u(\pi)+(i+1)\,u(0)-\int_0^{\pi}e^{it}u(t)\,dt\right|\leq C\bigvee_0^{\pi}(u) \tag{2.14}$$

for any functions of bounded variation $u:[0,\pi]\to\mathbb{C}$.

Consider the function $u:[0,\pi]\to\mathbb{R}$ with

$$u(t):=\begin{cases}0 \text{ if } 0\leq t<\pi\\ \\ 1 \text{ if } t=\pi.\end{cases}$$

Then u is of bounded variation, $\int_0^{\pi}e^{it}u(t)\,dt=0$, $\bigvee_0^{\pi}(u)=1$ and from (2.14) we get $C\geq\sqrt{2}$ showing that (2.14) is sharp and therefore (2.11) is sharp. □

Remark 3. *The case of Riemann integral, namely when $u(t)=t, t\in[a,b]\subseteq[0,2\pi]$, is as follows*

$$\left|f\left(e^{is}\right)-\frac{1}{b-a}\int_a^b f\left(e^{it}\right)dt\right|\leq 2^r H\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.15}$$

for any $s\in[a,b]$ provided that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1).

When u is an integral, then the following weighted integral inequality also holds.

Remark 4. *If $w:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ is Lebesgue integrable on $[a,b]$ and $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1), then*

$$\begin{aligned}&\left|f\left(e^{is}\right)\int_a^b w(t)\,dt-\int_a^b f\left(e^{it}\right)w(t)\,dt\right| \\ &\leq 2^r H\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r\int_a^b|w(t)|\,dt\end{aligned} \tag{2.16}$$

for any $s\in[a,b]$.

In particular, if $w(t)\geq 0$ for $t\in[a,b]$ and $\int_a^b w(t)\,dt>0$ then

$$\left|f\left(e^{is}\right)-\frac{1}{\int_a^b w(t)\,dt}\int_a^b f\left(e^{it}\right)w(t)\,dt\right|\leq 2^r H\max_{t\in[a,b]}\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.17}$$

for any $s\in[a,b]$.

THEOREM 4. *Assume that* $f:\mathcal{C}(0,1)\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$ *on the circle* $\mathcal{C}(0,1)$. *If* $[a,b]\subseteq[0,2\pi]$ *and the function* $u:[a,b]\to\mathbb{C}$ *is Lipschitzian with the constant* $K>0$ *on* $[a,b]$, *then*

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \leq 4LK\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]\leq 8LK\sin^2\left(\frac{b-a}{4}\right) \tag{2.18}$$

for any $s\in[a,b]$.

PROOF. It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is M-Lipschitzian, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq M\int_a^b|p(t)|\,dt. \tag{2.19}$$

Utilising the property (2.19), we have from (2.3) that

$$\begin{aligned}&\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right|\\&=\left|\int_a^b\left[f\left(e^{is}\right)-f\left(e^{it}\right)\right]du(t)\right|\\&\leq K\int_a^b\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|dt\leq KL\int_a^b\left|e^{is}-e^{it}\right|dt\end{aligned} \tag{2.20}$$

for any $s\in[a,b]$.

Since, by (2.6), $\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any $t,s\in\mathbb{R}$, then

$$\begin{aligned}&\int_a^b\left|e^{is}-e^{it}\right|dt\\&=2\int_a^b\left|\sin\left(\frac{s-t}{2}\right)\right|dt\\&=2\left[\int_a^s\sin\left(\frac{s-t}{2}\right)dt+\int_s^b\sin\left(\frac{t-s}{2}\right)dt\right]\\&=2\left[1-\cos\left(\frac{s-a}{2}\right)\right]+2\left[1-\cos\left(\frac{b-s}{2}\right)\right]\\&=4\left[\sin^2\left(\frac{s-a}{4}\right)+\sin^2\left(\frac{b-s}{4}\right)\right]\\&\leq 8\sin^2\left(\frac{b-a}{4}\right)\end{aligned} \tag{2.21}$$

for any $s\in[a,b]\subseteq[0,2\pi]$, and the inequality (2.18) is proved. □

The best inequality we can get from (2.18) is incorporated in

Corollary 2. *With the assumptions in Theorem 4 we have the inequality*

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right|\leq 8LK\sin^2\left(\frac{b-a}{8}\right). \tag{2.22}$$

The multiplicative constant 8 *cannot be replaced by a smaller quantity.*

Proof. We need to prove only the sharpness of the constant.

If we consider the function $f:\mathbb{C}\to\mathbb{C}$, $f(z)=z$, then obviously f is Lipschitzian with the constant $L=1$. Also, consider in (2.22) $a=0$ and $b=2\pi$ to get

$$\left|-[u(2\pi)-u(0)]-\int_0^{2\pi}e^{it}du(t)\right|\leq 4K. \tag{2.23}$$

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_0^{2\pi}e^{it}du(t)&=e^{it}u(t)\Big|_0^{2\pi}-i\int_0^{2\pi}e^{it}u(t)\,dt\\&=u(2\pi)-u(0)-i\int_0^{2\pi}e^{it}u(t)\,dt.\end{aligned}$$

which inserted in (2.23) produces the inequality

$$\left|-2[u(2\pi)-u(0)]+i\int_0^{2\pi}e^{it}u(t)\,dt\right|\leq 4K$$

which is equivalent with

$$\left|\int_0^{2\pi}e^{it}u(t)\,dt-\frac{2}{i}[u(2\pi)-u(0)]\right|\leq 4K \tag{2.24}$$

that holds for any K-Lipschitzian function $u:[0,2\pi]\to\mathbb{C}$ and is of interest in itself.

Now, assume that the inequality (2.24) holds with a constant $D>0$, namely

$$\left|\int_0^{2\pi}e^{it}u(t)\,dt-\frac{2}{i}[u(2\pi)-u(0)]\right|\leq DK \tag{2.25}$$

for any K-Lipschitzian function $u:[0,2\pi]\to\mathbb{C}$.

Consider $u:[0,2\pi]\to\mathbb{R}$, $u(t)=|t-\pi|$. Then, by the continuity property of the modulus we have that u is Lipschitzian with the constant $K=1$.

We also have that

$$\begin{aligned}\int_0^{2\pi}e^{it}u(t)\,dt&=\int_0^{2\pi}e^{it}|t-\pi|\,dt\\&=\int_0^{2\pi}|t-\pi|(\cos t+i\sin t)\,dt\\&=\int_0^{2\pi}|t-\pi|\cos t dt+i\int_0^{2\pi}|t-\pi|\sin t dt.\end{aligned}$$

Observe that, by symmetry reasons, $\int_0^{2\pi} |t-\pi| \sin t dt = 0$ and

$$\begin{aligned}\int_0^{2\pi} |t-\pi| \cos t dt &= 2\int_0^{\pi} (\pi - t) \cos t dt \\ &= 2\left[(\pi - t)\sin t|_0^{\pi} + \int_0^{\pi} \sin t dt\right] = 4\end{aligned}$$

and by (2.25) we get $D \geq 4$ which proves the desired sharpness of the constant 8 in (2.22). □

Remark 5. *If $u(t) = t, t \in [a,b]$, then we get from (2.18) and (2.22) the following inequalities for the Riemann integral*

$$\begin{aligned}&\left| f\left(e^{is}\right)(b-a) - \int_a^b f\left(e^{it}\right) dt \right| \\ &\leq 4L\left[\sin^2\left(\frac{s-a}{4}\right) + \sin^2\left(\frac{b-s}{4}\right)\right] \leq 8L\sin^2\left(\frac{b-a}{4}\right)\end{aligned} \tag{2.26}$$

for any $s \in [a,b]$ and

$$\left| f\left(e^{\frac{a+b}{2}i}\right)(b-a) - \int_a^b f\left(e^{it}\right) dt \right| \leq 8L\sin^2\left(\frac{b-a}{8}\right), \tag{2.27}$$

provided that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$.

Remark 6. *If $w : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is essentially bounded on $[a,b]$ and $f : \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$, then we have the following weighted integral inequality*

$$\begin{aligned}&\left| f\left(e^{is}\right)\int_a^b w(t)\,dt - \int_a^b f\left(e^{it}\right) w(t)\,dt \right| \\ &\leq 4L\|w\|_\infty \left[\sin^2\left(\frac{s-a}{4}\right) + \sin^2\left(\frac{b-s}{4}\right)\right] \\ &\leq 8L\|w\|_\infty \sin^2\left(\frac{b-a}{4}\right)\end{aligned} \tag{2.28}$$

for any $s \in [a,b]$ where $\|w\|_\infty := ess\sup_{t\in[a,b]} |w(t)|$.

In particular, we have

$$\left| f\left(e^{\frac{a+b}{2}i}\right)\int_a^b w(t)\,dt - \int_a^b f\left(e^{it}\right) w(t)\,dt \right| \leq 8L\|w\|_\infty \sin^2\left(\frac{b-a}{8}\right). \tag{2.29}$$

The case of monotonic integrators is as follows:

Theorem 5. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on the circle $\mathcal{C}(0,1)$. If $[a,b] \subseteq [0,2\pi]$ and the function $u : [a,b] \to \mathbb{R}$ is* *monotonic nondecreasing on* $[a,b]$, *then*

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \tag{2.30}$$
$$\leq 2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+L\int_a^b sgn(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \tag{2.31}$$
$$\leq 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]$$
$$+L\int_a^b sgn\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt.$$

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\int_a^b|p(t)|\,dv(t). \tag{2.32}$$

Utilising the property (2.32), we have from (2.3) that

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \tag{2.33}$$
$$=\left|\int_a^b\left[f\left(e^{is}\right)-f\left(e^{it}\right)\right]du(t)\right|$$
$$\leq\int_a^b\left|f\left(e^{is}\right)-f\left(e^{it}\right)\right|du(t)\leq L\int_a^b\left|e^{is}-e^{it}\right|du(t)$$

for any $s\in[a,b]$.

Since, by (2.6), $\left|e^{is}-e^{it}\right|=2\left|\sin\left(\frac{s-t}{2}\right)\right|$ for any $t,s\in\mathbb{R}$, then

$$\int_a^b\left|e^{is}-e^{it}\right|du(t) \tag{2.34}$$
$$=2\int_a^b\left|\sin\left(\frac{s-t}{2}\right)\right|du(t)$$
$$=2\left[\int_a^s\sin\left(\frac{s-t}{2}\right)du(t)+\int_s^b\sin\left(\frac{t-s}{2}\right)du(t)\right]$$

for any $s\in[a,b]\subseteq[0,2\pi]$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^s \sin\left(\frac{s-t}{2}\right) du(t) \\
&= \sin\left(\frac{s-t}{2}\right) u(t)\Big|_a^s + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\,dt \\
&= -\sin\left(\frac{s-a}{2}\right) u(a) + \frac{1}{2}\int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^b \sin\left(\frac{t-s}{2}\right) du(t) \\
&= \sin\left(\frac{t-s}{2}\right) u(t)\Big|_s^b - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right) u(t)\,dt \\
&= \sin\left(\frac{b-s}{2}\right) u(b) - \frac{1}{2}\int_s^b \cos\left(\frac{t-s}{2}\right) u(t)\,dt,
\end{aligned}$$

which, by (2.34), produce the equality

$$\begin{aligned}
&\int_a^b \left|e^{is} - e^{it}\right| du(t) \\
&= 2\left[\sin\left(\frac{b-s}{2}\right) u(b) - \sin\left(\frac{s-a}{2}\right) u(a)\right] \\
&+ \int_a^s \cos\left(\frac{s-t}{2}\right) u(t)\,dt - \int_s^b \cos\left(\frac{t-s}{2}\right) u(t)\,dt \\
&= 2\left[\sin\left(\frac{b-s}{2}\right) u(b) - \sin\left(\frac{s-a}{2}\right) u(a)\right] \\
&+ \int_a^b sgn(s-t)\cos\left(\frac{s-t}{2}\right) u(t)\,dt.
\end{aligned} \tag{2.35}$$

Utilising (2.33) we deduce the desired result (2.30). □

REMARK 7. *We remark that if $a = 0$ and $b = 2\pi$, then we get from (2.28) and (2.29) that*

$$\begin{aligned}
&\left|f\left(e^{is}\right)[u(2\pi) - u(0)] - \int_0^{2\pi} f\left(e^{it}\right) du(t)\right| \\
&\le 2L\sin\left(\frac{s}{2}\right)[u(2\pi) - u(0)] \\
&+ L\int_0^{2\pi} sgn(s-t)\cos\left(\frac{s-t}{2}\right) u(t)\,dt
\end{aligned} \tag{2.36}$$

for any $s \in [a, b]$.

In particular, we have

$$\left|f(-1)[u(2\pi)-u(0)]-\int_0^{2\pi} f\left(e^{it}\right)du(t)\right| \tag{2.37}$$
$$\leq \sqrt{2}L[u(2\pi)-u(0)]$$
$$+L\int_0^{2\pi} sgn(\pi-t)\sin\left(\frac{t}{2}\right)u(t)\,dt.$$

Corollary 3. *Assume that f and u are as in Theorem 5 then for any $[a,b]\subset[0,2\pi]$ with $0<b-a\leq\pi$ we have the sequence of inequalities*

$$\left|f\left(e^{is}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \tag{2.38}$$
$$\leq 2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right]$$
$$+L\int_a^b sgn(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt$$
$$\leq 2L\left[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)]+\sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\right]$$
$$=: B(s)$$

where

$$B(s)\leq 2L\times\begin{cases}\sin\left[\frac{1}{4}(b-a)+\frac{1}{2}\left|s-\frac{a+b}{2}\right|\right][u(b)-u(a)]\\[2ex] 2\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)\left[\frac{u(b)-u(a)}{2}+\left|u(s)-\frac{u(b)+u(a)}{2}\right|\right]\end{cases}$$

for any $s\in[a,b]$.

In particular, we have

$$\left|f\left(e^{\frac{a+b}{2}i}\right)[u(b)-u(a)]-\int_a^b f\left(e^{it}\right)du(t)\right| \tag{2.39}$$
$$\leq 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)]$$
$$+L\int_a^b sgn\left(\frac{a+b}{2}-t\right)\cos\left(\frac{\frac{a+b}{2}-t}{2}\right)u(t)\,dt$$
$$=: M.$$

where

$$M\leq 2L\sin\left(\frac{b-a}{4}\right)[u(b)-u(a)].$$

Proof. Since $0<b-a\leq\pi$, then $\frac{|s-t|}{2}\leq\frac{\pi}{2}$ for $s,t\in[a,b]$.

Utilising the fact that u is monotonic nondecreasing on $[a,b]$ and $\cos\left(\frac{|s-t|}{2}\right)\geq 0$ for $s,t\in[a,b]$, then

$$\int_a^s \cos\left(\frac{s-t}{2}\right)u(t)\,dt \leq u(s)\int_a^s \cos\left(\frac{s-t}{2}\right)dt = 2u(s)\sin\left(\frac{s-a}{2}\right) \tag{2.40}$$

and

$$\int_s^b \cos\left(\frac{s-t}{2}\right)u(t)\,dt \geq u(s)\int_s^b \cos\left(\frac{s-t}{2}\right)dt = 2u(s)\sin\left(\frac{b-s}{2}\right)$$

i.e.,

$$-\int_s^b \cos\left(\frac{s-t}{2}\right)u(t)\,dt \leq -2u(s)\sin\left(\frac{b-s}{2}\right). \tag{2.41}$$

Summing (2.40) with (2.41) we deduce that

$$\int_a^b sgn(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \leq 2u(s)\sin\left(\frac{s-a}{2}\right)-2u(s)\sin\left(\frac{b-s}{2}\right)$$

giving that

$$2L\left[\sin\left(\frac{b-s}{2}\right)u(b)-\sin\left(\frac{s-a}{2}\right)u(a)\right] + L\int_a^b sgn(s-t)\cos\left(\frac{s-t}{2}\right)u(t)\,dt \leq 2L\left[\sin\left(\frac{b-s}{2}\right)[u(b)-u(s)]+\sin\left(\frac{s-a}{2}\right)[u(s)-u(a)]\right]$$

which proves the second inequality in (2.38).

The bounds for $B(s)$ follows from the elementary property stating that

$$\alpha x+\beta y\leq \max\{\alpha,\beta\}(x+y)$$

where $\alpha,\beta,x,y\geq 0$. The details are omitted. □

3. A Quadrature Rule

We consider the following partition of the interval $[a,b]$

$$\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k\in[x_k,x_{k+1}]$ where $0\leq k\leq n-1$. Define $h_k := x_{k+1}-x_k$, $0\leq k\leq n-1$ and $\nu(\Delta_n)=\max\{h_k : 0\leq k\leq n-1\}$ the norm of the partition Δ_n.

For the continuous function $f:\mathcal{C}(0,1)\to\mathbb{C}$ and the function $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation on $[a,b]$, define the quadrature rule

$$O_n(f,u,\Delta_n,\xi):=\sum_{k=0}^{n-1}f\left(e^{i\xi_k}\right)[u(x_{k+1})-u(x_k)] \tag{3.1}$$

and the remainder $R_n(f,u,\Delta_n,\xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right)du(t)$ by $O_n(f,u,\Delta_n,\xi)$. Then we have

$$\int_a^b f\left(e^{it}\right)du(t)=O_n(f,u,\Delta_n,\xi)+R_n(f,u,\Delta_n,\xi). \tag{3.2}$$

The following result provides *a priory* bounds for $R_n(f,u,\Delta_n,\xi)$ in several instances of f and u as above.

Proposition 1. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z)-f(w)|\le H|z-w|^r$$

for any $w,z\in\mathcal{C}(0,1)$, where $H>0$ and $r\in(0,1]$ are given.

If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ with the norm $\nu(\Delta_n)\le\pi$ we have the error bound

$$\begin{aligned}
&|R_n(f,u,\Delta_n,\xi)| \\
&\le 2^rH\sum_{k=0}^{n-1}\sin^r\left[\frac{1}{4}(x_{k+1}-x_k)+\frac{1}{2}\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right]\bigvee_{x_k}^{x_{k+1}}(u) \\
&\le 2^rH\sum_{k=0}^{n-1}\sin^r\left[\frac{1}{2}(x_{k+1}-x_k)\right]\bigvee_{x_k}^{x_{k+1}}(u) \\
&\le H\sum_{k=0}^{n-1}(x_{k+1}-x_k)^r\bigvee_{x_k}^{x_{k+1}}(u)\le H\nu^r(\Delta_n)\bigvee_a^b(u)
\end{aligned} \tag{3.3}$$

for any intermediate points $\xi_k\in[x_k,x_{k+1}]$ where $0\le k\le n-1$.

Proof. Since $\nu(\Delta_n)\le\pi$, then on writing inequality (2.9) on each interval $[x_k,x_{k+1}]$ and for any intermediate points $\xi_k\in[x_k,x_{k+1}]$ where $0\le k\le n-1$, we have

$$\begin{aligned}
&\left|f\left(e^{i\xi_k}\right)[u(x_{k+1})-u(x_k)]-\int_{x_k}^{x_{k+1}}f\left(e^{it}\right)du(t)\right| \\
&\le 2^rH\sin^r\left[\frac{1}{4}(x_{k+1}-x_k)+\frac{1}{2}\left|\xi_k-\frac{x_k+x_{k+1}}{2}\right|\right]\bigvee_{x_k}^{x_{k+1}}(u) \\
&\le 2^rH\sin^r\left[\frac{1}{2}(x_{k+1}-x_k)\right]\bigvee_{x_k}^{x_{k+1}}(u)\le H(x_{k+1}-x_k)^r\bigvee_{x_k}^{x_{k+1}}(u)
\end{aligned} \tag{3.4}$$

where for the last inequality we have used the fact that $\sin x\le x$ for $x\in\left[0,\frac{\pi}{2}\right]$.

Summing over k from 0 to $n-1$ in (3.4) and utilizing the generalized triangle inequality, we deduce the first part of (3.3). The second part is obvious. □

Corollary 4. *Assume that* f, u *and* Δ_n *are as in Theorem 1. Define the midpoint trapezoid type quadrature rule by*

$$T_n(f,u,\Delta_n) := \sum_{k=0}^{n-1} f\left(e^{\frac{x_{k+1}+x_k}{2}i}\right)\left[u(x_{k+1}) - u(x_k)\right] \tag{3.5}$$

and the error $E_n(f,u,\Delta_n)$ *by*

$$\int_a^b f\left(e^{it}\right) du(t) = T_n(f,u,\Delta_n) + E_n(f,u,\Delta_n). \tag{3.6}$$

Then we have the error bounds

$$\begin{aligned} &|E_n(f,u,\Delta_n)| \\ &\leq 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}(x_{k+1}-x_k)\right] \bigvee_{x_k}^{x_{k+1}}(u) \\ &\leq \frac{1}{2^r} H \sum_{k=0}^{n-1} (x_{k+1}-x_k)^r \bigvee_{x_k}^{x_{k+1}}(u) \leq \frac{1}{2^r} H \nu^r(\Delta_n) \bigvee_a^b(u). \end{aligned} \tag{3.7}$$

The case of both integrator and integrand being Lipschitzian is incorporated in the following result:

Proposition 2. *Assume that* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the circle* $\mathcal{C}(0,1)$. *If* $[a,b] \subseteq [0,2\pi]$ *and the function* $u : [a,b] \to \mathbb{C}$ *is Lipschitzian with the constant* $K > 0$ *on* $[a,b]$, *then for any partition* $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *we have the error bound*

$$\begin{aligned} &|R_n(f,u,\Delta_n,\xi)| \\ &\leq 4LK \sum_{k=0}^{n-1} \left[\sin^2\left(\frac{\xi_k - x_k}{4}\right) + \sin^2\left(\frac{x_{k+1}-\xi_k}{4}\right)\right] \\ &\leq 8LK \sum_{k=0}^{n-1} \sin^2\left(\frac{x_{k+1}-x_k}{4}\right) \leq \frac{1}{2} LK \sum_{k=0}^{n-1} (x_{k+1}-x_k)^2 \\ &\leq \frac{1}{2} LK (b-a) \nu(\Delta_n) \end{aligned} \tag{3.8}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ *where* $0 \leq k \leq n-1$.

In particular, we have

$$\begin{aligned} |E_n(f,u,\Delta_n)| &\leq 8LK \sum_{k=0}^{n-1} \sin^2\left(\frac{x_{k+1}-x_k}{8}\right) \\ &\leq \frac{1}{8} LK \sum_{k=0}^{n-1} (x_{k+1}-x_k)^2 \leq \frac{1}{8} LK (b-a) \nu(\Delta_n). \end{aligned} \tag{3.9}$$

The proof follows by Theorem 4 and the details are omitted.

Proposition 3. *Assume that* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the circle* $\mathcal{C}(0,1)$. *If* $[a,b] \subseteq [0,2\pi]$ *and the function* $u : [a,b] \to \mathbb{R}$ *is* *monotonic nondecreasing on* $[a,b]$, *then for any partition* $\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ *with the norm* $\nu(\Delta_n) \leq \pi$ *we have the error bound*

$$\begin{aligned}(3.10)\quad & |R_n(f,u,\Delta_n,\xi)| \\ & \leq 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)u(x_{k+1}) - \sin\left(\frac{\xi_k - x_k}{2}\right)u(x_k)\right] \\ & + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} sgn(\xi_k - t)\cos\left(\frac{\xi_k - t}{2}\right)u(t)\,dt \\ & \leq 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{x_{k+1}-\xi_k}{2}\right)[u(x_{k+1}) - u(\xi_k)]\right. \\ & \left. + \sin\left(\frac{\xi_k - x_k}{2}\right)[u(\xi_k) - u(x_k)]\right] \\ & \leq 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(x_{k+1}-x_k) + \frac{1}{2}\left|\xi_k - \frac{x_k + x_{k+1}}{2}\right|\right][u(x_{k+1}) - u(x_k)] \\ & \leq 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(x_{k+1}-x_k)\right][u(x_{k+1}) - u(x_k)] \\ & \leq L\sum_{k=0}^{n-1}(x_{k+1}-x_k)[u(x_{k+1}) - u(x_k)] \leq \nu(\Delta_n)L[u(b) - u(a)]\end{aligned}$$

for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ *where* $0 \leq k \leq n-1$.

In particular, we have

$$\begin{aligned}(3.11)\quad & |E_n(f,u,\Delta_n)| \\ & \leq 2L\sum_{k=0}^{n-1}\sin\left(\frac{x_{k+1}-x_k}{4}\right)[u(x_{k+1}) - u(x_k)] \\ & + L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} sgn\left(\frac{x_k + x_{k+1}}{2} - t\right)\cos\left(\frac{\frac{x_k + x_{k+1}}{2} - t}{2}\right)u(t)\,dt \\ & \leq 2L\sum_{k=0}^{n-1}\sin\left(\frac{x_{k+1}-x_k}{4}\right)[u(x_{k+1}) - u(x_k)] \\ & \leq \frac{1}{2}L\sum_{k=0}^{n-1}(x_{k+1}-x_k)[u(x_{k+1}) - u(x_k)] \\ & \leq \frac{1}{2}L\nu(\Delta_n)[u(b) - u(a)].\end{aligned}$$

The proof follows by Corollary 3 and the details are omitted.

4. Applications for Functions of Unitary Operators

We recall that the bounded linear operator U on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [**16**, p. 275-p. 276]), if U is a unitary operator, then there exists a family of projections $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties

a) $E_\lambda \le E_\mu$ for $0 \le \lambda \le \mu \le 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the identity operator on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \le \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{4.1}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$f(U)x = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda x, \tag{4.2}$$

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{4.3}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2, \tag{4.4}$$

for any $x, y \in H$.

We consider the following partition of the interval $[a, b]$

$$\Delta_n : 0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := \lambda_{k+1} - \lambda_k$, $0 \le k \le n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Δ_n.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums

$$O_n(f, U, \Delta_n, \xi; x, y) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right) \left\langle \left(E_{\lambda_{k+1}} - E_{\lambda_k}\right) x, y\right\rangle \tag{4.5}$$

and

$$T_n(f, U, \Delta_n; x, y) := \sum_{k=0}^{n-1} f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2} i}\right) \left\langle \left(E_{\lambda_{k+1}} - E_{\lambda_k}\right) x, y\right\rangle \tag{4.6}$$

where $x, y \in H$.

Theorem 6. *With the above assumptions for* $U, \{E_\lambda\}_{\lambda\in[0,2\pi]}, \Delta_n$ *with* $\nu(\Delta_n) \le \pi$ *and if* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *satisfies the Hölder's type condition* $|f(z) - f(w)| \le H|z-w|^r$ *for any* $w, z \in \mathcal{C}(0,1)$, *where* $H > 0$ *and* $r \in (0,1]$ *are given, then we have the representation*

$$\langle f(U)x, y\rangle = O_n(f, U, \Delta_n, \xi; x, y) + R_n(f, U, \Delta_n, \xi; x, y) \tag{4.7}$$

with the error $R_n(f,U,\Delta_n,\xi;x,y)$ *satisfying the bounds*

$$
\begin{aligned}
(4.8)\qquad & |R_n(f,U,\Delta_n,\xi;x,y)| \\
& \le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k)+\frac{1}{2}\left|\xi_k-\frac{\lambda_k+\lambda_{k+1}}{2}\right|\right] \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{2}(\lambda_{k+1}-\lambda_k)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le H \sum_{k=0}^{n-1} (\lambda_{k+1}-\lambda_k)^r \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le H\nu^r(\Delta_n) \bigvee_0^{2\pi} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le H\nu^r(\Delta_n)\|x\|\|y\|
\end{aligned}
$$

for any $x,y\in H$ *and the intermediate points* $\xi_k\in[\lambda_k,\lambda_{k+1}]$ *where* $0\le k\le n-1$.

In particular we have

$$\langle f(U)x,y\rangle = T_n(f,U,\Delta_n;x,y)+E_n(f,U,\Delta_n;x,y) \tag{4.9}$$

with the error

$$
\begin{aligned}
(4.10)\qquad & |E_n(f,U,\Delta_n;x,y)| \\
& \le 2^r H \sum_{k=0}^{n-1} \sin^r\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k)\right] \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le \frac{1}{2^r} H \sum_{k=0}^{n-1} (\lambda_{k+1}-\lambda_k)^r \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\
& \le \frac{1}{2^r} H\nu^r(\Delta_n) \bigvee_0^{2\pi} \left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le \frac{1}{2^r} H\nu^r(\Delta_n)\|x\|\|y\|
\end{aligned}
$$

for any $x,y\in H$.

Proof. For given $x,y\in H$, define the function $u(\lambda):=\langle E_\lambda x,y\rangle,\lambda\in[0,2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u) =: \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le \|x\|\|y\|. \tag{4.11}$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px,x\rangle\ge 0$ for any $x\in H$, then the following inequality is a *generalization of the Schwarz inequality* in H

$$|\langle Px,y\rangle|^2 \le \langle Px,x\rangle\langle Py,y\rangle, \tag{4.12}$$

for any $x,y\in H$.

Now, if $d: 0=t_0<t_1<...<t_{n-1}<t_n=2\pi$ is an arbitrary partition of the interval $[0,2\pi]$, then we have by Schwarz's inequality for nonnegative operators

(4.12) that

$$\begin{aligned}(4.13)\qquad &\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\ &=\sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\}\\ &\le\sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=I.\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}(4.14)\qquad I&\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\ &\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\}\\ &=\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2}=\|x\|\,\|y\|\end{aligned}$$

for any $x,y\in H$.

On making use of (4.13) and (4.14) we deduce the desired result (4.11).

Now, applying Proposition 1 to the spectral representation (4.3) we deduce the desired result (4.7) with the error bound (4.8). The details are omitted. □

REMARK 8. *In the case when the partition reduces to the whole interval* $[0,2\pi]$, *then utilizing the inequality (2.7) we deduce the bound*

$$(4.15)\qquad\left|f\left(e^{is}\right)\langle x,y\rangle-\langle f(U)\,x,y\rangle\right|\le 2^rH\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le 2^rH\,\|x\|\,\|y\|$$

for any $s\in[0,2\pi]$ *and any vectors* $x,y\in H$.

In the case when the division is

$$\Delta_2:0=\lambda_0<\lambda_1=\pi<\lambda_2=2\pi$$

and we take the intermediate points $u\in[0,\pi]$ *and* $v\in[\pi,2\pi]$, *then we get from Theorem 6 that*

$$\begin{aligned}(4.16)\qquad &\left|f\left(e^{iu}\right)\langle E_\pi x,y\rangle+f\left(e^{iv}\right)\langle(1_H-E_\pi)\,x,y\rangle-\langle f(U)\,x,y\rangle\right|\\ &\le 2^rH\left[\sin^r\left[\frac{1}{4}\pi+\frac{1}{2}\left|u-\frac{\pi}{2}\right|\right]\bigvee_0^{\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right.\\ &\quad\left.+\sin^r\left[\frac{1}{4}\pi+\frac{1}{2}\left|v-\frac{3\pi}{2}\right|\right]\bigvee_\pi^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]\end{aligned}$$

for any vectors $x,y\in H$.

The best inequality we can get from (4.17) is obtained for $u = \frac{\pi}{2}$ *and* $v = \frac{3\pi}{2}$, *namely*

$$|f(i)\langle E_{\pi}x,y\rangle + f(-i)\langle(1_H - E_{\pi})x,y\rangle - \langle f(U)x,y\rangle| \leq 2^{\frac{r}{2}} H \bigvee_0^{2\pi}(\langle E_{(\cdot)}x,y\rangle) \leq 2^{\frac{r}{2}} H\|x\|\|y\| \tag{4.17}$$

for any vectors $x, y \in H$.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums depending only of one vector $x \in H$

$$\tilde{O}_n(f,U,\Delta_n,\xi;x) := \sum_{k=0}^{n-1} f\left(e^{i\xi_k}\right)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,x\right\rangle \tag{4.18}$$

and

$$\tilde{T}_n(f,U,\Delta_n;x,y) := \sum_{k=0}^{n-1} f\left(e^{\frac{\lambda_{k+1}+\lambda_k}{2}i}\right)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,x\right\rangle. \tag{4.19}$$

Theorem 7. *With the above assumptions for* $U, \{E_\lambda\}_{\lambda\in[0,2\pi]}, \Delta_n$ *with* $\nu(\Delta_n) \leq \pi$ *and, if* $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on the circle* $\mathcal{C}(0,1)$, *then we have the representation*

$$\langle f(U)x,x\rangle = \tilde{O}_n(f,U,\Delta_n,\xi;x) + \tilde{R}_n(f,U,\Delta_n,\xi;x) \tag{4.20}$$

with the error $\tilde{R}_n(f,U,\Delta_n,\xi;x)$ *satisfying the bounds*

$$\begin{aligned}
&\left|\tilde{R}_n(f,U,\Delta_n,\xi;x)\right| \\
&\leq 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\langle E_{\lambda_{k+1}}x,x\rangle - \sin\left(\frac{\xi_k-\lambda_k}{2}\right)\langle E_{\lambda_k}x,x\rangle\right] \\
&+ L\sum_{k=0}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}} sgn(\xi_k - t)\cos\left(\frac{\xi_k - t}{2}\right)\langle E_t x,x\rangle dt \\
&\leq 2L\sum_{k=0}^{n-1}\left[\sin\left(\frac{\lambda_{k+1}-\xi_k}{2}\right)\left[\left\langle\left(E_{\lambda_{k+1}} - E_{\xi_k}\right)x,x\right\rangle\right]\right. \\
&\left.+ \sin\left(\frac{\xi_k-\lambda_k}{2}\right)\left\langle\left(E_{\xi_k} - E_{\lambda_k}\right)x,x\right\rangle\right] \\
&\leq 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{4}(\lambda_{k+1}-\lambda_k) + \frac{1}{2}\left|\xi_k - \frac{\lambda_k+\lambda_{k+1}}{2}\right|\right]\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,x\right\rangle \\
&\leq 2L\sum_{k=0}^{n-1}\sin\left[\frac{1}{2}(\lambda_{k+1}-\lambda_k)\right]\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,x\right\rangle \\
&\leq L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}} - E_{\lambda_k}\right)x,x\right\rangle \leq \nu(\Delta_n)L\|x\|^2
\end{aligned} \tag{4.21}$$

for any $x \in H$ *and the intermediate points* $\xi_k \in [\lambda_k, \lambda_{k+1}]$ *where* $0 \leq k \leq n-1$.

In particular we have

$$\langle f(U)x,x\rangle = \tilde{T}_n(f,U,\Delta_n;x) + \tilde{E}_n(f,U,\Delta_n;x) \tag{4.22}$$

with the error

$$\begin{aligned}
&\left|\tilde{E}_n(f,U,\Delta_n;x)\right| \\
&\leq 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&+L\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} sgn\left(\frac{\lambda_k+\lambda_{k+1}}{2}-t\right)\cos\left(\frac{\frac{\lambda_k+\lambda_{k+1}}{2}-t}{2}\right)\langle E_t x,x\rangle\,dt \\
&\leq 2L\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\leq \frac{1}{2}L\sum_{k=0}^{n-1}(\lambda_{k+1}-\lambda_k)\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,x\right\rangle \\
&\leq \frac{1}{2}L\nu(\Delta_n)\|x\|^2
\end{aligned} \tag{4.23}$$

for any $x\in H$.

The proof follows by Proposition 3 applied for the monotonic nondecreasing function $u(t) := \langle E_t x,x\rangle, t\in[0,2\pi]$.

Remark 9. *We remark that if the partition reduces to the whole interval $[0,2\pi]$ then we get from (2.36) that*

$$\begin{aligned}
&\left|f\left(e^{is}\right)\|x\|^2-\langle f(U)x,x\rangle\right| \\
&\leq 2L\sin\left(\frac{s}{2}\right)\|x\|^2+L\int_0^{2\pi} sgn(s-t)\cos\left(\frac{s-t}{2}\right)\langle E_t x,x\rangle\,dt
\end{aligned} \tag{4.24}$$

for any $s\in[a,b]$ and $x\in H$.

In particular, we have

$$\begin{aligned}
&\left|f(-1)\|x\|^2-\langle f(U)x,x\rangle\right| \\
&\leq \sqrt{2}L\|x\|^2+L\int_0^{2\pi} sgn(\pi-t)\sin\left(\frac{t}{2}\right)\langle E_t x,x\rangle\,dt
\end{aligned} \tag{4.25}$$

for any $x\in H$.

Example 1. *In order to provide some simple examples for the inequalities above we choose two complex functions as follows.*

a) *Consider the power function $f:\mathbb{C}\backslash\{0\}\to\mathbb{C}$, $f(z)=z^m$ where m is a nonzero integer. Then, obviously, for any z,w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality*

$$|f(z)-f(w)|\leq|m||z-w|$$

which shows that f is Lipschitzian with the constant $L=|m|$ on the circle $\mathcal{C}(0,1)$. Then from (4.15), we get for any unitary operator U that

$$\left|e^{ims}\langle x,y\rangle-\langle U^m x,y\rangle\right|\le 2|m|\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le 2|m|\|x\|\|y\| \tag{4.26}$$

for any $s\in[0,2\pi]$ and $x,y\in H$.

Also, from (4.16) and the intermediate points $u\in[0,\pi]$ and $v\in[\pi,2\pi]$, we have for any unitary operator U

$$\begin{aligned}&\left|e^{imu}\langle E_\pi x,y\rangle+e^{imv}\langle(1_H-E_\pi)x,y\rangle-\langle U^m x,y\rangle\right|\\&\le 2|m|\left[\sin\left[\frac{1}{4}\pi+\frac{1}{2}\left|u-\frac{\pi}{2}\right|\right]\bigvee_0^{\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right.\\&\left.+\sin\left[\frac{1}{4}\pi+\frac{1}{2}\left|v-\frac{3\pi}{2}\right|\right]\bigvee_\pi^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]\end{aligned} \tag{4.27}$$

for any vectors $x,y\in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.27) is obtained for $u=\frac{\pi}{2}$ and $v=\frac{3\pi}{2}$, namely

$$\begin{aligned}&\left|i^m\langle E_\pi x,y\rangle+(-i)^m\langle(1_H-E_\pi)x,y\rangle-\langle U^m x,y\rangle\right|\\&\le\sqrt{2}|m|\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\sqrt{2}|m|\|x\|\|y\|,\end{aligned} \tag{4.28}$$

for any vectors $x,y\in H$.

b) *For $a\ne\pm1,0$ consider the function $f:\mathcal{C}(0,1)\to\mathbb{C}$, $f_a(z)=\frac{1}{1-az}$. Observe that*

$$|f_a(z)-f_a(w)|=\frac{|a||z-w|}{|1-az||1-aw|} \tag{4.29}$$

for any $z,w\in\mathcal{C}(0,1)$.

If $z=e^{it}$ with $t\in[0,2\pi]$, then we have

$$\begin{aligned}|1-az|^2&=1-2a\operatorname{Re}(\bar{z})+a^2|z|^2=1-2a\cos t+a^2\\&\ge 1-2|a|+a^2=(1-|a|)^2\end{aligned}$$

therefore

$$\frac{1}{|1-az|}\le\frac{1}{|1-|a||}\ \text{and}\ \frac{1}{|1-aw|}\le\frac{1}{|1-|a||} \tag{4.30}$$

for any $z,w\in\mathcal{C}(0,1)$.

Utilising (4.29) and (4.30) we deduce

$$|f_a(z)-f_a(w)|\le\frac{|a|}{(1-|a|)^2}|z-w| \tag{4.31}$$

for any $z,w\in\mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a=\frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

Applying the inequality (4.15), we get for any unitary operator U that

$$\left|\left(1-ae^{is}\right)^{-1}\langle x,y\rangle-\left\langle\left(1_H-aU\right)^{-1}x,y\right\rangle\right| \le \frac{2|a|}{\left(1-|a|\right)^2}\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\frac{2|a|}{\left(1-|a|\right)^2}\|x\|\,\|y\| \tag{4.32}$$

for any $s\in[0,2\pi]$ and $x,y\in H$.

Also, from (4.16) and the intermediate points $u\in[0,\pi]$ and $v\in[\pi,2\pi]$, we have for any unitary operator U

$$\begin{aligned}&\left|\left(1-ae^{iu}\right)^{-1}\langle E_\pi x,y\rangle+\left(1-ae^{iv}\right)^{-1}\langle\left(1_H-E_\pi\right)x,y\rangle\right.\\&\left.-\left\langle\left(1_H-aU\right)^{-1}x,y\right\rangle\right|\\&\le\frac{2|a|}{\left(1-|a|\right)^2}\left[\sin\left[\frac{1}{4}\pi+\frac{1}{2}\left|u-\frac{\pi}{2}\right|\right]\bigvee_0^{\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right.\\&\left.+\sin\left[\frac{1}{4}\pi+\frac{1}{2}\left|v-\frac{3\pi}{2}\right|\right]\bigvee_\pi^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]\end{aligned} \tag{4.33}$$

for any vectors $x,y\in H$, where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of U.

The best inequality we can get from (4.33) is obtained for $u=\frac{\pi}{2}$ and $v=\frac{3\pi}{2}$, namely

$$\begin{aligned}&\left|\left(1-ai\right)^{-1}\langle E_\pi x,y\rangle+\left(1+ai\right)^{-1}\langle\left(1_H-E_\pi\right)x,y\rangle-\left\langle\left(1_H-aU\right)^{-1}x,y\right\rangle\right|\\&\le\frac{\sqrt{2}|a|}{\left(1-|a|\right)^2}\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\frac{\sqrt{2}|a|}{\left(1-|a|\right)^2}\|x\|\,\|y\|\end{aligned} \tag{4.34}$$

for any vectors $x,y\in H$.

The interested reader may apply the above results for other divisions of the interval $[0,2\pi]$, for instance

$$\Delta_4:0=\lambda_0<\lambda_1=\frac{\pi}{2}<\lambda_2=\pi<\lambda_3=\frac{3\pi}{2}<\lambda_4=2\pi.$$

However, the details are omitted.

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