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BOUNDS FOR A ČEBYŠEV TYPE FUNCTIONAL IN TERMS OF RIEMANN-STIELTJES INTEGRAL

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Abstract. Upper and lower bounds for a Čebyšev type functional in terms of Riemann-Stieltjes integral are given. Applications for functions of selfadjoint operators in Hilbert spaces are also provided.

1. Introduction

In [16], the authors have considered the following functional:

$$D(f;u) := \int_a^b f(x)\,du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{1.1}$$

provided that the Riemann-Stieltjes integral $\int_a^b f(x)\,du(x)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

In [16], the following result in estimating the above functional has been obtained:

Theorem 1. *Let $f,u : [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(x) - u(y)| \le L|x-y| \quad \text{for any} \quad x,y \in [a,b] \quad (L>0) \tag{1.2}$$

and f is Riemann integrable on $[a,b]$.

If $m,M \in \mathbb{R}$ are such that

$$m \le f(x) \le M \quad \text{for any} \quad x \in [a,b], \tag{1.3}$$

then we have the inequality

$$|D(f;u)| \le \frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In [15], the following result complementing the above has been obtained:

Theorem 2. *Let $f,u : [a,b] \to \mathbb{R}$ be such that u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K>0$. Then we have*

$$|D(f;u)| \le \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{1.5}$$

The constant $\frac{1}{2}$ is sharp in the above sense.

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For a function $u : [a,b] \to \mathbb{R}$, define the associated functions Φ, Γ and Δ by:

$$\Phi(t) := \frac{(t-a)\,u(b) + (b-t)\,u(a)}{b-a} - u(t), \quad t \in [a,b]; \tag{1.6}$$
$$\Gamma(t) := (t-a)[u(b)-u(t)] - (b-t)[u(t)-u(a)], \quad t \in [a,b]$$

and

$$\Delta(t) := \frac{u(b)-u(t)}{b-t} - \frac{u(t)-u(a)}{t-a}, \quad t \in (a,b).$$

In [9], the following subsequent bounds for the functional $D(f;u)$ have been pointed out:

Theorem 3. *Let $f, u : [a,b] \to \mathbb{R}$.*

(i) *If f is of bounded variation and u is continuous on $[a,b]$, then*

$$|D(f;u)| \le \begin{cases} \sup_{t\in[a,b]} |\Phi(t)| \bigvee_a^b (f), \\ \frac{1}{b-a} \sup_{t\in[a,b]} |\Gamma(t)| \bigvee_a^b (f), \\ \frac{1}{b-a} \sup_{t\in(a,b)} [(t-a)(b-t)|\Delta(t)|] \bigvee_a^b (f). \end{cases} \tag{1.7}$$

(ii) *If f is $L-$Lipschitzian and u is Riemann integrable on $[a,b]$, then*

$$|D(f;u)| \le \begin{cases} L\int_a^b |\Phi(t)|\, dt, \\ \frac{L}{b-a}\int_a^b |\Gamma(t)|\, dt, \\ \frac{L}{b-a}\int_a^b (t-a)(b-t)|\Delta(t)|\, dt. \end{cases} \tag{1.8}$$

(iii) *If f is monotonic nondecreasing on $[a,b]$ and u is continuous on $[a,b]$, then*

$$|D(f;u)| \le \begin{cases} \int_a^b |\Phi(t)|\, df(t), \\ \frac{1}{b-a}\int_a^b |\Gamma(t)|\, df(t), \\ \frac{1}{b-a}\int_a^b (t-a)(b-t)|\Delta(t)|\, df(t). \end{cases} \tag{1.9}$$

The case of monotonic integrators is incorporated in the following two theorems [9]:

Theorem 4. *Let $f, u : [a,b] \to \mathbb{R}$ be such that f is $L-$Lipschitzian on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, then*

$$|D(f;u)| \le \frac{1}{2}L(b-a)[u(b)-u(a)-K(u)] \tag{1.10}$$
$$\le \frac{1}{2}L(b-a)[u(b)-u(a)],$$

where

$$K(u) := \frac{4}{(b-a)^2}\int_a^b u(x)\left(x - \frac{a+b}{2}\right)dx \ge 0. \tag{1.11}$$

The constant $\frac{1}{2}$ in both inequalities is sharp.

Theorem 5. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is monotonic nondecreasing on $[a,b]$, f is of bounded variation on $[a,b]$ and the Stieltjes integral $\int_a^b f(x)\,du(x)$ exists. Then*

$$|D(f;u)|\le[u(b)-u(a)-Q(u)]\bigvee_a^b(f)\le[u(b)-u(a)]\bigvee_a^b(f), \tag{1.12}$$

where

$$Q(u):=\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(x-\frac{a+b}{2}\right)u(x)\,dx\ge 0. \tag{1.13}$$

The first inequality in (1.12) is sharp.

In the case of convex integrators, the following result may be stated [11]:

Theorem 6. *Let $u:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$. Then*

$$\begin{aligned} 0&\le D(f;u) \\ &\le 2\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \\ &\le\begin{cases}\frac{1}{2}\left[u'_-(b)-u'_+(a)\right]\max\{|f(a)|,|f(b)|\}(b-a); \\ \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[u'_-(b)-u'_+(a)\right]\|f\|_p(b-a)^{\frac{1}{q}} \\ \qquad\qquad\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[u'_-(b)-u'_+(a)\right]\|f\|_1.\end{cases}\end{aligned} \tag{1.14}$$

The following result may be stated as [11]:

Theorem 7. *Let $u:[a,b]\to\mathbb{R}$ be a continuous convex function on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a function of bounded variation on $[a,b]$. Then*

$$|D(f;u)|\le\frac{1}{4}\left[u'_-(b)-u'_+(a)\right](b-a)\bigvee_a^b(f), \tag{1.15}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

For other related results for the functional $D(\cdot;\cdot)$, see [1]-[5], [7]-[14] and [18].

In this paper some new lower and upper bounds for $D(\cdot;\cdot)$ are provided. Applications for functions of selfadjoint operators on complex Hilbert spaces are also given.

2. Some New Bounds

The following lemma may be stated:

Lemma 1. *Let $g:[a,b]\to\mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:*

(i) *The function* $g-\frac{l+L}{2}\cdot\ell$*, where* $\ell(t)=t$*,* $t\in[a,b]$ *is* $\frac{1}{2}(L-l)$*−Lipschitzian;*
(ii) *We have the inequalities*

$$l\leq\frac{g(t)-g(s)}{t-s}\leq L\quad\text{for each}\ \ t,s\in[a,b]\quad\text{with}\ \ t\neq s;\tag{2.1}$$

(iii) *We have the inequalities*

$$l(t-s)\leq g(t)-g(s)\leq L(t-s)\quad\text{for each}\ \ t,s\in[a,b]\quad\text{with}\ \ t>s.\tag{2.2}$$

Following [18], we can introduce the definition of (l,L)-Lipschitzian functions:

Definition 1. *The function* $g:[a,b]\to\mathbb{R}$ *which satisfies one of the equivalent conditions (i) – (iii) from Lemma 1 is said to be* (l,L)*-Lipschitzian on* $[a,b]$*.*

If $L>0$ *and* $l=-L$*, then* $(-L,L)$*−Lipschitzian means* L*-Lipschitzian in the classical sense.*

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l,L)-Lipschitzian functions.

Proposition 1. *Let* $g:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. If* $-\infty<l=\inf_{t\in(a,b)}g'(t)$ *and* $\sup_{t\in(a,b)}g'(t)=L<\infty$*, then* g *is* (l,L)*-Lipschitzian on* $[a,b]$*.*

We have the following result:

Theorem 8. *Let* $u:[a,b]\to\mathbb{R}$ *be a convex function on* $[a,b]$ *and* $f:[a,b]\to\mathbb{R}$ *a* (l,L)*-Lipschitzian function on* $[a,b]$*. Then*

$$\begin{aligned}l\left[\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(t)\,dt\right]&\leq D(f;u)\\&\leq L\left[\frac{u(a)+u(b)}{2}(b-a)-\int_a^b u(t)\,dt\right].\end{aligned}\tag{2.3}$$

The inequalities in (2.3) are sharp.

Proof. Consider the auxiliary function $f_L:[a,b]\to\mathbb{R}$, $f_L=L\ell-f$, where ℓ is the identity function $\ell(t)=t$, $t\in[a,b]$. Since $f:[a,b]\to\mathbb{R}$ a (l,L)-Lipschitzian function on $[a,b]$ then $f(t)-f(s)\leq L(t-s)$ for each $t,s\in[a,b]$ with $t>s$ which shows that f_L is monotonic nondecreasing on $[a,b]$.

Utilizing the first inequality in (1.14) we have

$$0\leq D(L\ell-f,u)=LD(\ell,u)-D(f,u)$$

showing that

$$D(f,u)\leq LD(\ell,u).\tag{2.4}$$

A similar argument applied for the auxiliary function $f_l:[a,b]\to\mathbb{R}$, $f_L=f-l\ell$ produces the reverse inequality

$$lD(\ell,u)\leq D(f,u).\tag{2.5}$$

On the other hand, integrating by parts in the Riemann-Stieltjes integral we have

$$\begin{aligned} D(\ell,u) &= \int_a^b t\,du(t) - \frac{1}{b-a}[u(b)-u(a)]\int_a^b t\,dt \\ &= bu(b) - au(a) - \int_a^b u(t)\,dt - \frac{a+b}{2}[u(b)-u(a)] \\ &= \frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt, \end{aligned}$$

which together with (2.4) and (2.5) produce the desired result (2.3).

If we take $f_0(t) = t$, and $\varepsilon \in (0,1)$ then for each $t,s \in [a,b]$ with $t > s$ we have

$$(1-\varepsilon)(t-s) \le f_0(t) - f_0(s) = t - s \le (1+\varepsilon)(t-s)$$

which shows that f is a $(1-\varepsilon, 1+\varepsilon)$-Lipschitzian function on $[a,b]$.

Assume that there exists $A, B > 0$ such that

$$lABD(\ell,u) \le D(f,u) \le LBD(\ell,u) \tag{2.6}$$

for $u : [a,b] \to \mathbb{R}$ a convex function on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ a (l,L)-Lipschitzian function on $[a,b]$.

If we write the inequality (2.6) for f_0 and u strictly convex, we get

$$(1-\varepsilon)AD(\ell,u) \le D(\ell,u) \le (1+\varepsilon)BD(\ell,u)$$

and dividing by $D(\ell,u) > 0$ we get

$$(1-\varepsilon)A \le 1 \le (1+\varepsilon)B. \tag{2.7}$$

Letting $\varepsilon \to 0+$ in (2.7) we get $A \le 1 \le B$, which proves the sharpness of the inequality (2.3). □

Remark 1. *The double inequality in (2.3) is equivalent with*

$$\begin{aligned} &\left| D(f;u) - \frac{l+L}{2}\left(\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right)\right| \\ &\le \frac{1}{2}(L-l)\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right]. \end{aligned} \tag{2.8}$$

The constant $\frac{1}{2}$ *is best possible.*

Corollary 1. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. If* $-\infty < l = \inf_{t\in(a,b)} f'(t)$ *and* $\sup_{t\in(a,b)} f'(t) = L < \infty$*. If* $u : [a,b] \to \mathbb{R}$ *is a convex function on* $[a,b]$*, then the inequality (2.8) holds true.*

If $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$*, then*

$$|D(f;u)| \le \|f'\|_\infty \left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right]. \tag{2.9}$$

The inequality is sharp.

The proof follows from (2.8) by taking $L = \|f'\|_\infty$ and $l = -\|f'\|_\infty$.

For two Lebesgue integrable functions f and g we can define the Čebyšev functional:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

Corollary 2. *Let $w:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a (l,L)-Lipschitzian function on $[a,b]$. Then*

$$\frac{l}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)\,dt \le C(f,w) \le \frac{L}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)\,dt. \tag{2.10}$$

The inequalities in (2.10) are sharp.

Proof. Choose $u(t) := \int_a^t w(s)\,ds,\ t\in[a,b]$. Since $w:[a,b]\to\mathbb{R}$ is a monotonic nondecreasing function on $[a,b]$, then u is convex on $[a,b]$.

We also have

$$\begin{aligned}\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt &\\ = \frac{1}{2}(b-a)\int_a^b w(s)\,ds - \left[t\int_a^t w(s)\,ds\Big|_a^b - \int_a^b s w(s)\,ds\right] &\\ = \int_a^b \left(s-\frac{a+b}{2}\right) w(s)\,ds.&\end{aligned} \tag{2.11}$$

Writing the inequalities (2.3) for these functions we deduce the desired result (2.10). □

Remark 2. *The inequalities (2.10) are equivalent with*

$$\begin{aligned}&\left|C(f,w) - \frac{l+L}{2}\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)\,dt\right| \\ &\le \frac{1}{2}(L-l)\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)\,dt.\end{aligned} \tag{2.12}$$

The constant $\frac{1}{2}$ is best possible.

If $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < \infty$, then

$$|C(f,w)| \le \|f'\|_\infty \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) w(t)\,dt. \tag{2.13}$$

The inequality is sharp.

Definition 2. *For two constants δ,Δ with $\delta<\Delta$, we say that the function $g:[a,b]\to\mathbb{R}$ is (δ,Δ)-convex (see also* [6] *for more general concepts) if $g-\frac{1}{2}\delta\ell^2$ and $\frac{1}{2}\Delta\ell^2-g$ are convex functions on $[a,b]$.*

It is easy to see that, if g is twice differentiable on (a,b) and the second derivative satisfies the condition

$$\delta \le g''(t) \le \Delta \text{ for any } t\in(a,b),$$

then g is (δ,Δ)-convex.

The following result also holds:

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and for δ, Δ with $\delta < \Delta$, a (δ,Δ)-convex function $u : [a,b] \to \mathbb{R}$. Then we have the double inequality*

$$\delta \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \le D(f;u) \le \Delta \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt. \tag{2.14}$$

The inequalities are sharp.

Proof. Since the function f is monotonic nondecreasing and $u - \frac{1}{2}\delta\ell^2$ is convex, then from the first inequality in (1.14) we have

$$D\left(f; u - \frac{1}{2}\delta\ell^2\right) \ge 0,$$

which is equivalent with

$$\frac{1}{2}\delta D\left(f;\ell^2\right) \le D(f;u).$$

From the convexity of $\frac{1}{2}\Delta\ell^2 - g$ we also have

$$D(f;u) \le \frac{1}{2}\Delta D\left(f;\ell^2\right).$$

However

$$\begin{aligned} D\left(f;\ell^2\right) &= \int_a^b f(t)\,d\ell^2(t) - \frac{\ell^2(b) - \ell^2(a)}{b-a}\int_a^b f(t)\,dt \\ &= 2\int_a^b f(t)\,d(t) - (b+a)\int_a^b f(t)\,dt \\ &= 2\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt. \end{aligned}$$

If we take $u_0(t) := \frac{1}{2}t^2$, and $\varepsilon \in (0,1)$, then for $\delta = 1-\varepsilon$ and $\Delta = 1+\varepsilon$ we have that u_0 is $(1-\varepsilon, 1+\varepsilon)$-convex on $[a,b]$.

Assume that there exists the constants $P, Q > 0$ such that

$$\delta P \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \le D(f;u) \le \Delta Q \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt, \tag{2.15}$$

for $f : [a,b] \to \mathbb{R}$ a monotonic nondecreasing function on $[a,b]$ and (δ,Δ)-convex function $u : [a,b] \to \mathbb{R}$.

Since

$$D(f;u_0) = \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt$$

then by replacing $u_0, \delta = 1-\varepsilon$ and $\Delta = 1+\varepsilon$ in (2.15) we get

$$\begin{aligned} (1-\varepsilon) P \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt &\le \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,d \\ &\le (1+\varepsilon) Q \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt, \end{aligned} \tag{2.16}$$

which by division with $\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt$ that is positive for many functions f (for instance $f(t) = t - \frac{a+b}{2}$), we obtain

$$(1-\varepsilon) P \le 1 \le (1+\varepsilon) Q.$$

Letting $\varepsilon \to 0+$ we deduce $P \leq 1 \leq Q$, and the sharpness of the inequalities are proved. □

Remark 3. *Integrating by parts in the Riemann-Stieltjes integral we have*

$$\begin{aligned} D(f;u) &= f(b)u(b) - f(a)u(a) - \int_a^b u(t)\,df(t) \\ &- \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt \\ &= u(b)\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) + u(a)\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\ &- \int_a^b u(t)\,df(t). \end{aligned} \tag{2.17}$$

The inequality (2.3) is then equivalent with

$$\begin{aligned} & l\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right] \\ &\leq u(b)\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) + u(a)\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\ &- \int_a^b u(t)\,df(t) \\ &\leq L\left[\frac{u(a)+u(b)}{2}(b-a) - \int_a^b u(t)\,dt\right]. \end{aligned} \tag{2.18}$$

while (2.14) is equivalent with

$$\begin{aligned} & \delta\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \\ &\leq u(b)\left(f(b) - \frac{1}{b-a}\int_a^b f(t)\,dt\right) + u(a)\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right) \\ &- \int_a^b u(t)\,df(t) \\ &\leq \Delta\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt. \end{aligned} \tag{2.19}$$

3. Applications for Selfadjoint Operators

Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are summed up in the following fundamental result concerning the spectral decomposition of bounded selfadjoint operators in Hilbert spaces, see for instance [17, p. 256]

Theorem 10 (Spectral Representation Theorem). *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{m-0} = 0, E_M = 1_H$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda. \tag{3.2}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon \tag{3.3}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases} \tag{3.4}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{3.5}$$

where the integral is of Riemann-Stieltjes type.

Corollary 3. *With the assumptions of Theorem 10 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H \tag{3.6}$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\,\rangle \quad \text{for all } x, y \in H. \tag{3.7}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\,\rangle \quad \text{for all } x \in H. \tag{3.8}$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H. \tag{3.9}$$

Utilising the Spectral Representation Theorem we can prove the following inequalities for functions of selfadjoint operators:

Theorem 11. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda | \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. Assume that the function $f : I \to \mathbb{R}$ is differentiable on the interior of I denoted $\mathring{I}$ and $[m, M] \subset \mathring{I}$. If the derivative f' is (δ, Δ)-Lipschitzian with $\delta < \Delta$, then*

$$\begin{aligned}(3.10)\qquad &\frac{1}{2}\delta (M1_H - A)(A - m1_H) \\ &\leq \frac{1}{M-m}[f(M)(A - m1_H) + f(m)(M1_H - A] - f(A) \\ &\leq \frac{1}{2}\Delta (M1_H - A)(A - m1_H)\end{aligned}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ the spectral family of A and $x \in H$. Utilising the inequality (2.10) for the (δ, Δ)-Lipschitzian function f' and the monotonic nondecreasing function $w(t) = \langle E_t x, x\rangle$, $t \in [m - \varepsilon, M]$ for a small positive ε, we have

$$\begin{aligned}(3.11)\qquad &\frac{\delta}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\left(t - \frac{m-\varepsilon+M}{2}\right)\langle E_t x, x\rangle dt \\ &\leq \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M} f'(t)\langle E_t x, x\rangle dt \\ &- \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M} f'(t)\, dt \cdot \frac{1}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\langle E_t x, x\rangle dt \\ &\leq \frac{\Delta}{M-m+\varepsilon}\int_{m-\varepsilon}^{M}\left(t - \frac{a+b}{2}\right) w(t)\, dt.\end{aligned}$$

Letting $\varepsilon \to 0+$ in (3.11) we get

$$\begin{aligned}(3.12)\qquad &\delta\int_{m-0}^{M}\left(t - \frac{m+M}{2}\right)\langle E_t x, x\rangle dt \\ &\leq \int_{m-0}^{M} f'(t)\langle E_t x, x\rangle dt - \frac{1}{M-m}\int_{m-0}^{M} f'(t)\, dt \cdot \int_{m-0}^{M}\langle E_t x, x\rangle dt \\ &\leq \Delta\int_{m-0}^{M}\left(t - \frac{a+b}{2}\right) w(t)\, dt\end{aligned}$$

for any $x \in H$.

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}(3.13)\quad & \int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)\langle E_t x,x\rangle\,dt \\ &= \frac{1}{2}\int_{m-0}^{M}\langle E_t x,x\rangle\,d\left(\left(t-\frac{m+M}{2}\right)^2\right) \\ &= \frac{1}{2}\left[\langle E_t x,x\rangle\left(t-\frac{m+M}{2}\right)^2\bigg|_{m-0}^{M}-\int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)^2 d\left(\langle E_t x,x\rangle\right)\right] \\ &= \frac{1}{2}\left[\|x\|^2\left(\frac{M-m}{2}\right)^2-\int_{m-0}^{M}\left(t-\frac{m+M}{2}\right)^2 d\left(\langle E_t x,x\rangle\right)\right] \\ &= \frac{1}{2}\left[\int_{m-0}^{M}\left[\left(\frac{M-m}{2}\right)^2-\left(t-\frac{m+M}{2}\right)^2\right]d\left(\langle E_t x,x\rangle\right)\right] \\ &= \frac{1}{2}\int_{m-0}^{M}(M-t)(t-m)\,d\left(\langle E_t x,x\rangle\right)=\frac{1}{2}\left\langle\left(M1_H-A\right)\left(A-m1_H\right)x,x\right\rangle\end{aligned}$$

for any $x\in H$.

We also have

$$\begin{aligned}(3.14)\quad \int_{m-0}^{M}f'(t)\langle E_t x,x\rangle\,dt &= f(t)\langle E_t x,x\rangle\big|_{m-0}^{M}-\int_{m-0}^{M}f(t)\,d\left(\langle E_t x,x\rangle\right) \\ &= f(M)\|x\|^2-\int_{m-0}^{M}f(t)\,d\left(\langle E_t x,x\rangle\right) \\ &= \int_{m-0}^{M}\left[f(M)-f(t)\right]d\left(\langle E_t x,x\rangle\right) \\ &= \left\langle\left[f(M)1_H-f(A)\right]x,x\right\rangle\end{aligned}$$

and, similarly

$$(3.15)\quad \int_{m-0}^{M}\langle E_t x,x\rangle\,dt=\left\langle\left(M1_H-A\right)x,x\right\rangle$$

for any $x\in H$.

Utilising (3.14) and (3.15) we have

$$\begin{aligned}(3.16)\quad & \int_{m-0}^{M}f'(t)\langle E_t x,x\rangle\,dt-\frac{1}{M-m}\int_{m-0}^{M}f'(t)\,dt\cdot\int_{m-0}^{M}\langle E_t x,x\rangle\,dt \\ &= \left\langle\left[f(M)1_H-f(A)\right]x,x\right\rangle-\frac{f(M)-f(m)}{M-m}\left\langle\left(M1_H-A\right)x,x\right\rangle \\ &= \left\langle\left[\frac{(M-m)f(M)1_H-\left[f(M)-f(m)\right]\left(M1_H-A\right)}{M-m}-f(A)\right]x,x\right\rangle \\ &= \left\langle\left[\frac{f(m)\left(M1_H-A\right)+f(M)\left(A-m1_H\right)}{M-m}-f(A)\right]x,x\right\rangle\end{aligned}$$

for any $x\in H$.

From (3.12) we deduce the desired result (3.10). □

From Theorem 6, we have for $h:[a,b]\to\mathbb{R}$ a convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$,

$$0\le D(g;h)\le 2\cdot\frac{h'_-(b)-h'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)g(t)\,dt. \tag{3.17}$$

Since, by (2.17) we have

$$\begin{aligned}0&\le D(g;h)\\&=h(b)\left(g(b)-\frac{1}{b-a}\int_a^b g(t)\,dt\right)+h(a)\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g(a)\right)\\&-\int_a^b h(t)\,df(t)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&\int_a^b\left(t-\frac{a+b}{2}\right)g(t)\,dt\\&=\frac{1}{2}\int_a^b g(t)\,d\left[\left(t-\frac{a+b}{2}\right)^2\right]\\&=\frac{1}{2}\left[g(t)\left(t-\frac{a+b}{2}\right)^2\Big|_a^b-\int_a^b\left(t-\frac{a+b}{2}\right)^2dg(t)\right]\\&=\frac{1}{2}\left[[g(b)-g(a)]\left(\frac{b-a}{2}\right)^2-\int_a^b\left(t-\frac{a+b}{2}\right)^2dg(t)\right]\\&=\frac{1}{2}\int_a^b\left[\left(\frac{b-a}{2}\right)^2-\left(t-\frac{a+b}{2}\right)^2\right]dg(t)\\&=\frac{1}{2}\int_a^b(b-t)(t-a)\,dg(t),\end{aligned} \tag{3.19}$$

then by (3.17) we have

$$\begin{aligned}0&\le h(b)\left(g(b)-\frac{1}{b-a}\int_a^b g(t)\,dt\right)+h(a)\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g(a)\right)\\&-\int_a^b h(t)\,df(t)\\&\le\frac{h'_-(b)-h'_+(a)}{b-a}\int_a^b(b-t)(t-a)\,dg(t)\end{aligned} \tag{3.20}$$

We can state the following result as well:

Theorem 12. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Assume that the function $f:I\to\mathbb{R}$ is convex on the interior of I*

denoted $\mathring{I}$ *and* $[m,M]\subset\mathring{I}$. *Then*

$$0\leq\frac{1}{M-m}\left[f\left(M\right)\left(A-m1_{H}\right)+f\left(m\right)\left(M1_{H}-A\right]-f\left(A\right)\right.$$
$$\leq\frac{f_{-}^{\prime}\left(M\right)-f_{+}^{\prime}\left(m\right)}{M-m}\left(M1_{H}-A\right)\left(A-m1_{H}\right). \tag{3.21}$$

The proof follows by (3.20) by choosing $h=f$ and $g=\langle E_{t}x,x\rangle$, $t\in\mathbb{R}$, where $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$ is the spectral family of A.

Consider the exponential function $f:\mathbb{R}\rightarrow\mathbb{R}$, and let A be a bonded self-adjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp\,(A)\}$ and $M=\max\{\lambda\,|\lambda\in Sp\,(A)\}$. Then by (3.10) we have

$$\frac{1}{2}\exp\left(m\right)\left(M1_{H}-A\right)\left(A-m1_{H}\right)$$
$$\leq\frac{1}{M-m}\left[\exp\left(M\right)\left(A-m1_{H}\right)+\exp\left(m\right)\left(M1_{H}-A\right]-\exp\left(A\right)\right.$$
$$\leq\frac{1}{2}\exp\left(M\right)\left(M1_{H}-A\right)\left(A-m1_{H}\right). \tag{3.22}$$

Consider the function $f:[m,M]\rightarrow\mathbb{R}$, $f\left(t\right)=-\ln t$ and $[m,M]\subset\left(0,\infty\right)$. Then by (3.10) we have

$$\frac{1}{2M^{2}}\left(M1_{H}-A\right)\left(A-m1_{H}\right)$$
$$\leq\ln\left(A\right)-\frac{1}{M-m}\left[\ln\left(M\right)\left(A-m1_{H}\right)+\ln\left(m\right)\left(M1_{H}-A\right]\right.$$
$$\leq\frac{1}{2m^{2}}\left(M1_{H}-A\right)\left(A-m1_{H}\right). \tag{3.23}$$

If we take the power function $f:[m,M]\rightarrow\mathbb{R}$, $f\left(t\right)=t^{p},p\geq2$ and $[m,M]\subset[0,\infty)$ then by (3.10) we have

$$\frac{1}{2}p\left(p-1\right)m^{p-2}\left(M1_{H}-A\right)\left(A-m1_{H}\right)$$
$$\leq\frac{1}{M-m}\left[M^{p}\left(A-m1_{H}\right)+m^{p}(M1_{H}-A\right]-A^{p}$$
$$\leq\frac{1}{2}p\left(p-1\right)M^{p-2}\left(M1_{H}-A\right)\left(A-m1_{H}\right). \tag{3.24}$$

Consider the convex function $f:\mathbb{R}\rightarrow\mathbb{R}$, $f\left(t\right)=\left|t-\frac{m+M}{2}\right|$. Utilizing the inequality (3.21) we have

$$0\leq\frac{M-m}{2}-\left|A-\frac{m+M}{2}\right|\leq\frac{2}{M-m}\left(M1_{H}-A\right)\left(A-m1_{H}\right). \tag{3.25}$$

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