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## SOME INEQUALITIES OBTAINED FOR POSITIVE POWER SERIES

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ABSTRACT. The aim of this paper is to provide some inequalities starting from several classical inequalities like Young's inequality, Bergstrom's inequality, Radon's inequality, Heinz's inequality, by using power series.

### 1. Introduction

In order to prove several inequalities starting from several classical inequalities like Young's inequality, Bergstrom's inequality, Radon's inequality, Heinz's inequality, by using power series we need to recall the following results.

If  $x_i \in \mathbb{R}_+$  then a particularization of a theorem given in [10] can be formulated as below and will be used in next section.

**Theorem 1.** ([10]) *If  $n \in \mathbb{N}$ ,  $n \geq 2$ ,  $x_1, x_2, \dots, x_n \in \mathbb{R}_+$ , and  $a_1, a_2, \dots, a_n \in \mathbb{R} \setminus \{0\}$  with  $a_1 + a_2 + \dots + a_n \neq 0$  then,*

$$\begin{aligned} & \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} - \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n} = \\ & = \frac{1}{a_1 + a_2 + \dots + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j}. \end{aligned}$$

The scalar Young inequality says that if  $a, b \geq 0$  and  $0 \leq \nu \leq 1$  then we have

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b$$

with equality if and only if  $a = b$ .

The scalar Heinz's inequality says that if  $a, b \geq 0$  and  $0 \leq \nu \leq 1$  then,

$$a^\nu b^{1-\nu} + a^{1-\nu} b^\nu \leq a + b.$$

The next result is a reverse of an inequality obtained by Kittaneh and Manasrah, see [5] or [2], who obtained a refinement of Heinz inequality.

**Theorem 2.** ([2]) *If  $a, b \geq 0$  and  $0 \leq \nu \leq 1$ , then*

$$(2.3) \quad (a+b)^2 \leq (a^\nu b^{1-\nu} + a^{1-\nu} b^\nu)^2 + 2s_0(a-b)^2,$$

where  $s_0 = \max\{\nu, 1-\nu\}$ .

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**Theorem 3.** ([2]) *If  $a, b \geq 0$  and  $0 \leq \nu \leq 1$  then*

$$(2.2) \quad (\nu a + (1 - \nu)b)^2 \leq (a^\nu b^{1-\nu})^2 + s_0^2(a - b)^2,$$

where  $s_0 = \max\{\nu, 1 - \nu\}$ .

In the next section we will use the following inequality, see [6]:

**Proposition 1.** ([6]) *If  $\{x_1, x_2, \dots, x_p\}$ ,  $x_i \in \mathbb{R}^+$  and  $p$  are real, positive numbers and  $m \in \mathbb{N}$  then we have:*

$$\sum_{i=1}^p x_i^m - (p-1)a^m \leq \left( \sum_{i=1}^p x_i - (p-1)a \right)^m,$$

where  $a = \min\{x_1, x_2, \dots, x_p\}$ .

It is necessary also to recall a refinement of the Kittaneh-Manasrah inequality given by N. Minculete in [7], in some special cases as an application:

**Proposition 2.** *For  $0 < a, b \leq 1$  and  $\lambda \in (0, 1)$  we have:*

$$\begin{aligned} r(\sqrt{a} - \sqrt{b})^2 + A(\lambda)ab \log^2 \left( \frac{a}{b} \right) &\leq \lambda a + (1 - \lambda)b - a^\lambda b^{1-\lambda} \leq \\ &\leq (1 - r)(\sqrt{a} - \sqrt{b})^2 + B(\lambda)ab \log^2 \left( \frac{a}{b} \right), \end{aligned}$$

where  $r = \min\{\lambda, 1 - \lambda\}$ ,  $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$  and  $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$ .

The last result which will be used below was given in [1].

**Theorem 4.** ([1]) *If  $n \in \mathbb{N}^* - \{1\}$ ,  $a, b, x_k \in \mathbb{R}_+$ ,  $k \in \{1, \dots, n\}$ ,  $X_n = \sum_{k=1}^n x_k$  and  $m, t, u \in [1, \infty)$ , such that  $aX_n^t > b \max_{1 \leq k \leq n} x_k^t$ , then:*

$$(3) \quad \sum_{k=1}^n \frac{x_k^m}{(aX_n^t - bx_k^t)^u} \geq \frac{n^{-m+tu+1}}{(an^t - b)^u} X_n^{m-tu}.$$

## 2. Some inequalities deduced using a power series method

Using Theorem 1, see [10] we will give below two inequalities for power series.

**Proposition 3.** *Let  $x_i > 0$ , for all  $i = \overline{1, n}$  and  $a_i \in \mathbb{R}_+ - \{0\}$  with  $\sum_{i=1}^n a_i \neq 0$ . If  $x_i < 1$  for all  $i = \overline{1, n}$  then the following inequality holds:*

$$\begin{aligned} \sum_{i=1}^n \frac{1}{a_i} \cdot \frac{1}{1 - x_i^2} &\geq \frac{1}{\sum_{i=1}^n a_i} \cdot \sum_{1 \leq i < j \leq n} \left( \frac{a_i}{a_j} \cdot \frac{1}{1 - x_j^2} + \frac{a_j}{a_i} \cdot \frac{1}{1 - x_i^2} - 2 \cdot \frac{1}{1 - x_i x_j} \right) + \\ &+ \frac{1}{\sum_{i=1}^n a_i} \cdot \frac{n^2}{1 - \left( \frac{\sum_{i=1}^n x_i}{n} \right)^2}. \end{aligned}$$

*Proof.* Replacing  $x_i$  by  $x_i^l$  when  $x_i > 0$  for  $i = \overline{1, n}$ , in equality

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} - \frac{(x_1 + x_2 + \dots + x_n)^2}{(a_1 + a_2 + \dots + a_n)} = \frac{1}{a_1 + a_2 + \dots + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j},$$

see [10], we obtain,

$$\frac{x_1^{2l}}{a_1} + \frac{x_2^{2l}}{a_2} + \dots + \frac{x_n^{2l}}{a_n} - \frac{(x_1^l + x_2^l + \dots + x_n^l)^2}{(a_1 + a_2 + \dots + a_n)} = \frac{1}{a_1 + a_2 + \dots + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i x_j^l - a_j x_i^l)^2}{a_i a_j}$$

or

$$\begin{aligned} & \frac{x_1^{2l}}{a_1} + \frac{x_2^{2l}}{a_2} + \dots + \frac{x_n^{2l}}{a_n} = \\ & = \frac{(x_1^l + x_2^l + \dots + x_n^l)^2}{(a_1 + a_2 + \dots + a_n)} + \frac{1}{a_1 + a_2 + \dots + a_n} \sum_{1 \leq i < j \leq n} \left( \frac{a_i}{a_j} x_j^{2l} + \frac{a_j}{a_i} x_i^{2l} - 2x_j^l x_i^l \right). \end{aligned}$$

Using as in [8], inequality

$$\left( \frac{x_1 + x_2 + \dots + x_n}{n} \right)^k \leq \frac{1}{n} (x_1^k + x_2^k + \dots + x_n^k)$$

which takes place when  $x_i \geq 0$ ,  $k \in \mathbb{N}^*$  we have,

$$\begin{aligned} \frac{x_1^{2l}}{a_1} + \frac{x_2^{2l}}{a_2} + \dots + \frac{x_n^{2l}}{a_n} & \geq \frac{n^2}{a_1 + a_2 + \dots + a_n} \left( \frac{\sum_{i=1}^n x_i}{n} \right)^{2l} + \\ & + \frac{1}{a_1 + a_2 + \dots + a_n} \sum_{1 \leq i < j \leq n} \left( \frac{a_i}{a_j} x_j^{2l} + \frac{a_j}{a_i} x_i^{2l} - 2x_j^l x_i^l \right). \end{aligned}$$

Summing when  $l \in \{1, 2, \dots, p\}$  and then considering in last inequality  $p \rightarrow \infty$ , we find the inequality from the conclusion, taking into account that  $0 < x_i < 1$ ,  $i \in \{1, 2, \dots, n\}$  involves  $\frac{x_1 + x_2 + \dots + x_n}{n} < 1$  and that warrants the convergence of the respective series.

■

**Theorem 5.** Let the power series  $\sum_{n=1}^{\infty} a'_n x^n$  with  $a'_n \geq 0$ ,  $(\forall) n \in \mathbb{N}^*$  which is convergent and has the sum  $f(x)$ , when  $x \in (-R, R)$ , where  $R = \lim_{n \rightarrow \infty} \frac{a'_n}{a'_{n+1}}$  and  $R \neq 0$ . If  $0 < x_i < \sqrt{R}$ ,  $i \in \{1, \dots, n\}$  then it holds

$$\begin{aligned} \sum_{i=1}^n \frac{1}{a_i} \cdot f(x_i^2) & \geq \frac{1}{\sum_{i=1}^n a_i} \cdot \sum_{1 \leq i < j \leq n} \left[ \frac{a_i}{a_j} \cdot f(x_j^2) + \frac{a_j}{a_i} \cdot f(x_i^2) - 2f(x_i x_j) \right] + \\ & + \frac{n^2}{\sum_{i=1}^n a_i} f \left( \left( \frac{\sum_{i=1}^n x_i}{n} \right)^2 \right). \end{aligned}$$

*Proof.* By the same reason we get,

$$\begin{aligned} \frac{a'_1 x_1^{2l}}{a_1} + \frac{a'_1 x_2^{2l}}{a_2} + \dots + \frac{a'_1 x_n^{2l}}{a_n} & \geq \frac{n^2}{a_1 + a_2 + \dots + a_n} a'_1 \left( \frac{\sum_{i=1}^n x_i}{n} \right)^{2l} + \\ & + \frac{1}{a_1 + a_2 + \dots + a_n} \sum_{1 \leq i < j \leq n} \left( \frac{a_i}{a_j} a'_1 x_j^{2l} + \frac{a_j}{a_i} a'_1 x_i^{2l} - 2a'_1 x_j^l x_i^l \right). \end{aligned}$$

Summing when  $l \in \{1, 2, \dots, p\}$  we can notice that the power series obtained are convergent because  $x_i^2 \in (0, R)$ ,  $x_i x_j \in (0, R)$  and  $\left(\frac{\sum_{i=1}^n x_i}{n}\right)^2 < R$ .

■

The next result is based on the inequality from Proposition 1.

**Theorem 6.** *Let the power series  $\sum_{n=1}^{\infty} a'_n x^n$  with  $a'_n \geq 0$ ,  $(\forall) n \in \mathbb{N}^*$  which is convergent and has the sum  $f(x)$ , when  $x \in (-R, R)$ , where  $R = \lim_{n \rightarrow \infty} \frac{a'_n}{a'_{n+1}}$  and  $R \neq 0$ . If  $\{x_1, x_2, \dots, x_p\}$ ,  $x_i \in \mathbb{R}^+$  are  $p$  real, positive numbers with  $0 < x_i < R$ ,  $i \in \{1, \dots, p\}$  and  $\sum_{i=1}^p x_i < (p-1)a + R$  then we have:*

$$\sum_{i=1}^p f(x_i) - (p-1)f(a) \leq f\left(\sum_{i=1}^p x_i - (p-1)a\right).$$

*Proof.* Using the inequality,

$$\sum_{i=1}^p x_i^m - (p-1)a^m \leq \left(\sum_{i=1}^p x_i - (p-1)a\right)^m,$$

and summing then like below,

$$\sum_{k=0}^m \left(\sum_{i=1}^p a'_k x_i^k - (p-1)a'_k a^k\right) \leq \sum_{k=0}^m a'_k \left(\sum_{i=1}^p x_i - (p-1)a\right)^k$$

we obtain

$$\sum_{i=1}^p \sum_{k=0}^m a'_k x_i^k - (p-1) \sum_{k=0}^m a'_k a^k \leq \sum_{k=0}^m a'_k \left(\sum_{i=1}^p x_i - (p-1)a\right)^k$$

and then when  $m$  tends to infinity we have the inequality from the conclusion. ■

**Remark 1.** *Taking into account the expansions of some well-known power series like  $e^x$ ,  $\cosh x$ ,  $\sinh x$  (for the last two  $x$  must be  $x < 1$ ) we have for the numbers  $\{x_1, x_2, \dots, x_n\}$ ,  $x_i \in \mathbb{R}^+$  which are  $n$  real, positive numbers the inequalities:*

$$\begin{aligned} \sum_{i=1}^n \frac{1}{a_i} \cdot \cosh(x_i^2) &\geq \frac{1}{\sum_{i=1}^n a_i} \cdot \sum_{1 \leq i < j \leq n} \left[ \frac{a_i}{a_j} \cdot \cosh(x_j^2) + \frac{a_j}{a_i} \cdot \cosh(x_i^2) - 2 \cosh(x_i x_j) \right] + \\ (1) \quad &+ \frac{n^2}{\sum_{i=1}^n a_i} \cosh\left(\left(\frac{\sum_{i=1}^n x_i}{n}\right)^2\right). \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^n \frac{1}{a_i} \cdot \sinh(x_i^2) &\geq \frac{1}{\sum_{i=1}^n a_i} \cdot \sum_{1 \leq i < j \leq n} \left[ \frac{a_i}{a_j} \cdot \sinh(x_j^2) + \frac{a_j}{a_i} \cdot \sinh(x_i^2) - 2 \sinh(x_i x_j) \right] + \\ (2) \quad &+ \frac{n^2}{\sum_{i=1}^n a_i} \sinh\left(\left(\frac{\sum_{i=1}^n x_i}{n}\right)^2\right). \end{aligned}$$

$$\sum_{i=1}^n \frac{1}{a_i} \cdot \exp(x_i^2) \geq \frac{1}{\sum_{i=1}^n a_i} \cdot \sum_{1 \leq i < j \leq n} \left[ \frac{a_i}{a_j} \cdot \exp(x_j^2) + \frac{a_j}{a_i} \cdot \exp(x_i^2) - 2 \exp(x_i x_j) \right] +$$

$$(3) \quad + \frac{n^2}{\sum_{i=1}^n a_i} \exp \left( \left( \frac{\sum_{i=1}^n x_i}{n} \right)^2 \right).$$

The following result presents two inequalities for power series starting from inequalities from Theorem 2 and 3.

**Theorem 7.** *Let the power series  $\sum_{n=1}^{\infty} a'_n x^n$  with  $a'_n \geq 0$ ,  $(\forall) n \in \mathbb{N}^*$  which is convergent and has the sum  $f(x)$ , when  $x \in (-R, R)$ , where  $R = \lim_{n \rightarrow \infty} \frac{a'_n}{a'_{n+1}}$  and  $R \neq 0$ .*

*(i) If  $0 \leq a < \sqrt{R}$ ,  $0 \leq b < \sqrt{R}$  and  $0 \leq \nu \leq 1$  then*

$$f(a^2) + f(b^2) \leq f(a^{2\nu} b^{2(1-\nu)}) + f(a^{2(1-\nu)} b^{2\nu}) + 2s_0 (f(a^2) + f(b^2) - 2f(ab)),$$

and

$$4f\left(\left(\frac{a+b}{2}\right)^2\right) \leq f(a^{2\nu} b^{2(1-\nu)}) + f(a^{2(1-\nu)} b^{2\nu}) + 2s_0 (f(a^2) + f(b^2)) + 2(1-s_0)f(ab),$$

where  $s_0 = \max\{\nu, 1-\nu\}$ .

*(ii) If  $0 < a < \sqrt{R}$ ,  $0 < b < \sqrt{R}$  and  $0 \leq \nu \leq 1$  then*

$$\begin{aligned} \nu^2 f(a^2) + (1-\nu)^2 f(b^2) + 2\nu(1-\nu)f(ab) &\leq \\ &\leq f(a^{2\nu} b^{2(1-\nu)}) + s_0^2 (f(a^2) + f(b^2) - 2f(ab)), \end{aligned}$$

where  $s_0 = \max\{\nu, 1-\nu\}$ .

*Proof.* We will use the same method as in previous theorems. (i) Therefore from inequality (2.3) where we replace  $a$  and  $b$  by  $a^l$  and  $b^l$  and multiply by  $a'_l$ ,  $l \in \{1, 2, \dots, n\}$  we have,

$$\sum_{l=0}^n a'_l (a^l + b^l)^2 \leq \sum_{l=0}^n a'_l (a^{l\nu} b^{l(1-\nu)} + a^{l(1-\nu)} b^{l\nu})^2 + \sum_{l=0}^n 2s_0 a'_l (a^l - b^l)^2.$$

By computation we obtain,

$$\sum_{l=0}^n a'_l (a^{2l} + b^{2l}) \leq \sum_{l=0}^n a'_l (a^{2l\nu} b^{2l(1-\nu)} + a^{2l(1-\nu)} b^{2l\nu}) + 2s_0 \sum_{l=0}^n a'_l (a^{2l} + b^{2l} - 2a^l b^l)$$

and when  $n$  tends to infinity, we find the first inequality. For the second one, using the generalized means inequality, we find that

$$4 \sum_{l=0}^n a'_l \left(\frac{a+b}{2}\right)^{2l} \leq \sum_{l=0}^n a'_l (a^{2l\nu} b^{2l(1-\nu)} + a^{2l(1-\nu)} b^{2l\nu} + 2a^l b^l) + 2s_0 \sum_{l=0}^n a'_l (a^{2l} + b^{2l} - 2a^l b^l),$$

and when  $n$  tends to infinity, we obtain the second inequality.

(ii) We will use inequality (2.2) from Theorem 2.1(Theorem 4), see [2] with  $a^l$  instead of  $a$  and  $b^l$  instead of  $b$ , we multiply by  $a'_l$ ,  $l \in \{1, 2, \dots, n\}$  (2.2) and then summing when  $l = \overline{1, n}$  we will obtain:

$$\sum_{l=0}^n a'_l [\nu^2 a^{2l} + (1-\nu)^2 b^{2l} + 2\nu(1-\nu)a^l b^l] \leq \sum_{l=0}^n a'_l [a^{2l\nu} b^{2l(1-\nu)} + s_0^2 (a^{2l} + b^{2l} - 2a^l b^l)].$$

From hypothesis,  $0 < a < \sqrt{R}$  and  $0 < b < \sqrt{R}$  it follows when  $n$  tends to infinity the desired inequality.

■

**Remark 2.** Taking into account the expansions of some well-known power series like  $e^x$ ,  $\cosh x$ ,  $\sinh x$ , (for the last two  $x < 1$ ) if  $\{x_1, x_2, \dots, x_p\}$ ,  $x_i \in \mathbb{R}_+$  are  $p$  real, positive numbers then we have:

$$(4) \quad \sum_{i=1}^p \exp(x_i) - (p-1) \exp(a) \leq \exp\left(\sum_{i=1}^p x_i - (p-1)a\right).$$

$$(5) \quad \sum_{i=1}^p \cosh(x_i) - (p-1) \cosh(a) \leq \cosh\left(\sum_{i=1}^p x_i - (p-1)a\right).$$

$$(6) \quad \sum_{i=1}^p \sinh(x_i) - (p-1) \sinh(a) \leq \sinh\left(\sum_{i=1}^p x_i - (p-1)a\right).$$

**Remark 3.** Taking into account the expansions of some well-known power series like  $e^x$ ,  $\cosh x$ ,  $\sinh x$ , (for the last two  $x < 1$ ) we have for  $a, b > 0$  the variant for  $\sinh$ ,  $\cosh$  of some generalizations of Young's and Heinz's inequalities:

$$(7) \quad \begin{aligned} & \sinh(a^2) + \sinh(b^2) \leq \\ & \leq \sinh(a^{2\nu} b^{2(1-\nu)}) + \sinh(a^{2(1-\nu)} b^{2\nu}) + 2s_0 (\sinh(a^2) + \sinh(b^2) - 2\sinh(ab)), \end{aligned}$$

The following result will give an inequality obtained for real power series by using the inequality from Proposition 2.

**Theorem 8.** Let the power series  $\sum_{n=1}^{\infty} a'_n x^n$  with  $a'_n \geq 0$ ,  $(\forall) n \in \mathbb{N}^*$  which is convergent and has the sum  $f(x)$ , when  $x \in (-R, R)$ , where  $R = \lim_{n \rightarrow \infty} \frac{a'_n}{a'_{n+1}}$  and  $R \neq 0$ .

For  $0 < a, b < R$  and  $\lambda \in (0, 1)$  the following inequality holds:

$$\begin{aligned} r[f(a) + f(b) - 2f(a^{\frac{1}{2}} b^{\frac{1}{2}})] + A(\lambda)S(ab) \log^2\left(\frac{a}{b}\right) & \leq \\ & \leq \lambda f(a) + (1-\lambda)f(b) - f(a^\lambda b^{1-\lambda}) \leq \\ & \leq (1-r)[f(a) + f(b) - 2f(a^{\frac{1}{2}} b^{\frac{1}{2}})] + B(\lambda)S(ab) \log^2\left(\frac{a}{b}\right), \end{aligned}$$

where  $r = \min\{\lambda, 1-\lambda\}$ ,  $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ ,  $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$  and

$$S(x) = x(f'(x) + xf''(x))$$

*Proof.* We put  $a^l$  instead of  $a$  and  $b^l$  instead  $b$  in inequality from Proposition 2, see [7] and obtain,

$$\begin{aligned} r a'_l (\sqrt{a^l} - \sqrt{b^l})^2 + a'_l \log^2\left(\frac{a}{b}\right) A(\lambda) l^2 a^l b^l & \leq \\ & \leq \lambda a'_l a^l + (1-\lambda) a'_l b^l - a'_l a^{\lambda l} b^{l(1-\lambda)} \leq \\ & \leq (1-r) a'_l (\sqrt{a^l} - \sqrt{b^l})^2 + a'_l \log^2\left(\frac{a}{b}\right) B(\lambda) l^2 a^l b^l. \end{aligned}$$

When  $l = \overline{1, n}$  we have,

$$r \sum_{l=0}^n a'_l (\sqrt{a^l} - \sqrt{b^l})^2 + \log^2\left(\frac{a}{b}\right) A(\lambda) \sum_{l=0}^n a'_l l^2 a^l b^l \leq$$

$$\begin{aligned}
&\leq \lambda \sum_{l=0}^n a'_l a^l + (1-\lambda) \sum_{l=0}^n a'_l b^l - \sum_{l=0}^n a'_l a^{\lambda l} b^{l(1-\lambda)} \leq \\
&\leq (1-r) \sum_{l=0}^n a'_l (\sqrt{a^l} - \sqrt{b^l})^2 + \log^2\left(\frac{a}{b}\right) B(\lambda) \sum_{l=0}^n a'_l l^2 a^l b^l.
\end{aligned}$$

Therefore if  $n$  tends to infinity, we have

$$\begin{aligned}
&r[f(a) + f(b) - 2f(a^{\frac{1}{2}}b^{\frac{1}{2}})] + A(\lambda)S(ab) \log^2\left(\frac{a}{b}\right) \leq \\
&\leq \lambda f(a) + (1-\lambda)f(b) - f(a^\lambda b^{1-\lambda}) \leq \\
&\leq (1-r)[f(a) + f(b) - 2f(a^{\frac{1}{2}}b^{\frac{1}{2}})] + B(\lambda)S(ab) \log^2\left(\frac{a}{b}\right),
\end{aligned}$$

where  $S(x)$  is the sum of the series  $\sum_{l=1}^{\infty} a'_l l^2 x^l$ . This series has the same convergence radius,  $R$  as the power series  $\sum_{l=1}^{\infty} a'_l x^l$  which has the sum  $f(x)$ . In order to compute this sum, we denote by  $D(x)$  the sum of the convergent series  $\sum_{l=1}^{\infty} a'_l l^2 x^{l-1}$  with the same radius  $R$ . Then

$$S(x) = xD(x)$$

and we denote by  $F(x)$ ,

$$\int D(x)dx = \sum_{l=1}^{\infty} a'_l l x^l.$$

We also denote by  $K(x)$  the sum of the convergent series  $\sum_{l=1}^{\infty} a'_l l x^{l-1}$  with the same radius  $R$  and we notice that  $F(x) = xK(x)$ . Because

$$\int K(x)dx = \sum_{l=1}^{\infty} a'_l x^l = f(x)$$

we have  $K(x) = f'(x)$  and thus  $F(x) = x f'(x)$ . By derivation we have  $D(x) = F'(x) = f'(x) + x f''(x)$  and then  $S(x) = x(f'(x) + x f''(x))$ .

■

Last result was obtained for power series starting from the inequality from Theorem 4.

**Theorem 9.** *Let the power series  $\sum_{n=1}^{\infty} a'_n x^n$  with  $a'_n \geq 0$ ,  $(\forall) n \in \mathbb{N}^*$  which is convergent and has the sum  $f(x)$ , when  $x \in (-R, R)$ , where  $R = \lim_{n \rightarrow \infty} \frac{a'_n}{a'_{n+1}}$  and  $R \neq 0$ .*

*If  $n \in \mathbb{N}^* - \{1\}$ ,  $a, b, x_k \in \mathbb{R}_+^*$ ,  $k \in \{1, \dots, n\}$ ,  $X_n = \sum_{k=1}^n x_k$  and  $t, u \in [1, \infty)$ , such that  $aX_n^t > b \max_{1 \leq k \leq n} x_k^t$ , and  $x_k < R$ ,  $k \in \{1, \dots, n\}$  then:*

$$\sum_{k=1}^n \frac{f(x_k)}{(aX_n^t - bx_k^t)^u} \geq \frac{n^{tu+1}}{(an^t - b)^u X_n^{tu}} f\left(\frac{X_n}{n}\right).$$

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