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TRAPEZOIDAL TYPE INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL VIA ČEBYŠEV FUNCTIONAL WITH APPLICATIONS

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Abstract. Some new inequalities for the functional

$$E_T(f,u)$$
$$:= f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right)$$
$$-\int_a^b f(t)\,du(t),$$

under various assumptions for the functions f and u are given. Applications for functions of selfadjoint operators and unitary operators on complex Hilbert spaces are also provided.

1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [28] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m,M,n,N such that

$$m\le f(t)\le M \quad\text{and}\quad n\le g(t)\le N \quad\text{for a.e. } t\in[a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [5], states that

$$|C(f,g)|\le\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2, \tag{1.4}$$

provided that f',g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f,g:[a,b]\to\mathbb{R}$ are assumed to be absolutely continuous and $f',g'\in L_\infty[a,b]$ while $\|f'\|_\infty=ess\sup_{t\in[a,b]}|f'(t)|$.

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A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [39]:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [32] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f,g are absolutely continuous and $f',g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [3] have proved the following results:

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \tag{1.7}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$ or $p=1$ and $q=\infty$, and

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} ess \sup_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|, \tag{1.8}$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p>1$, $\frac{1}{p}+\frac{1}{q}=1$; $p=1$, $q=\infty$ or $p=\infty, q=1$).

Notice that for $q=\infty, p=1$ in (1.7) we obtain

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\leq \|g\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \end{aligned} \tag{1.9}$$

and if g satisfies (1.3), then

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\leq \left\|g - \frac{n+N}{2}\right\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt. \end{aligned} \tag{1.10}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [30], [35] and [40] and the references therein.

For some recent inequalities for Riemann-Stieltjes integral see [7]-[12] and [31].

In this paper some bounds for the functional

$$\begin{aligned}
&E_T(f,u) \\
&:= f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right) \\
&-\int_a^b f(t)\,du(t),
\end{aligned}$$

under various assumptions for the functions f and u are obtained. Applications for functions of selfadjoint operators and unitary operators on complex Hilbert spaces are also provided.

2. Some Preliminary Results

We start with the following representation:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function and $u:[a,b]\to\mathbb{C}$ a function of bounded variation. Then we have the equalities*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left[\frac{f(b)(t-a)+f(a)(b-t)}{b-a}-f(t)\right]du(t) \\
&=\frac{1}{b-a}\left[f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)\right. \\
&\left.+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t) \\
&=\frac{1}{b-a}\int_a^b f'(t)\,u(t)\,dt-\frac{1}{b-a}\int_a^b f'(t)\,dt\frac{1}{b-a}\int_a^b u(t)\,dt \\
&=C(f',g).
\end{aligned} \tag{2.1}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f'(t)\,u(t)\,dt-\frac{1}{b-a}\int_a^b f'(t)\,dt\frac{1}{b-a}\int_a^b u(t)\,dt \\
&=\frac{1}{b-a}\left[f(t)\,u(t)|_a^b-\int_a^b f(t)\,u(t)\,dt\right] \\
&-\frac{f(b)-f(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b u(t)\,dt \\
&=\frac{f(b)\,u(b)-f(a)\,u(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,u(t)\,d \\
&-\frac{f(b)-f(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b u(t)\,dt \\
&=\frac{1}{b-a}\left[f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)\right. \\
&\left.+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t),
\end{aligned}$$

which proves the second equality in (2.1).

Integrating again by parts, we have

$$\begin{aligned}
& u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt \\
& = u(b)-\frac{1}{b-a}\left[u(t)\,t\big|_a^b-\int_a^b t du(t)\right] \\
& = \frac{u(b)(b-a)-u(b)\,b+u(a)\,a+\int_a^b t du(t)}{b-a} \\
& = \frac{\int_a^b t du(t)-a\left[u(b)-u(a)\right]}{b-a}=\frac{1}{b-a}\int_a^b (t-a)\,du(t)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b u(t)\,dt-u(a) \\
& = \frac{1}{b-a}\left[u(t)\,t\big|_a^b-\int_a^b t du(t)\right]-u(a) \\
& = \frac{u(b)\,b-u(a)\,a-\int_a^b t du(t)-u(a)(b-a)}{b-a} \\
& = \frac{b\left[u(b)-u(a)\right]-\int_a^b t du(t)}{b-a}=\frac{1}{b-a}\int_a^b (b-t)\,du(t).
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{1}{b-a}\left[f(b)\left(u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right)\right. \\
& \left.+f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt-u(a)\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t) \\
& = \frac{1}{b-a}\left[f(b)\frac{1}{b-a}\int_a^b (t-a)\,du(t)\right. \\
& \left.+f(a)\frac{1}{b-a}\int_a^b (b-t)\,du(t)\right]-\frac{1}{b-a}\int_a^b f(t)\,du(t) \\
& = \frac{1}{b-a}\int_a^b\left[\frac{f(b)(t-a)+f(a)(b-t)}{b-a}-f(t)\right]du(t)
\end{aligned}$$

and the first equality in (2.1) is also proved. □

Now, for $\gamma,\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}\,|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0\ \text{ for each }\ t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}\,|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right|\ \text{ for each }\ t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma, \Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \tag{2.2}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$,*we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a,b] \to \mathbb{C} \mid\ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \textit{ for each } t \in [a,b] \}. \end{aligned} \tag{2.3}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a,b] \to \mathbb{C} \mid\ & \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b] \}. \end{aligned} \tag{2.4}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \tag{2.5}$$

Lemma 2. *Let* $f, g : [a,b] \to \mathbb{C}$ *be Lebesgue measurable functions. Then*

$$|C(f,g)| \leq \frac{1}{b-a} \begin{cases} \inf_{\lambda \in \mathbb{C}} \|g - \gamma\|_1 \cdot ess\sup_{t \in [a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| & \begin{array}{l} g \in L_1[a,b], \\ f \in L_\infty[a,b] \end{array} \\ \\ \inf_{\lambda \in \mathbb{C}} \|g - \gamma\|_q \cdot \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|^p dt \right)^{\frac{1}{p}} & \begin{array}{l} g \in L_q[a,b], \\ f \in L_p[a,b], \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\ \\ \inf_{\gamma \in \mathbb{C}} \|g - \gamma\|_\infty \cdot \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt & \begin{array}{l} g \in L_\infty[a,b], \\ f \in L_1[a,b] \end{array} \end{cases} \tag{2.6}$$

Follows by the Sonin's identity for complex valued functions

$$C(f,g)=\frac{1}{b-a}\int_a^b (g(t)-\gamma)\left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right)dt$$

and by the integral Hölder inequality.

Corollary 2. *Let $f,g:[a,b]\to\mathbb{C}$ be Lebesgue measurable functions. If $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, and $g\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, then*

$$\begin{aligned}(2.7)\qquad &|C(f,g)|\\ &\leq\frac{1}{2}|\Gamma-\gamma|\begin{cases} ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f\in L_\infty[a,b]\\[2ex] \left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l}f\in L_p[a,b],\\ p>1,\end{array}\\[2ex] \frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt & f\in L_1[a,b].\end{cases}\end{aligned}$$

Another important corollary is as follows:

Corollary 3. *Let $f,g:[a,b]\to\mathbb{C}$ be Lebesgue measurable functions. If g is of bounded variation, then*

$$\begin{aligned}(2.8)\qquad &|C(f,g)|\\ &\leq\frac{1}{2}\bigvee_a^b(g)\begin{cases} ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f\in L_\infty[a,b]\\[2ex] \left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l}f\in L_p[a,b],\\ p>1,\end{array}\\[2ex] \frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt & f\in L_1[a,b].\end{cases}\end{aligned}$$

Proof. Since g is of bounded variation, then

$$(2.9)\qquad\begin{aligned}\left|g(t)-\frac{g(a)+g(b)}{2}\right| &\leq \frac{1}{2}[|g(b)-g(t)|+|g(t)-g(a)|]\\ &\leq\frac{1}{2}\bigvee_a^b(g)\end{aligned}$$

for any $t\in[a,b]$.

We have

$$\begin{aligned}\left\|g(\cdot)-\frac{g(a)+g(b)}{2}\right\|_\infty &= ess\sup_{t\in[a,b]}\left|g(t)-\frac{g(a)+g(b)}{2}\right|\\ &\leq\frac{1}{2}\bigvee_a^b(g)\end{aligned}$$

and

$$\begin{aligned}\left\Vert g\left(\cdot\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right\Vert_{q} &= \left(\int_{a}^{b}\left\vert g\left(t\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right\vert^{q}dt\right)^{1/q}\\ &\leq \frac{1}{2}\bigvee_{a}^{b}\left(g\right)\left(\int_{a}^{b}dt\right)^{1/q}=\frac{1}{2}\left(b-a\right)^{1/q}\bigvee_{a}^{b}\left(g\right)\end{aligned}$$

for $q\geq1$.

Utilising (2.6) we get (2.8). □

For functions h that are *Lipschitzian in the middle point* with the constant $L_{\frac{a+b}{2}}$ and the exponent $s>0$, i.e., satisfying the condition

$$\left\vert h\left(t\right)-h\left(\frac{a+b}{2}\right)\right\vert \leq L_{\frac{a+b}{2}}\left\vert t-\frac{a+b}{2}\right\vert^{s}$$

for any $t\in\left[a,b\right]$, we have the following result as well.

Another important corollary is as follows:

Corollary 4. *Let $f,g:\left[a,b\right]\rightarrow\mathbb{C}$ be Lebesgue measurable functions. If g is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $s>0$, then*

$$\left\vert C\left(f,g\right)\right\vert \leq\frac{1}{2^{s}}L_{\frac{a+b}{2}} \tag{2.10}$$
$$\times\begin{cases}\frac{(b-a)^{s}}{s+1}\cdot ess\sup_{t\in[a,b]}\left\vert f(t)-\frac{1}{b-a}\int_{a}^{b}f\left(s\right)ds\right\vert & f\in L_{\infty}\left[a,b\right]\\[2ex] \frac{(b-a)^{s-\frac{1}{p}}}{(sq+1)^{1/q}}\cdot\left(\int_{a}^{b}\left\vert f(t)-\frac{1}{b-a}\int_{a}^{b}f\left(s\right)ds\right\vert^{p}dt\right)^{\frac{1}{p}} & \begin{array}{l}f\in L_{p}\left[a,b\right],\\ p>1,\\ \frac{1}{p}+\frac{1}{q}=1\end{array}\\[2ex] \left(b-a\right)^{s-1}\cdot\int_{a}^{b}\left\vert f(t)-\frac{1}{b-a}\int_{a}^{b}f\left(s\right)ds\right\vert dt & f\in L_{1}\left[a,b\right].\end{cases}$$

Proof. We have, for $q\geq1$, that

$$\begin{aligned}\left\Vert g-g\left(\frac{a+b}{2}\right)\right\Vert_{[a,b],q} &= \left(\int_{a}^{b}\left\vert g\left(t\right)-g\left(\frac{a+b}{2}\right)\right\vert^{p}dt\right)^{1/q}\\ &\leq\left(\int_{a}^{b}L_{\frac{a+b}{2}}^{p}\left\vert t-\frac{a+b}{2}\right\vert^{sq}dt\right)^{1/p}\\ &=L_{\frac{a+b}{2}}\left(\int_{a}^{b}\left\vert t-\frac{a+b}{2}\right\vert^{sq}dt\right)^{1/q}.\end{aligned} \tag{2.11}$$

Observe that

$$\begin{aligned}
&\left(\int_a^b \left|t-\frac{a+b}{2}\right|^{sq} dt\right)^{1/q} \\
&=\left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)^{sq} dt+\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{sq} dt\right)^{1/q} \\
&=\left(2\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{sq} dt\right)^{1/q}=\left(2\left.\frac{\left(t-\frac{a+b}{2}\right)^{sq+1}}{sq+1}\right|_{\frac{a+b}{2}}^b\right)^{1/q} \\
&=\left(2\frac{\left(\frac{b-a}{2}\right)^{sq+1}}{sq+1}\right)^{1/q}=\left(\frac{(b-a)^{sq+1}}{2^{sq}(sq+1)}\right)^{1/q}=\frac{(b-a)^{s+1/q}}{2^s(sq+1)^{1/q}}.
\end{aligned}$$

Then by (2.11) we have

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],q}\le L_{\frac{a+b}{2}}\frac{(b-a)^{s+1/q}}{2^s(sq+1)^{1/q}}.$$

Also

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty}\le L_{\frac{a+b}{2}}\frac{(b-a)^{s}}{2^s}.$$

By utilizing the inequality (2.6) we have

$$\begin{aligned}
&|C(f,g)| \\
&\le\frac{1}{b-a}\begin{cases} L_{\frac{a+b}{2}}\frac{(b-a)^{s+1}}{2^s(s+1)}\cdot ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f\in L_\infty[a,b] \\ L_{\frac{a+b}{2}}\frac{(b-a)^{s+1/q}}{2^s(sq+1)^{1/q}}\cdot\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ L_{\frac{a+b}{2}}\frac{(b-a)^{s}}{2^s}\cdot\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt & f\in L_1[a,b]\end{cases} \\
&=\frac{1}{2^s}L_{\frac{a+b}{2}}\begin{cases} \frac{(b-a)^{s}}{s+1}\cdot ess\sup_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| & f\in L_\infty[a,b] \\ \frac{(b-a)^{s}}{(sq+1)^{1/q}}\cdot\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b], \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1\end{array} \\ (b-a)^{s-1}\cdot\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt & f\in L_1[a,b]\end{cases}
\end{aligned}$$

and the corollary is proved. □

Remark 1. *In the case when g is Lipschitzian with the constant $L > 0$, then*

$$|C(f,g)| \leq \frac{1}{2}L \times \begin{cases} \frac{1}{2}(b-a)\cdot ess\sup_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| & f \in L_\infty[a,b] \\ \frac{(b-a)^{1-\frac{1}{p}}}{(q+1)^{1/q}} \cdot \left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f \in L_p[a,b], \\ p > 1, \\ \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt & f \in L_1[a,b]. \end{cases} \tag{2.12}$$

3. Error Bounds for a Generalized Trapezoid Rule

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the generalized trapezoid formula

$$f(b)\left(u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right) + f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right)$$

we consider the error functional

$$\begin{aligned} E_T(f,u) &:= f(b)\left(u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right) + f(a)\left(\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right) \\ &- \int_a^b f(t)\,du(t). \end{aligned} \tag{3.1}$$

For some recent results concerning this functional see [24] and [36].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be absolutely continuous and $u : [a,b] \to \mathbb{C}$ of bounded variation.*

(i) *If $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, and $u \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, then*

$$|E_T(f,u)| \leq \frac{1}{2}|\Gamma - \gamma| \times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]} \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| & f' \in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}} \left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f \in L_p[a,b], \\ p > 1, \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| dt & f \in L_1[a,b]. \end{cases} \tag{3.2}$$

(ii) *If* $\varphi,\Phi\in\mathbb{C}$, $\varphi\neq\Phi$, *and* $f'\in\bar{\Delta}_{[a,b]}(\varphi,\Phi)$, *then*

$$|E_T(f,u)|\leq\frac{1}{2}|\Phi-\varphi| \times\begin{cases}(b-a)\,ess\sup_{t\in[a,b]}\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| & u\in L_\infty[a,b]\\ (b-a)^{\frac{1}{q}}\left(\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & u\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ \int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|dt & u\in L_1[a,b].\end{cases}\tag{3.3}$$

Proof. From Lemma 1 we have the representation

$$E_T(f,u)=(b-a)\,C(f',u).\tag{3.4}$$

(i) If $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, and $u\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, then by Lemma 2 we have

$$|C(f',u)| \leq\frac{1}{2}|\Gamma-\gamma|\begin{cases}ess\sup_{t\in[a,b]}\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right| & f'\in L_\infty[a,b]\\ \left(\frac{1}{b-a}\int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & f\in L_p[a,b],\ p>1,\\ \frac{1}{b-a}\int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|dt & f\in L_1[a,b],\end{cases}$$

which implies the desired result (3.2).

(ii) If $\varphi,\Phi\in\mathbb{C}$, $\varphi\neq\Phi$, and $f'\in\bar{\Delta}_{[a,b]}(\varphi,\Phi)$, then by Lemma 2 we have

$$|C(f',u)| \leq\frac{1}{2}|\Phi-\varphi|\begin{cases}ess\sup_{t\in[a,b]}\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| & u\in L_\infty[a,b]\\ \left(\frac{1}{b-a}\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & u\in L_p[a,b],\ p>1,\\ \frac{1}{b-a}\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|dt & u\in L_1[a,b],\end{cases}$$

which implies the desired result (3.3). □

The following result also holds:

Theorem 2. *Let* $f:[a,b]\to\mathbb{C}$ *be absolutely continuous and* $u:[a,b]\to\mathbb{C}$ *of bounded variation.*

(i) *We have*

$$|E_T(f,u)| \le \frac{1}{2}\bigvee_a^b(u) \tag{3.5}$$
$$\times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]}\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right| & f'\in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}}\left(\int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|dt & f\in L_1[a,b]. \end{cases}$$

(ii) *If f' is of bounded variation, then*

$$|E_T(f,u)| \le \frac{1}{2}\bigvee_a^b(f') \tag{3.6}$$
$$\times \begin{cases} (b-a)\, ess\sup_{t\in[a,b]}\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| & u\in L_\infty[a,b] \\ (b-a)^{\frac{1}{q}}\left(\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} u\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|dt & u\in L_1[a,b]. \end{cases}$$

The proof follows by the identity (3.4) and from Corollary 3. We omit the details.

The case of Lipschitzian functions is as follows:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be absolutely continuous and $u:[a,b]\to\mathbb{C}$ of bounded variation.*

(i) *If u is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $s>0$, then*

$$|E_T(f,u)| \le \frac{1}{2^s}L_{\frac{a+b}{2}} \tag{3.7}$$
$$\times \begin{cases} \frac{(b-a)^s}{s+1} ess\sup_{t\in[a,b]}\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right| & f'\in L_\infty[a,b] \\ \frac{(b-a)^{s-\frac{1}{p}}}{(sq+1)^{1/q}}\left(\int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \int_a^b\left|f'(t)-\frac{f(b)-f(a)}{b-a}\right|dt & f\in L_1[a,b]. \end{cases}$$

(ii) *If f' is Lipschitzian in the middle point with the constant $K_{\frac{a+b}{2}}$ and the exponent $v>0$, then*

$$|E_T(f,u)| \le \frac{1}{2^v} K_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^v}{v+1} ess\sup_{t\in[a,b]} \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| & u \in L_\infty[a,b] \\ \frac{(b-a)^{v-\frac{1}{p}}}{(vq+1)^{1/q}} \left(\int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} u\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| dt & u \in L_1[a,b]. \end{cases} \tag{3.8}$$

The proof follows by Corollary 4.

Remark 2. *If u is Lipschitzian with the constant $L>0$, then*

$$|E_T(f,u)| \le \frac{1}{2} L \times \begin{cases} \frac{1}{2}(b-a)\, ess\sup_{t\in[a,b]} \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| & f' \in L_\infty[a,b] \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{1/q}} \left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} f\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right| dt & f \in L_1[a,b]. \end{cases} \tag{3.9}$$

If f' is Lipschitzian with the constant $K>0$, then

$$|E_T(f,u)| \le \frac{1}{2} K \times \begin{cases} \frac{1}{2}(b-a)\, ess\sup_{t\in[a,b]} \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| & u \in L_\infty[a,b] \\ \frac{(b-a)^{\frac{1}{q}}}{(v+1)^{1/q}} \left(\int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{\frac{1}{p}} & \begin{array}{l} u\in L_p[a,b], \\ p>1, \frac{1}{p}+\frac{1}{q}=1 \end{array} \\ \int_a^b \left|u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds\right| dt & u \in L_1[a,b]. \end{cases} \tag{3.10}$$

4. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{4.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [29, p. 256]:

Theorem 4 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) *$E_\lambda \le E_{\lambda'}$ for $\lambda \le \lambda'$;*
b) *$E_{m-0} = 0, E_M = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;*
c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda'_k)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{4.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5. *With the assumptions of Theorem 4 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{4.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [23].

Lemma 3. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any $x, y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]^2 \leq \langle (E_\beta - E_\alpha)\, x, x\rangle \langle (E_\beta - E_\alpha)\, y, y\rangle\,, \tag{4.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ *denotes the total variation of the function* $\left\langle E_{(\cdot)}x, y\right\rangle$ *on* $[\alpha, \beta]$.

Remark 3. *For $\alpha = m - \varepsilon$ with $\varepsilon > 0$ and $\beta = M$ we get from (3.1) the inequality*

$$\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \langle (I - E_{m-\varepsilon})\, x, x\rangle^{1/2} \langle (I - E_{m-\varepsilon})\, y, y\rangle^{1/2} \tag{4.5}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\|\,\|y\|\,, \tag{4.6}$$

where $\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ *denotes the limit* $\lim_{\varepsilon\to 0+}\left[\bigvee_{m-\varepsilon}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 5. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda\,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda\,|\lambda \in Sp(A)\} =: \max Sp(A)$. If $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and $f : I \to \mathbb{C}$ is absolutely continuous on $[m, M] \subset \mathring{I}$ (the interior of I), then*

$$\begin{aligned}
&\left|\left\langle\left[\frac{f(M)(A - m1_H) + f(m)(M1_H - A)}{M - m} - f(A)\right]x, y\right\rangle\right| \\
&\leq \frac{1}{2}\bigvee_{m-0}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\begin{cases} (M-m)\, ess\sup_{t\in[m,M]}\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| \\ \\ (M-m)^{\frac{1}{q}}\left(\int_m^M\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_m^M\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt\end{cases} \\
&\leq \frac{1}{2}\|x\|\,\|y\|\begin{cases} (M-m)\, ess\sup_{t\in[m,M]}\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| \\ \\ (M-m)^{\frac{1}{q}}\left(\int_m^M\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_m^M\left|f'(t) - \frac{f(M)-f(m)}{M-m}\right| dt\end{cases}
\end{aligned} \tag{4.7}$$

for any $x, y \in H$.

Proof. Utilising the representation (2.1) and the inequality (3.5) we have

$$\left|\int_{m-\varepsilon}^{M}\left[\frac{f(M)(t-m+\varepsilon)+f(m-\varepsilon)(M-t)}{M-m+\varepsilon}-f(t)\right]d\langle E_t x,y\rangle\right|$$
$$\leq \frac{1}{2}\bigvee_{m-\varepsilon}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\begin{cases}(M-m+\varepsilon)\, ess\sup_{t\in[m-\varepsilon,M]}\left|f'(t)-\frac{f(M)-f(m-\varepsilon)}{M-m+\varepsilon}\right|\\ \\ (M-m+\varepsilon)^{\frac{1}{q}}\left(\int_{m-\varepsilon}^{M}\left|f'(t)-\frac{f(M)-f(m-\varepsilon)}{M-m+\varepsilon}\right|^{p}dt\right)^{\frac{1}{p}}\\ \\ \int_{m-\varepsilon}^{M}\left|f'(t)-\frac{f(M)-f(m-\varepsilon)}{M-m+\varepsilon}\right|dt\end{cases}$$

for small $\varepsilon>0$ and for any $x,y\in H$.

Taking the limit over $\varepsilon\to 0+$ and using the continuity of f and the Spectral Representation Theorem, we deduce the desired result (4.7). □

For recent results concerning inequalities for functions of selfadjoint operators, see [1], [14], [15], [16], [17], [18], [19], [23], [33], [37], [38], [41] and the books [21], [22] and [27].

5. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U:H\to H$ on a Hilbert space H satisfying

$$U^{*}U=UU^{*}=1_{H}$$

where U^{*} is the adjoint of U, and $1_{H}:H\to H$ is the identity operator. This property is equivalent to the following:

- (i) U preserves the inner product $\langle\cdot,\cdot\rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux,Uy\rangle=\langle x,y\rangle$ and
- (ii) U is surjective.

The following result is well known [29, p. 275 - p. 276]:

Theorem 6 (Spectral Representation Theorem)**.** *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, called the spectral family of U, with the following properties*

- a) $P_\lambda\leq P_{\lambda'}$ *for* $\lambda\leq\lambda'$;
- b) $P_0=0, P_{2\pi}=I$ *and* $P_{\lambda+0}=P_\lambda$ *for all* $\lambda\in[0,2\pi)$;
- c) *We have the representation*

$$U=\int_0^{2\pi}\exp(i\lambda)\,dP_\lambda.$$

More generally, for every continuous complex-valued function φ defined on the unit circle $\mathcal{C}(0,1)$ and for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left\|\varphi(U)-\sum_{k=1}^{n}\varphi\left(\exp\left(i\lambda_k'\right)\right)\left[P_{\lambda_k}-P_{\lambda_{k-1}}\right]\right\|\leq\varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{5.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 6. *With the assumptions of Theorem 6 for U, P_λ and φ we have the representations*

$$\varphi(U)x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)x, y\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{5.2}$$

In particular,

$$\langle \varphi(U)x, x\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2\, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

The following result holds:

Theorem 7. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U. Let f be a differentiable complex-valued function defined on an open disk containing the unit circle $\mathcal{C}(0,1)$. Then we have*

$$\begin{aligned} &|\langle [2\pi f(1) - f(U)]x, y\rangle| \\ &\le \frac{1}{2}\bigvee_0^{2\pi}\left(\langle P_{(\cdot)}x, y\rangle\right)\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right|; \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}}; \\ \int_0^{2\pi}\left|f'(e^{it})\right| dt; \end{cases} \\ &\le \frac{1}{2}\|x\|\|y\|\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right|; \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}}; \\ \int_0^{2\pi}\left|f'(e^{it})\right| dt, \end{cases} \end{aligned} \tag{5.3}$$

for all $x, y \in H$.

Proof. Utilising the representation (2.1), the inequality (3.5) and the fact that f is differentiable as a complex function, we have

$$\left|\int_0^{2\pi}\left[\frac{f\left(e^{i2\pi}\right)(t-0)+f\left(e^{0}\right)(2\pi-t)}{2\pi}-f\left(e^{it}\right)\right]d\left\langle P_\lambda x,y\right\rangle\right| \tag{5.4}$$
$$\le \frac{1}{2}\bigvee_0^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|ie^{it}f'(e^{it})-\frac{f\left(e^{i2\pi}\right)-f\left(e^{0}\right)}{2\pi}\right| \\ \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|ie^{it}f'(e^{it})-\frac{f\left(e^{i2\pi}\right)-f\left(e^{0}\right)}{2\pi}\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_0^{2\pi}\left|ie^{it}f'(e^{it})-\frac{f\left(e^{i2\pi}\right)-f\left(e^{0}\right)}{2\pi}\right| dt \end{cases}$$

for all $x,y\in H$.

The inequality (5.4) is equivalent with

$$\left|\int_0^{2\pi}\left[f(1)-f\left(e^{it}\right)\right]d\left\langle P_\lambda x,y\right\rangle\right|$$
$$\le \frac{1}{2}\bigvee_0^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\begin{cases} 2\pi ess\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right| \\ \\ (2\pi)^{\frac{1}{q}}\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}} \\ \\ \int_0^{2\pi}\left|f'(e^{it})\right| dt \end{cases}$$

and the desired result (5.3) is proved. □

Remark 4. *Consider the exponential function* $f:\mathbb{C}\to\mathbb{C},\ f(z)=\exp z:=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$. *Then* $f'(z)=\exp z$ *and*

$$\begin{aligned}\left|f'(e^{it})\right| &= \left|\exp(\cos t+i\sin t)\right|=\exp(\cos t)\left|\exp(i\sin t)\right| \\ &= \exp(\cos t)\end{aligned}$$

for $t\in[0,2\pi]$.

Observe that

$$\sup_{t\in[0,2\pi]}\left|f'(e^{it})\right|=e$$

and for $p\ge 1$

$$\left(\int_0^{2\pi}\left|f'(e^{it})\right|^p dt\right)^{\frac{1}{p}}=\left(\int_0^{2\pi}\exp(p\cos t)\,dt\right)^{\frac{1}{p}}=\left[2\pi I_0(p)\right]^{1/p}$$

where I_0 *is the modified Bessel function of the first kind, i.e., we recall that*

$$I_0(z):=\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{z}{2}\right)^{2m},\ z\in\mathbb{C}.$$

Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U. Then we have by (5.3)

$$
\begin{aligned}
&\left|\left\langle\left[2\pi e-\exp\left(U\right)\right]x,y\right\rangle\right| \\
&\leq \pi\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\begin{cases} e; \\ \\ \left(I_0\left(p\right)\right)^{\frac{1}{p}}; \quad p>1 \\ \\ I_0\left(1\right); \end{cases} \\
&\leq \pi\left\|x\right\|\left\|y\right\|\begin{cases} e; \\ \\ \left(I_0\left(p\right)\right)^{\frac{1}{p}}; \quad p>1 \\ \\ I_0\left(1\right), \end{cases}
\end{aligned}
\tag{5.5}
$$

for all $x,y\in H$.

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